APPENDIX 1: Matrices and bases

Just as the notion of inner product space is a very important abstraction of real and complex 3-space, or more generally of N-space, so the various operator-theoretic ideas associated with inner product spaces can be thought of as natural extensions of corresponding results for matrices. Understanding how wavelets are constructed and how they can be used is best understood, however, by being able to switch between the two sets of ideas, so it is useful to look at the general ideas in the context of $\mathbb{C}^N$ and matrices, especially as matrix ideas are probably more familiar at this stage. By the time you've waded through all the matrix notation, however, you'll probably be convinced abstraction brings out the ideas best without cluttering everything up with lots of symbols! In this supplementary material we'll stick to the complex field, but everything goes through virtually unchanged for the real field, omitting the conjugate signs when they occur.

It will be useful to be able to regard vectors as matrices:

$$v \;=\; (v_1,\, v_2,\, \dots\, ,\, v_N) \quad \longleftrightarrow \quad v \;=\; \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix}$$

identifying a vector v in $\mathbb{C}^N$ with the column vector v in $\mathbb{C}^{N\times 1}$. To allow matrix multiplication on the right we shall then identify the linear operator $\mathcal{A} : \mathbb{C}^M \;\to \mathbb{C}^N$ with the $N \times M$ matrix

$$A \;=\; \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1M} \\ a_{21} & a_{22} & \cdots & a_{2M} \\ \vdots & \vdots & \dots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NM} \end{bmatrix}$$

Recall how this identification works: for $v \;=\; (v_1,\, v_2,\, \dots\, ,\, v_M)$ in $\mathbb{C}^M$,

$$\mathcal{A}(v) \;\sim\; \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1M} \\ a_{21} & a_{22} & \cdots & a_{2M} \\ \vdots & \vdots & \dots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NM} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_M \end{bmatrix}.$$

To interpret it in terms of bases and inner products, let $\varepsilon^{(1)},\, \varepsilon^{(2)},\, \dots\, ,\, \varepsilon^{(M)}$ be the standard basis for $\mathbb{C}^M$ and $\varepsilon^{(1)},\, \varepsilon^{(2)},\, \dots\, ,\, \varepsilon^{(N)}$ the corresponding one for $\mathbb{C}^N$. Then

$$\mathcal{A}(\varepsilon^{(m)}) \;=\; \sum_{n=1}^{N} \big(\mathcal{A}(\varepsilon^{(m)}),\, \varepsilon^{(n)}\big)\varepsilon^{(n)} \;=\; \sum_{n=1}^{N} a_{nm}\varepsilon^{(n)}\,,$$

while

$$\mathcal{A}(v) \;=\; \sum_{m=1}^{M} v_m \mathcal{A}(\varepsilon^{(m)}) \;=\; \sum_{n=1}^{N} \Big\{ \sum_{m=1}^{M} a_{nm}\, v_m \Big\} \varepsilon^{(n)} \,.$$

The **adjoint**,

$$A^* \;=\; \overline{A}^t \;=\; \begin{bmatrix} \overline{a_{11}} & \overline{a_{21}} & \dots & \overline{a_{N1}} \\ \overline{a_{12}} & \overline{a_{22}} & \dots & \overline{a_{N2}} \\ \vdots & \vdots & & \vdots \\ \overline{a_{1M}} & \overline{a_{2M}} & \dots & \overline{a_{NM}} \end{bmatrix},$$

of the matrix A has an important operator-theoretic interpretation also, but now as a mapping $\mathcal{A}^* : \mathbb{C}^N \to \mathbb{C}^M$ defined for $\varepsilon^{(n)}$ in $\mathbb{C}^N$ by

$$\mathcal{A}^*(\varepsilon^{(n)}) \;=\; \sum_{m=1}^{M} \overline{a_{nm}}\, \varepsilon^{(m)} \,,$$

and for arbitrary z in $\mathbb{C}^N$ by

$$\mathcal{A}^*(z) \;=\; \sum_{m=1}^{M} \Big\{ \sum_{n=1}^{N} \overline{a_{nm}}\, z_n \Big\} \varepsilon^{(m)} \,.$$

For then,

$$\big(v,\, \mathcal{A}^*(z)\big) = \sum_{m=1}^{M} v_m \,\big(\varepsilon^{(m)},\, \mathcal{A}^*(z)\big) = \sum_{m=1}^{M} v_m \overline{\Big\{ \sum_{n=1}^{N} \overline{a_{nm}}\, z_n \Big\}}$$

$$= \sum_{m=1}^{M} \sum_{n=1}^{N} a_{nm}\, v_m\, \overline{z_n} \;=\; \big(\mathcal{A}(v),\, z\big)$$

for all v in $\mathbb{C}^M$ and z in $\mathbb{C}^N$.

It has been said that wavelets is simply a 'change of basis'; that is true to the same extent that Fourier analysis is only a 'change of basis'. Nonetheless, it will be very instructive to set up the relevant matrix theory from this 'change of basis' point of view . Choose vectors

$$\phi^{(1)} \;=\; \big(\phi_1^{(1)},\, \phi_2^{(1)},\, \dots\,,\, \phi_N^{(1)}\big), \qquad \phi^{(2)} \;=\; \big(\phi_1^{(2)},\, \phi_2^{(2)},\, \dots\,,\, \phi_N^{(2)}\big),$$

$$\dots\,, \quad \phi^{(M)} \;=\; \big(\phi_1^{(M)},\, \phi_2^{(M)},\, \dots\,,\, \phi_N^{(M)}\big)$$

in $\mathbb{C}^N$ which need not be a basis or have any other structure, and denote by Φ the $N \times M$ matrix

$$\Phi \;=\; \left[\, \phi^{(1)} \quad \phi^{(2)} \quad \cdots \quad \phi^{(M)} \,\right] \;=\; \begin{bmatrix} \phi_1^{(1)} & \phi_1^{(2)} & \cdots & \phi_1^{(M)} \\ \phi_2^{(1)} & \phi_2^{(2)} & \cdots & \phi_2^{(M)} \\ \vdots & \vdots & \cdots & \vdots \\ \phi_N^{(1)} & \phi_N^{(2)} & \cdots & \phi_N^{(M)} \end{bmatrix}$$

formed by taking the $\phi^{(m)}$ as columns in Φ. The operator $\mathcal{S}$ associated with this matrix is a mapping $\mathcal{S} : \mathbb{C}^M \;\to\; \mathbb{C}^N$ such that

$$\mathcal{S} : \varepsilon^{(m)} \;\longrightarrow\; \phi^{(m)}\,, \qquad \mathcal{S}(v) \;=\; \sum_{m\,=\,1}^{M} \; v_m\, \phi^{(m)};$$

it is said to be a **Synthesizing Operator** because it builds vectors in $\mathbb{C}^N$ using the $\phi^{(m)}$ as basic building blocks. On the other hand, the adjoint of $\mathcal{S}$ is a mapping $\mathcal{S}^* : \mathbb{C}^N \;\to\; \mathbb{C}^M$ such that

$$\mathcal{S}^*(z) \;=\; \sum_{m\,=\,1}^{M} \left\{ \sum_{n\,=\,1}^{N} \; \overline{\phi_n^{(m)}}\; z_n \right\} \varepsilon^{(m)} \;=\; \sum_{m\,=\,1}^{M} \; \big(z,\, \phi^{(m)}\big)\, \epsilon^{(m)}\,;$$

it is called an **Analyzing Operator** because it measures how well z correlates with each of the $\phi^{(m)}$.

Taking the composition $\mathcal{S} \circ \mathcal{S}^*$ of $\mathcal{S}^*$ and $\mathcal{S}$ we thus obtain the operator

$$\mathcal{S} \circ \mathcal{S}^* :\; \mathbb{C}^N \;\longrightarrow\; \mathbb{C}^N\,, \qquad \big(\mathcal{S} \circ \mathcal{S}^*\big)(z) \;=\; \sum_{m\,=\,1}^{M} \; \big(z,\, \phi^{(m)}\big)\, \phi^{(m)}$$

which we recognize immediately as the series expansion of z with respect to the $\phi^{(m)}$. Notice also that the matrix associated with $\mathcal{S} \circ \mathcal{S}^*$ of $\mathcal{S}^*$ is the matrix product $\Phi\Phi^*$. In particular, we get **Perfect Reconstruction** when

$$z \;=\; \big(\mathcal{S} \circ \mathcal{S}^*\big)(z) \;=\; \sum_{m\,=\,1}^{M} \; \big(z,\, \phi^{(m)}\big)\, \phi^{(m)}\,,$$

i.e., when $\mathcal{S} \circ \mathcal{S}^* \;=\; I$, equivalently when $\Phi\Phi^* \;=\; I$. This is at the heart, not only of wavelets, but also of many areas of mathematics. Let's look at some examples.

(1) *Orthonormal Basis* Recall that a basis $\zeta^{(1)}, \zeta^{(2)}, \ldots, \zeta^{(N)}$ for $\mathbb{C}^N$ is said to be an orthonormal basis when

$$E(\zeta^{(n)}) \;=\; \big(\zeta^{(n)},\, \zeta^{(n)}\big) \;=\; 1, \qquad \big(\zeta^{(j)},\, \zeta^{(k)}\big) \;=\; 0,$$

for all n and $j \neq k$. Now denote by Z the matrix

$$Z \;=\; [\,\zeta^{(1)} \quad \zeta^{(2)} \quad \cdots \quad \zeta^{(M)}\,]$$

as before. Then the matrix associated with $\mathcal{S}^* \circ \mathcal{S} : \mathbb{C}^N \to \mathbb{C}^N$ is the matrix product

$$Z^* Z \;=\; [\,\big(\zeta^{(k)},\, \zeta^{(j)}\big)\,]_{j,k}\;.$$

Consequently, if we recall that an $N \times N$ matrix U is said to be unitary when $U^*U = I$, or equivalently when $UU^* = I$, we have proved

Theorem 1. *The vectors $\zeta^{(1)}, \zeta^{(2)}, \ldots, \zeta^{(N)}$ form an orthonormal basis for $\mathbb{C}^N$ if and only if the associated matrix*

$$Z \;=\; [\,\zeta^{(1)} \quad \zeta^{(2)} \quad \cdots \quad \zeta^{(N)}\,] \;=\; \begin{bmatrix} \zeta_1^{(1)} & \zeta_1^{(2)} & \cdots & \zeta_1^{(N)} \\ \zeta_2^{(1)} & \zeta_2^{(2)} & \cdots & \zeta_2^{(N)} \\ \vdots & \vdots & \cdots & \vdots \\ \zeta_N^{(1)} & \zeta_N^{(2)} & \cdots & \zeta_N^{(M)} \end{bmatrix}$$

is unitary, and then we have perfect reconstruction

$$z \;=\; \sum_{n\,=\,1}^{N} \big(z,\, \zeta^{(n)}\big)\, \zeta^{(n)}\,.$$

(2) *Discrete Fourier Transform*: Set $w = e^{2\pi i/N}$, so that w is an N^{th} root of unity. Then we saw earlier that the Fourier basis

$$\xi^{(0)} \;=\; \frac{1}{\sqrt{N}}\Big(1,\, 1,\, \ldots,\, 1\Big), \qquad \xi^{(1)} \;=\; \frac{1}{\sqrt{N}}\Big(1,\, w,\, \ldots,\, w^{N-1}\Big),$$

$$\xi^{(2)} \;=\; \frac{1}{\sqrt{N}}\Big(1,\, w^2,\, \ldots,\, w^{2(N-1)}\Big), \quad \ldots, \quad \xi^{(N-1)} \;=\; \frac{1}{\sqrt{N}}\Big(1,\, w^{N-1},\, \ldots,\, w^{(N-1)^2}\Big)$$

is an orthonormal basis for $\mathbb{C}^N$. The DFT as we defined it is the mapping $\mathcal{F} : \mathbb{C}^N \to \mathbb{C}^N$ given by

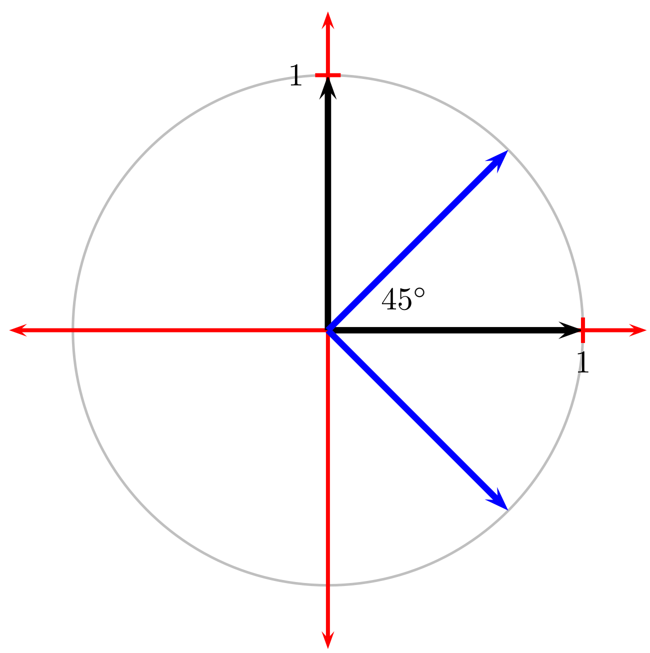
$$\mathcal{F} : x \;\longrightarrow\; X \;=\; \sum_{n\,=\,0}^{N-1} \big(x,\, \xi^{(n)}\big)\, \varepsilon^{(n)}\,,$$

in other words the analyzing mapping adjoint to the change of basis operator $\varepsilon^{(n)} \to \xi^{(n)}$. And that's exactly as it should be because the idea behind taking the Fourier transform is to check how well a given signal correlates with the 'pure tones' $\xi^{(n)}$. In addition, the Fourier representation

$$x \;=\; \sum_{n=0}^{N-1} \big(x,\,\xi^{(n)}\big)\,\xi^{(n)}$$

is simply perfect reconstruction with respect to the orthonormal Fourier basis.

(3) *Haar type bases, block bases*: Now let's start with some matrices which are obviously unitary, even orthogonal, and construct some orthonormal bases that will be the prototype of (discrete) wavelet bases. First some obvious geometry: the figure



shows that

$$\varepsilon^{(1)} \;=\; (1,\,0), \quad \varepsilon^{(2)} \;=\; (1,\,0); \qquad h^{(1)} \;=\; \frac{1}{\sqrt{2}}(1,\,1), \quad \widetilde{h}^{(1)} \;=\; \frac{1}{\sqrt{2}}(1,\,-1)$$

shown respectively in black and blue, are two pairs of orthonormal bases for $\mathbb{R}^2$ and $\mathbb{C}^2$. The first is the standard basis, of course, while the second is the simplest case of the (discrete) Haar basis obtained by rotating the standard basis through 45°. This is the starting point for wavelet theory, but it is very interesting to note that these two bases are uniformly uncorrelated with each other as much as they can be. Some very important work in error-correcting coding now used extensively in cell-phone communications begins here!

We can extend the Haar basis to a block basis for $\mathbb{R}^4$ and $\mathbb{C}^4$ in a natural way because the matrix

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

is obviously unitary (even orthogonal) so its columns provide an orthonormal basis

$$\begin{aligned} h^{(1)} &= \frac{1}{\sqrt{2}}(1,\, 1,\, 0,\, 0), \qquad & \widetilde{h}^{(1)} &= \frac{1}{\sqrt{2}}(1,\, -1,\, 0,\, 0), \\ h^{(2)} &= \frac{1}{\sqrt{2}}(0,\, 0,\, 1,\, 1), \qquad & \widetilde{h}^{(2)} &= \frac{1}{\sqrt{2}}(0,\, 0,\, 1,\, -1) \end{aligned}$$

for $\mathbb{R}^4$ and $\mathbb{C}^4$. It obviously extends to bases for all $\mathbb{R}^{2N}$ and $\mathbb{C}^{2N}$. You've probably used this already when averaging and differencing a signal. Indeed Perfect Reconstruction gives the decomposition

$$z \;=\; \underbrace{\sum_{n=1}^{N} \big(z,\, h^{(n)}\big) h^{(n)}}_{coarse\ details} + \underbrace{\sum_{n=1}^{N} \big(z,\, \widetilde{h}^{(n)}\big) \widetilde{h}^{(n)}}_{fine\ details}$$

of any z in $\mathbb{C}^{2N}$.

(4) *Frames*: Suppose

$$\phi^{(1)},\ \phi^{(2)},\ \ldots,\ \phi^{(M)}$$

are vectors in $\mathbb{C}^N$ with $M > N$. Then these cannot be a basis for $\mathbb{C}^N$ and the matrix

$$\Phi \;=\; \big[\, \phi^{(1)} \quad \phi^{(2)} \quad \cdots \quad \phi^{(M)} \,\big] \;=\; \begin{bmatrix} \phi_1^{(1)} & \phi_1^{(2)} & \cdots & \phi_1^{(M)} \\ \phi_2^{(1)} & \phi_2^{(2)} & \cdots & \phi_2^{(M)} \\ \vdots & \vdots & \cdots & \vdots \\ \phi_N^{(1)} & \phi_N^{(2)} & \cdots & \phi_N^{(M)} \end{bmatrix}$$

is not square - it's $N \times M$, of course - and so can't be unitary or have an inverse. Nonetheless, we can still ask if there exist vectors

$$\psi^{(1)},\ \psi^{(2)},\ \ldots,\ \psi^{(M)}$$

so that the corresponding matrix

$$\Psi \;=\; \big[\, \psi^{(1)} \quad \psi^{(2)} \quad \cdots \quad \psi^{(M)} \,\big] \;=\; \begin{bmatrix} \psi_1^{(1)} & \psi_1^{(2)} & \cdots & \psi_1^{(M)} \\ \psi_2^{(1)} & \psi_2^{(2)} & \cdots & \psi_2^{(M)} \\ \vdots & \vdots & \cdots & \vdots \\ \psi_N^{(1)} & \psi_N^{(2)} & \cdots & \psi_N^{(M)} \end{bmatrix}$$

has the property $\Psi\Phi^* = I$. In this case each z in $\mathbb{C}^N$ admits a representation

$$z \;=\; \sum_{m=1}^{M} \big(z,\,\phi^{(m)}\big)\,\psi^{(m)}\,.$$

The vectors $\phi^{(1)},\ \phi^{(2)},\ \ldots,\ \phi^{(M)}$ are then said to form a **frame**, while the $\psi^{(1)},\ \psi^{(2)},\ \ldots,\ \psi^{(M)}$ form the **dual frame**. (There is a more general definition.)This a very active area of current research in both the finite and infinite cases.

As an interesting example, consider the case of $M = 2N$ where

$$\phi^{(m)} \;=\; \frac{1}{\sqrt{2}}\varepsilon^{(m)}\,, \qquad \phi^{(N+m)} \;=\; \frac{1}{\sqrt{2}}\xi^{(m)}$$

for $1 \le m \le N$. In other words we have concatenated two very different bases for $\mathbb{C}^N$, one localized in *time* the other localized in *frequency*. Each z in $\mathbb{C}^N$ admits 3 different representations:

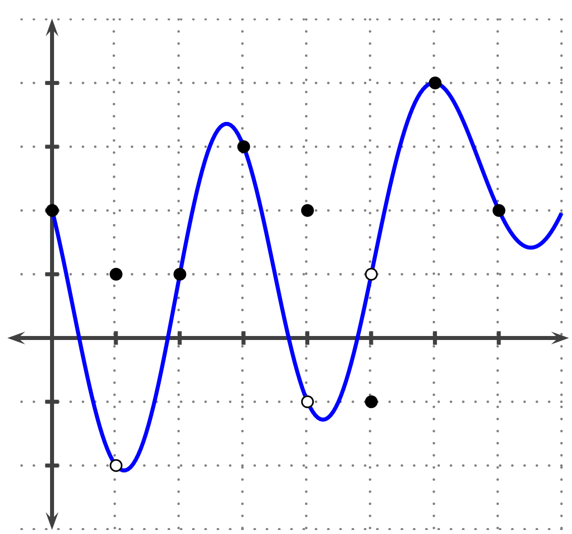
$$z \;=\; \sum_{m=1}^{N} \big(z,\,\varepsilon^{(m)}\big)\,\varepsilon^{(m)}$$

$$z \;=\; \sum_{m=1}^{N} \big(z,\,\xi^{(m)}\big)\,\xi^{(m)}\,,$$

and

$$z \;=\; \frac{1}{2}\Big\{\sum_{m=1}^{N} \big(z,\,\varepsilon^{(m)}\big)\,\varepsilon^{(m)} + \sum_{m=1}^{N} \big(z,\,\xi^{(m)}\big)\,\xi^{(m)}\Big\}\,.$$

Which gives the most compact representation of z? This becomes particularly interesting when z is a *composite* signal



in $\mathbb{C}^8$ where neither the standard basis nor the Fourier basis can be expected to give the most compact representation by itself.

(5) **Problem**: (Uniformly uncorrelated bases) *Find vectors* $\phi^{(jk)}$, $1 \leq j,\, k \leq 4$, *such that each family*

$$\{\phi^{(11)},\ \phi^{(12)},\ \phi^{(13)},\ \phi^{(14)}\}, \qquad \{\phi^{(21)},\ \phi^{(22)},\ \phi^{(23)},\ \phi^{(24)}\},$$
$$\{\phi^{(31)},\ \phi^{(32)},\ \phi^{(33)},\ \phi^{(34)}\}, \qquad \{\phi^{(41)},\ \phi^{(42)},\ \phi^{(43)},\ \phi^{(44)}\},$$

is an orthonormal basis for $\mathbb{C}^4$, *but*

$$\big|\big(\phi^{(jk)},\, \phi^{(\ell m)}\big)\big| = \frac{1}{2}$$

for all $j \neq \ell$ *and all* k, m.

For instance, we might take $\phi^{(1k)} = \varepsilon^{(k)}$ and $\phi^{(2k)} = \xi^{(k)}$ since these two have the uniformly uncorrelated property. But can we find two additional orthonormal bases while still preserving the uniformly uncorrelated property? In the real case this is closely connected with some sophisticated error-correcting codes and signal communication techniques.