APPENDIX 2: Fast Fourier Transform

Matrix Factorizations

Recall the Fourier matrix as an $N \times N$ matrix:

$$\mathcal{F}_N = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & \overline{w} & \overline{w}^2 & \dots & \overline{w}^{N-2} & \overline{w}^{N-1} \\ 1 & \overline{w}^2 & \overline{w}^4 & \dots & \overline{w}^{2(N-2)} & \overline{w}^{2(N-1)} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & \overline{w}^{N-1} & \overline{w}^{2(N-1)} & \dots & \overline{w}^{(N-1)(N-2)} & \overline{w}^{(N-1)^2} \end{bmatrix}$$

We are attempting to represent it as the product of matrices with 2 non-zero entries in each row - at least in the case $N = 2^M$. Just how everything changes for different values of N, say N prime, could be a matter for investigation also. As $\mathcal{F}_N$ is symmetric, we could replace $\overline{w}$ by w if we wanted. This would make the notation and calculations simpler perhaps.

(a) $n = 4$ case. $w = e^{2\pi i/4} = i = \sqrt{-1}$.

$$\mathcal{F}_4 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$$

Idea of FFT is to represent $\mathcal{F}$ as a product of matrix operations in which the rows of the matrices contain only 2 non-zero elements, thereby reducing number of calculations.

$$\mathcal{F}_4 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -i \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & i \end{bmatrix} \times \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

But what's the meaning of this decomposition? The usual explanation is in terms of 'butterflies', but let's compute what the individual matrices in this product actually do to an element of $\ell^2(\mathbb{Z}_4)$:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \\ a-c \\ b-d \end{bmatrix}$$

(b) $n = 8$ case: $w = e^{2\pi i/8} = e^{\pi i/4}$ so that

$$\overline{w}^2 = -i, \quad \overline{w}^3 = -i\overline{w}, \quad \overline{w}^4 = -1, \quad \overline{w}^5 = i\overline{w}, \quad \overline{w}^6 = i, \quad \overline{w}^7 = i\overline{w}.$$

Thus

$$\mathcal{F}_8 = \frac{1}{(\sqrt{2})^3}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \overline{w} & -i & -i\overline{w} & -1 & -\overline{w} & i & i\overline{w} \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & -i\overline{w} & i & \overline{w} & -1 & i\overline{w} & -i & -\overline{w} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\overline{w} & -i & i\overline{w} & -1 & \overline{w} & i & -i\overline{w} \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & i\overline{w} & i & -\overline{w} & -1 & -i\overline{w} & -i & \overline{w} \end{bmatrix}$$

$$\mathcal{F}_8 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \overline{w} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -i\overline{w} \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\overline{w} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & i\overline{w} \end{bmatrix}$$

$$\times \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -i \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & i \end{bmatrix}$$

$$\times \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}.$$