Course:	Introduction to Stochastic Processes
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Lecture 6
Branching processes

6.1 A bit of history

In the mid 19th century several aristocratic families in Victorian England realized that their family names could become extinct. Was it just unfounded paranoia, or did something real prompt them to come to this conclusion? They decided to ask around, and Sir Francis Galton

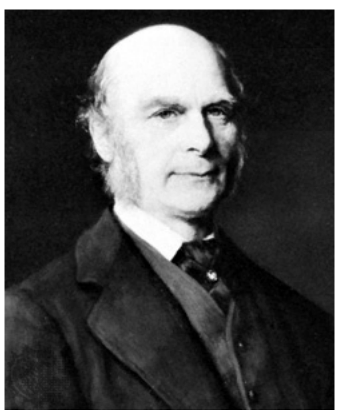

Figure 1. Sir **Francis Galton** (a "polymath, anthropologist, eugenicist, tropical explorer, geographer, inventor, meteorologist, proto-geneticist, psychometrician and statistician" and half-cousin of Charles Darwin) left, and Rev. **Henry William Watson** (a "mathematician, author of mathematics books, ordained priest and a Cambridge Apostle") right.

posed the following question (1873, *Educational Times*):

> *How many male children (on average) must each generation of a family have in order for the family name to continue in perpetuity?*

An answer came from Reverend Henry William Watson soon after, and the two wrote a joint paper entitled *On the probability of extinction of families* in

1874. By the end of this lecture, you will be able to give a precise answer to Galton's question.

While Galton and Watson seem to have received all the laurels for the discovery of branching processes, it turns out they were not the first to introduce them. A similar model was proposed, independently, by Bienaymé in 1845.

Figure 2. Irénée-Jules Bienaymé, "French statistician and a disciple of Laplace"

6.2 The model

The model proposed by Watson was the following:

1. A population starts with one individual at time $n = 0$: $Z_0 = 1$.
2. After one unit of time (at time $n = 1$) the sole individual produces Z_1 identical clones of itself and dies. Z_1 is an $\mathbb{N}_0$-valued random variable.
3. a) If Z_1 happens to be equal to 0 the population is dead and nothing happens at any future time $n \geq 2$.

   b) If $Z_1 > 0$, a unit of time later, each of Z_1 individuals gives birth to a random number of children and dies. The first one has $Z_{1,1}$ children, the second one $Z_{1,2}$ children, etc. The last, Z_1^{th} one, gives birth to Z_{1,Z_1} children. We assume that the distribution of the number of children is the same for each individual in every generation and independent of either the number of individuals in the generation and of the number of children the others have. This distribution, shared by all $Z_{n,i}$ and Z_1, is called the *offspring distribution*. The total number of individuals in the second generation is now

$$Z_2 = \sum_{k=1}^{Z_1} Z_{1,k}.$$

4. The third, fourth, etc. generations are produced in the same way. If it ever happens that $Z_n = 0$, for some n, then $Z_m = 0$ for all $m \geq n$ - the population is extinct. Otherwise,

$$Z_{n+1} = \sum_{k=1}^{Z_n} Z_{n,k}.$$

Definition 6.2.1. A stochastic process with the properties described above is called a **(simple) branching process**.

The actual (physical) mechanism that produces the next generation from the present one can differ from application to application. It is, however, the offspring distribution alone that determines the evolution of a branching process. With this new formalism, we can pose Galton's question more precisely:

> *Under what conditions on the offspring distribution will the process $\{Z_n\}_{n\in\mathbb{N}_0}$ never go extinct, i.e., when is it true that*
>
> $$\mathbb{P}[Z_n \geq 1 \textit{ for all } n \in \mathbb{N}_0] = 1? \tag{6.2.1}$$

How branching processes differ from random walks. We construct a random walk by starting at $X_0 = 0$, tossing a coin, moving up or down according to the outcome of the coin, and repeating the procedure. What makes things especially simple is the fact that it is always the same coin, no matter where we are.

Branching processes are a bit more complicated in two ways:

1. the "coin" is allowed to be more complicated and needs to be replaced by a "super-die" with possible outcomes $0, 1, 2, \dots$ which happen with probabilities $p_0, p_1, \dots$ (the offspring distribution).

2. to determine Z_{n+1} from Z_n, more than one super-die needs to be thrown. In fact, each of Z_n individuals can be thought of as throwing their own, personal, super-die. Moreover, the number of super-dies that will have to be thrown at time n is random and not known in advance.

Mathematically, we deal with this situation by "pre-throwing" sufficiently many super-dies and using their results when needed. Since we have no way of knowing what Z_n is going to be, the only way to guarantee that enough super-dies will have been thrown is to throw infinitely many of them, for each n. More precisely, we assume that we are given a double sequence $\{Z_{n,i}\}_{i\in\mathbb{N}, n\in\mathbb{N}}$ of independent random variables each having the offspring distribution. Most of these will go unused (all but finitely many per row), but it does not matter (this is only a mental game, after all). More concretely, you can think of $Z_{n,i}$ as the number of children the i^{th} individual in the n^{th} generation would have had, had she been born.

To determine the total number of children of the entire generation n (consisting of Z_n individuals) we simply add them all up:

$$Z_{n+1} = \sum_{i=1}^{Z_n} Z_{n,i}.$$

6.3 An approach via generating functions

Having defined and constructed a branching process with offspring distribution $\{Z_n\}_{n\in\mathbb{N}_0}$, let us analyze its probabilistic structure. The first question we need to answer is the following: *What is the distribution of Z_n, for $n \in \mathbb{N}_0$?* It is clear that Z_n must be $\mathbb{N}_0$-valued, so its distribution is completely described by its pmf, which is, in turn, completely determined by its generating function. While an explicit expression for the pmf of Z_n may not be available, its generating function can always be computed:

Proposition 6.3.1. *Let $\{Z_n\}_{n\in\mathbb{N}_0}$ be a branching process, and let the generating function of its offspring distribution $\{p_n\}_{n\in\mathbb{N}_0}$ be given by $P(s)$. Then the generating function of Z_n is the n-fold composition of P with itself, i.e.,*

$$P_{Z_n}(s) = \underbrace{P(P(\dots P(s)\dots))}_{n\ P\text{'s}}, \text{ for } n \geq 1.$$

Proof. For $n = 1$, the distribution of Z_1 is exactly $\{p_n\}_{n\in\mathbb{N}_0}$, so $P_{Z_1} = P(s)$. Suppose that the statement of the proposition holds for some $n \in \mathbb{N}$. Then

$$Z_{n+1} = \sum_{i=1}^{Z_n} Z_{i,n}$$

can be viewed as a random sum of Z_n independent random variables with pmf $\{p_n\}_{n\in\mathbb{N}_0}$, where the number of summands Z_n is independent of $\{Z_{n,i}\}_{i\in\mathbb{N}}$. By Proposition 5.16 in the lecture on generating functions, we have seen that the generating function $P_{Z_{n+1}}$ of Z_{n+1} is a composition of the generating function $P(s)$ of each of the summands and the generating function P_{Z_n} of the random time Z_n. Therefore,

$$P_{Z_{n+1}}(s) = P_{Z_n}(P(s)) = \underbrace{P(P(\dots P(P(s))\dots)))}_{n+1\ \text{P's}},$$

and the full statement of the Proposition follows by induction. □

Let us use Proposition 6.3.1 in some simple examples.

Example 6.3.2. Let $\{Z_n\}_{n\in\mathbb{N}_0}$ be a branching process with offspring distribution $\{p_n\}_{n\in\mathbb{N}_0}$. In the first three examples no randomness occurs and the population growth can be described exactly. In the other examples, more interesting things happen.

1. $p_0 = 1$, $p_n = 0$, $n \in \mathbb{N}$:

   In this case $Z_0 = 1$ and $Z_n = 0$ for all $n \in \mathbb{N}$. This infertile population dies after the first generation.

2. $p_0 = 0, p_1 = 1, p_n = 0$, $n \geq 2$:

   Each individual produces exactly one child before he/she dies. The population size is always 1: $Z_n = 1$, $n \in \mathbb{N}_0$.

3. $p_0 = 0, p_1 = 0, \dots, p_k = 1, p_n = 0$, $n \geq k$, for some $k \geq 2$:

   Here, there are k kids per individual, so the population grows exponentially: $P(s) = s^k$, so $P_{Z_n}(s) = ((\dots(s^k)^k \dots)^k)^k = s^{k^n}$. Therefore, $Z_n = k^n$, for $n \in \mathbb{N}$.

4. $p_0 = p, p_1 = q = 1 - p, p_n = 0$, $n \geq 2$:

   Each individual tosses a (biased) coin and has one child if the outcome is *heads*, or dies childless if the outcome is *tails*. The generating function of the offspring distribution is $P(s) = p + qs$. Therefore,
   $$P_{Z_n}(s) = \underbrace{(p + q(p + q(p + q(\dots(p + qs)))))}_{n \text{ pairs of parentheses}}.$$

   The expression above can be simplified considerably. One needs to realize two things:

   a) After all the products above are expanded, the resulting expression must be of the form $A + Bs$, for some A, B. If you inspect the expression for P_{Z_n} even more closely, you will see that the coefficient B next to s is just q^n.

   b) P_{Z_n} is a generating function of a probability distribution, so $A + B = 1$.

   Therefore,
   $$P_{Z_n}(s) = (1 - q^n) + q^n s.$$

   Of course, the value of Z_n will be equal to 1 if and only if all of the coin-tosses of its ancestors turned out to be *heads*. The probability of that event is q^n. So we didn't need Proposition 6.3.1 after all.

This example can be interpreted alternatively as follows. Each individual has exactly one child, but its gender is determined at random - male with probability q and female with probability p. Assuming that all females change their last name when they marry, and assuming that all of them marry, Z_n is just the number of individuals carrying the family name after n generations.

5. $p_0 = p^2, p_1 = 2pq, p_2 = q^2, p_n = 0, n \geq 3$:

   In this case each individual has exactly two children and their gender is female with probability p and male with probability q, independently of each other. The generating function P of the offspring distribution $\{p_n\}_{n\in\mathbb{N}}$ is given by $P(s) = (p+qs)^2$. Then

$$P_{Z_n} = \underbrace{(p+q(p+q(\dots p+qs)^2 \dots)^2)^2}_{n \text{ pairs of parentheses}}.$$

   Unlike the example above, it is not so easy to simplify the above expression.

Proposition 6.3.1 can be used to compute the mean and variance of the population size Z_n, for $n \in \mathbb{N}$.

Proposition 6.3.3. *Let $\{p_n\}_{n\in\mathbb{N}_0}$ be a pmf of the offpsring distribution of a branching process $\{Z_n\}_{n\in\mathbb{N}_0}$. If $\{p_n\}_{n\in\mathbb{N}_0}$ admits an expectation, i.e., if $\mu = \sum_{k=0}^{\infty} kp_k < \infty$, then*

$$\mathbb{E}[Z_n] = \mu^n. \tag{6.3.1}$$

If the variance of $\{p_n\}_{n\in\mathbb{N}_0}$ is also finite, i.e., if $\sigma^2 = \sum_{k=0}^{\infty}(k-\mu)^2 p_k < \infty$, then

$$\begin{aligned} \operatorname{Var}[Z_n] &= \sigma^2\mu^n(1+\mu+\mu^2+\dots+\mu^n) \\ &= \begin{cases} \sigma^2\mu^n \frac{1-\mu^{n+1}}{1-\mu}, & \mu \neq 1, \\ \sigma^2(n+1), & \mu = 1 \end{cases} \end{aligned} \tag{6.3.2}$$

Proof. Since the distribution of Z_1 is just $\{p_n\}_{n\in\mathbb{N}_0}$, it is clear that $\mathbb{E}[Z_1] = \mu$ and $\operatorname{Var}[Z_1] = \sigma^2$. We proceed by induction and assume that the formulas (6.3.1) and (6.3.2) hold for $n \in \mathbb{N}$. By Proposition 6.3.1, the generating function $P_{Z_{n+1}}$ is given as a composition $P_{Z_{n+1}}(s) = P_{Z_n}(P(s))$. Therefore, if we use the identity $\mathbb{E}[Z_{n+1}] = P'_{Z_{n+1}}(1)$, we get

$$\begin{aligned} P'_{Z_{n+1}}(1) &= P'_{Z_n}(P(1))P'(1) = P'_{Z_n}(1)P'(1) = \mathbb{E}[Z_n]\mathbb{E}[Z_1] = \mu^n\mu \\ &= \mu^{n+1}. \end{aligned}$$

A similar (but more complicated and less illuminating) argument can be used to establish (6.3.2). □

6.4 Extinction probability

We now turn to the central question (the one posed by Galton). We define **extinction** to be the following event:

$$E = \{\omega \in \Omega : Z_n(\omega) = 0 \text{ for some } n \in \mathbb{N}\}.$$

It is the property of the branching process that $Z_m = 0$ for all $m \geq n$ whenever $Z_n = 0$. Therefore, we can write E as an *increasing* union of sets E_n, where

$$E_n = \{\omega \in \Omega : Z_n(\omega) = 0\}.$$

Therefore, the sequence $\{\mathbb{P}[E_n]\}_{n\in\mathbb{N}}$ is non-decreasing and

$$\mathbb{P}[E] = \lim_{n\in\mathbb{N}} \mathbb{P}[E_n].$$

The number $p_E = \mathbb{P}[E]$ is called the **extinction probability**. Using generating functions, and, in particular, the fact that $\mathbb{P}[E_n] = \mathbb{P}[Z_n = 0] = P_{Z_n}(0)$ we get

$$p_E = \mathbb{P}[E] = \lim_{n\in\mathbb{N}} P_{Z_n}(0) = \lim_{n\in\mathbb{N}} \underbrace{P(P(\dots P(0)\dots))}_{n \text{ P's}}.$$

It is amazing that this probability can be computed, even if the explicit form of the generating function P_{Z_n} is not known.

Proposition 6.4.1. *The extinction probability p_E is the smallest non-negative solution of the equation*

$$x = P(x), \text{ called the } \textbf{\textit{extinction equation}},$$

where P is the generating function of the offspring distribution.

Proof. Let us show first that p_E is *a* solution of the equation $x = P(x)$. Indeed, P is a continuous function, so $P(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} P(x_n)$ for every convergent sequence $\{x_n\}_{n\in\mathbb{N}_0}$ in $[0,1]$ with $x_n \to x_\infty$. Let us take a particular sequence given by

$$x_n = \underbrace{P(P(\dots P(0)\dots))}_{n \text{ P's}}.$$

Then

1. $p_E = \mathbb{P}[E] = \lim_{n\in\mathbb{N}} x_n$, and
2. $P(x_n) = x_{n+1}$.

Therefore,

$$p_E = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} P(x_n) = P(\lim_{n\to\infty} x_n) = P(p_E),$$

and so p_E solves the equation $P(x) = x$.

The fact that p_E is the *smallest* solution of $x = P(x)$ on $[0,1]$ is a bit trickier to get. Let p' be another solution of $x = P(x)$ on $[0,1]$. Since $0 \leq p'$ and P is a non-decreasing function, we have

$$P(0) \leq P(p') = p'.$$

We can apply the function P to both sides of the inequality above to get

$$P(P(0)) \leq P(P(p')) = P(p') = p'.$$

Continuing in the same way we get

$$P[E_n] = \underbrace{P(P(\dots P(0)\dots))}_{n\ \mathrm{P}'} \leq p',$$

we get $p_E = \lim_{n\in\mathbb{N}} \mathbb{P}[E_n] \leq \lim_{n\in\mathbb{N}} p' = p'$, so p_E is not larger than any other solution p' of $x = P(x)$. $\square$

Example 6.4.2. Let us compute extinction probabilities in the cases from Example 6.3.2.

1. $p_0 = 1$, $p_n = 0$, $n \in \mathbb{N}$:

   No need to use any theorems. $p_E = 1$ in this case.

2. $p_0 = 0, p_1 = 1, p_n = 0$, $n \geq 2$:

   Like above, the situation is clear - $p_E = 0$.

3. $p_0 = 0, p_1 = 0, \dots, p_k = 1, p_n = 0$, $n \geq k$, for some $k \geq 2$:

   No extinction here - $p_E = 0$.

4. $p_0 = p, p_1 = q = 1 - p, p_n = 0$, $n \geq 2$:
   Since $P(s) = p + qs$, the extinction equation is $s = p + qs$. If $p = 0$, the equation looks like $s = s$ and every s is a solution. The smallest solution is, therefore, $s = 0$, so no extinction occurs. If $p > 0$, the only solution is $s = 1$ - the extinction is guaranteed. It is interesting to observe that there is a jump in the extinction probability as p changes from 0 to a positive number.

5. $p_0 = p^2, p_1 = 2pq, p_2 = q^2, p_n = 0$, $n \geq 3$:

   Here $P(s) = (p + qs)^2$, so the extinction equation reads

   $$s = (p + qs)^2.$$

This is a quadratic in s and its solutions are $s_1 = 1$ and $s_2 = \frac{p^2}{q^2}$, if we assume that $q > 0$. When $p < q$, the smaller of the two is s_2. When $p \geq q$, $s = 1$ is the smallest solution. Therefore

$$p_E = \min(1, \frac{p^2}{q^2}).$$

6.5 Criticality in Branching

The examples above show that one can often find a particular value of some parameter of the offspring distribution at which the behavior of the branching process changes in a qualitative way. It turns out that this parameter is nothing but the expected value $\mu = \mathbb{E}[Z_1]$.

Proposition 6.5.1. *Suppose that the offspring distribution is such that $p_0 > 0$, and let $\mu = \mathbb{E}[Z_1]$ be its expectation. Then*

1. *If $\mu \leq 1$ the population always goes extinct, i.e. $p_E = 1$.*
2. *If $\mu > 1$, the probability of extinction p_E is a number in $(0, 1)$.*

Proof. The main insight comes from the following three pictures.

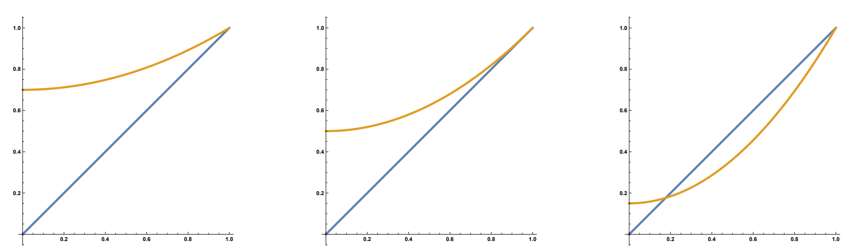

Figure 3. The generating function $P(s)$ (orange) and the identity function s (blue) in the subcritical ($\mu < 1$), critical ($\mu = 1$) and supercritical ($\mu > 1$) regimes, from left to right.

Since $\mu = \mathbb{E}[Z_1] = P'(1)$, the three situations above correspond to the cases where the derivatives of the two curves (orange and blue) stand in three different relationships (orange $<$ blue, orange $=$ blue, and orange $>$ blue). In the first two cases, the two curves intersect only at $s = 1$. Indeed, the derivative of the orange curve (being convex) is always below the derivative (1) of the blue curve, making the difference $P(s) - s$ decreasing, and therefore positive, for $s \in [0, 1)$. In the third case, the orange curve starts from $p_0 > 0$

at $s = 0$ and ends up a little bit below 1 for s close to 1. Therefore the first time is crosses the blue line is somewhere in $(0, 1)$. $\square$

6.6 Problems

Problem 6.6.1. Let $\{Z_n\}_{n\in\mathbb{N}_0}$ be a branching process with a geometric offspring distribution, with parameter $p \in (0, 1)$.

1. Compute the generating function of Z_2.
2. Compute the expected value and the variance of Z_n, for each $n \in \mathbb{N}$.
3. Compute the extinction probability.
4. Compute the extinction probability if we start from $Z_0 = 100$, instead of $Z_0 = 1$.

Problem 6.6.2. In a branching process, the offspring distribution is given by its characteristic function

$$P(s) = as^2 + bs + c$$

where $a, b, c > 0$ and $a + b + c = 1$.

(i) Find the exticion probability for this branching process.

(ii) Give a necessary and sufficient condition for sure extinction.

Problem 6.6.3. Bacteria reproduce by cell division. In a unit of time, a bacterium will either die (with probability $\frac{1}{4}$), stay the same (with probability $\frac{1}{4}$), or split into 2 parts (with probability $\frac{1}{2}$). The population starts with 100 bacteria at time $n = 0$.

1. Write down the expression for the generating function of the distribution of the size of the population at time $n \in \mathbb{N}_0$. (*Note:* you can use n-fold composition of functions.)
2. Compute the extinction probability for the population.
3. Given that there are 1000 bacteria in the population at time 50, what is the expected number of bacteria at time 51?

Problem 6.6.4. A branching process starts from 10 individuals, and each reproduces according to the probability distribution $(p_0, p_1, p_2, \dots)$, where $p_0 = 1/4$, $p_1 = 1/4$, $p_2 = 1/2$, $p_n = 0$, for $n > 2$. The extinction probability for the whole population is equal to

(a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{20}$ (d) $\frac{1}{200}$ (e) $\frac{1}{1024}$

Problem 6.6.5. Let Z be a branching process with the offspring distribution given by

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline & p_0 & p_1 & p_2 \end{array}.$$

If $p_0 < p_1 < p_2$, then the extinction probability is

(a) 1 (b) p_0/p_2 (c) p_0/p_1 (d) p_1/p_2 (e) 0

Problem 6.6.6. A (solitaire) game starts with 3 silver dollars in the pot. At each turn the number of silver dollars in the pot is counted (call it K) and the following procedure is repeated K times: a die is thrown, and according to the outcome the following four things can happen

- If the outcome is 1 or 2 the player takes 1 silver dollar from the pot.
- If the outcome is 3 nothing happens.
- If the outcome is 4 the player puts 1 extra silver dollar in the pot (you can assume that the player has an unlimited supply of silver dollars).
- If the outcome is 5 or 6, the player puts 2 extra silver dollars in the pot.

If there are no silver dollars on in the pot, the game stops.

1. Compute the expected number of silver dollars in the pot after turn $n \in \mathbb{N}$.
2. Compute the probability that the game will stop eventually.
3. (*) Let m_n be the maximal possible number of silver dollars in the pot after the n-th turn? What is the probability that the actual number of silver dollars in the pot after n turns is equal to $m_n - 1$?

Problem 6.6.7. Let $\{Z_n\}_{n \in \mathbb{N}_0}$ be a simple branching process which starts from one individual. Each individual has exactly $m \in \mathbb{N}$ children, each of whom survives until reproductive age with probability $0 < p < 1$, and dies before he/she is able to reproduce with probability $q = 1 - p$, independently of his/her siblings. The children that reach reproductive age reproduce according to the same rule.

1. Write down the generating function for the offspring distribution.
2. For what values of p and m will the population go extinct with certainty (probability 1).

Problem 6.6.8. Let $\{Z_n\}_{n\in\mathbb{N}_0}$ be a branching process (with $Z_0 = 1$) and with the offspring distribution

0	1	2	3	4
a	0	0	b	c

where $a, b, c > 0$ and $a + b + c = 1$.

1. Compute $\mathbb{E}[Z_n]$ for each $n \in \mathbb{N}$.
2. For what combinations of values of a, b and c is the extinction probability p_E strictly smaller than 1?
3. Compute $\mathbb{P}[Z_{n+1} = 5|Z_n = 2]$.
4. Compute $\mathbb{P}[Z_{n+1} = 7|Z_n = 3]$

Problem 6.6.9. It is a well-known fact(oid) that armadillos always have identical quadruplets (four offspring). Each of the 4 little armadillos has a $1/3$ chance of becoming a doctor, a lawyer or a scientist, independently of its 3 siblings. A doctor armadillo will reproduce further with probability $2/3$, a lawyer with probability $1/3$ and a scientist with probability $1/4$, again, independently of everything else. If it reproduces at all, an armadillo reproduces only once in its life, and then leaves the armadillo scene. (For the purposes of this problem assume that armadillos reproduce asexually.) Let us call the armadillos who have offspring *fertile*.

1. What is the distribution of the number of fertile offspring? Write down its generating function.
2. What is the generating function for the number of great-grandchildren an armadillo will have? What is its expectation? (*Note:* do not expand powers of sums)
3. Let the armadillo population be modeled by a branching process, and let's suppose that it starts from exactly one individual at time 0. Is it certain that the population will go extinct sooner or later?

Problem 6.6.10. *Branching in alternating environments*. Suppose that a branching process $\{Z_n\}_{n\in\mathbb{N}_0}$ is constructed in the following way: it starts with one individual. The individuals in odd generations reproduce according to an offspring distribution with generating function $P_{odd}(s)$ and those in even generations according to an offspring distribution with generating function $P_{even}(s)$. All independence assumptions are the same as in the classical case.

1. Find an expression for the generating function P_{Z_n} of Z_n.
2. Derive the extinction equation.

Problem 6.6.11. (*) The purpose of this problem is to describe a class of offspring distributions for which an explicit expression for $P_{Z_n}(s)$ can be obtained.

An $\mathbb{N}_0$-valued distribution is said to be of *fractional-linear type* if its generating function P has the following form

$$P(s) = \frac{as+b}{1-cs}, \tag{6.6.1}$$

for some constants $a, b, c \geq 0$. In order for P to be a generating function of a probability distribution we must have $P(1) = 1$, i.e. $a+b+c = 1$, which will be assumed throughout the problem.

1. What (familiar) distributions correspond to the following special cases (in each case identify the distribution and its parameters):

   (a) $c = 0$, (b) $a = 0$, (c) $b = 0$.

2. Let A be the following 2×2 matrix

   $$A = \begin{bmatrix} a & b \\ -c & 1 \end{bmatrix} \text{ and let } A^n = \begin{bmatrix} a^{(n)} & b^{(n)} \\ c^{(n)} & d^{(n)} \end{bmatrix}$$

   be its n^{th} power (using matrix multiplication, of course). Explain why

   $$\underbrace{P(P(\dots P(s)\dots)}_{n \text{ Ps}} = \frac{a^{(n)}s + b^{(n)}}{c^{(n)}s + d^{(n)}}. \tag{6.6.2}$$

3. Take a, b and c that correspond to a geometric distribution with parameter $p = 1/2$, and compute $(2A)^n$ for all n. Use your result to write down the generating function of Z_n in the linear-fractional form (6.6.1). **Note:** We are computing $(2A)^n$ instead of A^n simply because it is easier. In order to compute $(2A)^n$, compute it for $n = 1$, $n = 2$ and $n = 3$ and try to spot a pattern.

4. Find the extinction probability as a function of a, b and c in the general case (don't forget to use the identity $a + b + c = 1$!).

5. Suppose that, given any two numbers $r_1, r_2 \in (0, 1)$, you can mint two biased coins in such a way that the probabilities of obtaining heads are r_1 and r_2, respectively. How would you use these coins, and what values of r_1 and r_2 would you choose, to produce (simulate) a random variable of fractional-linear type with parameters a, b and c?

Problem 6.6.12. (*) For a branching process $\{Z_n\}_{n\in\mathbb{N}_0}$, denote by S the total number of individuals that ever lived, i.e., set

$$S = \sum_{n=0}^{\infty} Z_n = 1 + \sum_{n=1}^{\infty} Z_n.$$

(i) Assume that the offspring distribution has the generating function given by

$$P(s) = p + qs.$$

Find the generating function P_S in this case.

(ii) Assume that the offspring distribution has the generating function given by

$$P(s) = p/(1 - qs).$$

Find P_S in this case.

(iii) Find the general expression for $\mathbb{E}[S]$ and calculate this expectation in the special cases (i) and (ii).