Course: Theory of Probability II
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Lecture 17
Brownian motion as a Markov process

Brownian motion is one of the "universal" examples in probability. So far, it featured as a continuous version of the simple random walk and served as an example of a continuous-time martingale. It can also be considered as one of the fundamental Markov processes. We start by explaining what that means.

The Strong Markov Property of the Brownian Motion

Definition 17.1 (Markov property)**.** A stochastic process $\{X_t\}_{t\in[0,\infty)}$, defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,\infty)}, \mathcal{P})$ is said to be an $\{\mathcal{F}_t\}_{t\in[0,\infty)}$-**Markov process** if, for all $B \in \mathcal{B}(\mathbb{R})$, and $t, h \geq 0$, we have

$$\mathbb{P}[X_{t+h} \in B|\mathcal{F}_t] = \mathbb{P}[X_{t+h} \in B|\sigma(X_t)], \text{ a.s.}$$

Example 17.2 (Brownian motion is a Markov Process)**.** Let $\{B_t\}_{t\in[0,\infty)}$ be a Brownian motion, and let $\mathcal{F}^B_t = \sigma(B_s, s \leq t)$ be its natural filtration. The independence of increments implies that $B_{t+h} - B_t$ is independent of $\mathcal{F}^B_t$, for $t, h \geq 0$. Therefore, for a bounded Borel function $f : \mathbb{R} \to \mathbb{R}$, we have

$$\mathbb{E}[f(B_{t+h})|\mathcal{F}^B_t] = \mathbb{E}[f\big(B_t + (B_{t+h} - B_t)\big)|\mathcal{F}^B_t] = \hat{f}(B_t), \text{ a.s.}$$

where $\hat{f}(x) = \mathbb{E}[f(x + B_{t+h} - B_t)]$. In particular, we have

$$\mathbb{E}[f(B_{t+h})|\mathcal{F}^B_t] = \mathbb{E}[f(B_{t+h})|\sigma(B_t)], \text{ a.s.,}$$

and, by setting $f = \mathbf{1}_A$, we conclude that B is an $\mathcal{F}^B$-Markov process.

A similar statement, only when the deterministic time t is replaced by a stopping time τ is typically referred to as the **strong Markov property**. While the main goal of the section is to state and prove it, we make a quick detour and introduce another important notion, a slight generalization of what it means to be a Brownian motion.

Definition 17.3 ($\{\mathcal{F}_t\}_{t\in[0,\infty)}$-Brownian motion)**.** Let $\{\mathcal{F}_t\}_{t\in[0,\infty)}$ be a filtration. An $\{\mathcal{F}_t\}_{t\in[0,\infty)}$-adapted process $\{B_t\}_{t\in[0,\infty)}$ is said to be an $\{\mathcal{F}_t\}_{t\in[0,\infty)}$-**Brownian motion**, if

1. $B_t - B_s \sim N(0, t-s)$, for $0 \le s \le t < \infty$,
2. $B_t - B_s$ is independent of $\mathcal{F}_s$, for all $0 \le s \le t < \infty$, and
3. for all $\omega \in \Omega$, $t \mapsto B_t(\omega)$ is a continuous functions.

The following proposition gives an (at first glance unexpected) characterization of the $\{\mathcal{F}_t\}_{t\in[0,\infty)}$-Brownian property. It is a special case of a very common theme in stochastics. It features a complex-valued martingale; that simply means that both its real and imaginary parts are martingales.

Proposition 17.4 (A characterization of the $\{\mathcal{F}_t\}_{t\in[0,\infty)}$-Brownian motion)**.** *An $\{\mathcal{F}_t\}_{t\in[0,\infty)}$-adapted process $\{X_t\}_{t\in[0,\infty)}$ with continuous paths is an $\{\mathcal{F}_t\}_{t\in[0,\infty)}$-Brownian motion if and only if the complex-valued process $\{Y^r_t\}_{t\in[0,\infty)}$, given by*

$$Y^r_t = e^{irX_t + \frac{1}{2}r^2 t} \text{ for } t \ge 0,\ r \in \mathbb{R}, \tag{17.1}$$

is an $\{\mathcal{F}_t\}_{t\in[0,\infty)}$-martingale, for each $r \in \mathbb{R}$. In particular, an $\{\mathcal{F}_t\}_{t\in[0,\infty)}$-Brownian motion is an $\{\mathcal{F}_t\}_{t\in[0,\infty)}$-Markov process and also an $\{\mathcal{F}_t\}_{t\in[0,\infty)}$-martingale.

Proof. The martingality of Y^r (for each r) implies that the conditional characteristic function of the increment $X_{t+h} - X_t$, given $\mathcal{F}_t$, is centered normal with variance h, for all $t, h \ge 0$. By Proposition 10.18, $X_{t+h} - X_t$ is independent of $\mathcal{F}_t$, and we conclude that X is an $\{\mathcal{F}_t\}_{t\in[0,\infty)}$-Brownian motion.

The Markov property follows from the fact that the conditional distribution of X_{t+h}, given $\mathcal{F}_t$, is normal with mean X_t (and variance h), so it only depends on X_t. The martingale property, similarly, is the consequence of the fact that $X_{t+h} - X_h$ is centered, conditionally on $\mathcal{F}_t$. □

We know already that each Brownian motion is an $\{\mathcal{F}^B_t\}_{t\in[0,\infty)}$-Brownian motion. There are other filtrations, though, that share this property. A less interesting (but quite important) example is the natural filtration of a d-dimensional Brownian motion[1], for $d > 1$. Then, each of the components is an $\{\mathcal{F}^{(B^1,\dots,B^d)}\}_{t\in[0,\infty)}$-Brownian motion. A more unexpected example is the following:

[1] a d-dimensional Brownian motion $(B^1, \dots, B^d)$ is simply a process, taking values in $\mathbb{R}^d$, each of whose components is a Brownian motion in its own right, independent of all the others.

Proposition 17.5 (Brownian motion is an $\{\mathcal{F}^{B+}_t\}_{t\in[0,\infty)}$-Brownian motion)**.** *Let $\{B_t\}_{t\in[0,\infty)}$ be a Brownian motion, and let $\{\mathcal{F}^{B+}_t\}_{t\in[0,\infty)}$ be the right-continuous augmentation of its natural filtration $\{\mathcal{F}^B_t\}_{t\in[0,\infty)}$. Then $\{B_t\}_{t\in[0,\infty)}$ is an $\{\mathcal{F}^{B+}_t\}_{t\in[0,\infty)}$-Brownian motion, and an $\{\mathcal{F}^{B+}_t\}_{t\in[0,\infty)}$-Markov process.*

Proof. Thanks to Proposition 17.4, it suffices to show that (17.1) holds with $\mathcal{F}_t$ replaced by $\mathcal{F}_t^{B+}$. For that, we start with the fact that, for $\varepsilon < h$,

$$\mathbb{E}[e^{irX_{t+h}}|\mathcal{F}^B_{t+\varepsilon}] = e^{irX_{t+\varepsilon}-\frac{1}{2}r^2(h-\varepsilon)}, \text{ a.s., for all } r \in \mathbb{R}.$$

We condition both sides with respect to $\mathcal{F}^B_{t+}$ and use the tower property to conclude that

$$\mathbb{E}[e^{irX_{t+h}}|\mathcal{F}^B_{t+}] = \mathbb{E}[e^{irX_{t+\varepsilon}-\frac{1}{2}r^2(h-\varepsilon)}|\mathcal{F}^B_{t+}], \text{ a.s., for all } r \in \mathbb{R}.$$

We let $\varepsilon \searrow 0$ and use the dominated convergence theorem and the right continuity of X to get

$$\mathbb{E}[e^{irX_{t+h}}|\mathcal{F}^B_{t+}] = \mathbb{E}[e^{irX_t-\frac{1}{2}r^2h}|\mathcal{F}^B_{t+}] = e^{irX_t-\frac{1}{2}r^2h}, \text{ a.s., for all } r \in \mathbb{R}. \quad \square$$

Corollary 17.6 (Blumenthal's 0-1 law)**.** *For $t \geq 0$, the σ-algebras $\mathcal{F}_t^{B+}$ and $\mathcal{F}_t^B$ are a.s.-equal, i.e.,*

$$\textit{for each } A \in \mathcal{F}_t^{B+} \textit{ there exists } A' \in \mathcal{F}_t^B \textit{ such that } \mathbb{P}[A \Delta A'] = 0.$$

Proof. Thanks to the $\{\mathcal{F}_t^{B+}\}_{t\in[0,\infty)}$-Brownian property of the Brownian motion, given $t \geq 0$ and a nonnegative Borel function f, we have

$$\begin{aligned}\mathbb{E}[f(B_{s_1}, B_{s_2}, \dots, B_{s_n})|\mathcal{F}^B_{t+}] &= \mathbb{E}[f(B_{s_1}, B_{s_2}, \dots, B_{s_n})|\sigma(B_t)] \\ &= \mathbb{E}[f(B_{s_1}, B_{s_2}, \dots, B_{s_n})|\mathcal{F}^B_t], \text{ a.s.,}\end{aligned} \tag{17.2}$$

for all $s_n > s_{n-1} > \dots, s_1 > t$. Trivially,

$$\mathbb{E}[\mathbf{1}_A|\mathcal{F}^B_{t+}] = \mathbb{E}[\mathbf{1}_A|\mathcal{F}^B_t], \text{ a.s., for all } A \in \mathcal{F}_t. \tag{17.3}$$

Let $\mathcal{A}$ denote the set of all events A in $\mathcal{F}_\infty = \vee_{t\geq 0}\mathcal{F}_t^B$ such that $\mathbb{E}[\mathbf{1}_A|\mathcal{F}^B_t] = \mathbb{E}[\mathbf{1}_A|\mathcal{F}^B_{t+}]$, a.s.. By (17.2) and (17.3), $\mathcal{A}$ contains all sets of the form $A_1 \cap A_2$, for $A \in \mathcal{F}_t$ and $A_2 \in \sigma(B_s, s \geq t)$. The $\pi-\lambda$-theorem can now be used to conclude that $\mathbb{E}[\mathbf{1}_A|\mathcal{F}^B_{t+}] = \mathbb{E}[\mathbf{1}_A|\mathcal{F}^B_t]$, for *all* $A \in \mathcal{F}_\infty$, and, in particular, for all $A \in \mathcal{F}^B_{t+}$, i.e., that

$$\mathbf{1}_A = \mathbf{1}_{A'} \text{ a.s., where } \mathbf{1}_{A'} = \mathbb{E}[\mathbf{1}_A|\mathcal{F}^B_t]. \quad \square$$

Corollary 17.7. *With probability* 1*, the Brownian path takes both strictly positive and strictly negative values in each neighborhood of* 0*.*

Proof. Suppose, to the contrary, that $\mathbb{P}[A] > 0$, where

$$A = \{\exists\, \varepsilon > 0,\ B_t \geq 0 \text{ for } t \in [0, \varepsilon]\}.$$

Check that $A \in \mathcal{F}_0^{B+}$, so that $\mathbb{P}[A] = 1$. Since $\{-B_t\}_{t\in[0,\infty)}$ is also a Brownian motion, we have $\mathbb{P}[B] = 1$, where

$$B = \{\exists\, \varepsilon > 0,\ B_t \leq 0 \text{ for } t \in [0, \varepsilon]\}.$$

Consequently,

$$\mathbb{P}[\{\exists\, \varepsilon > 0,\ B_t = 0 \text{ for } t \in [0, \varepsilon]\} = 1,$$

and so, there exists $\varepsilon_0 > 0$ such that

$$\mathbb{P}[\{B_t = 0 \text{ for } t \in [0, \varepsilon_0]\} > 0.$$

In particular, $\mathbb{P}[B_{\varepsilon_0} = 0] > 0$, which is in contradiction with the fact that B_{ε_0} is normally distributed with variance $\varepsilon_0 > 0$.

We conclude that $\mathbb{P}[A] = 0$, i.e., with probability 1, for each $\varepsilon > 0$ there exists $t \in [0, \varepsilon)$ such that $B_t < 0$. Similarly, with probability 1, for each $\varepsilon > 0$ there exists $t \in [0, \varepsilon)$ such that $B_t > 0$, and the claim of the proposition follows. □

Proposition 17.8 (The strong Markov property of the Brownian motion)**.** *Let $\{B_t\}_{t\in[0,\infty)}$ be a Brownian motion and let $\tau < \infty$, a.s., be an $\{\mathcal{F}_t^{B+}\}_{t\in[0,\infty)}$-stopping time. Then the process $\{W_t\}_{t\in[0,\infty)}$, given by*

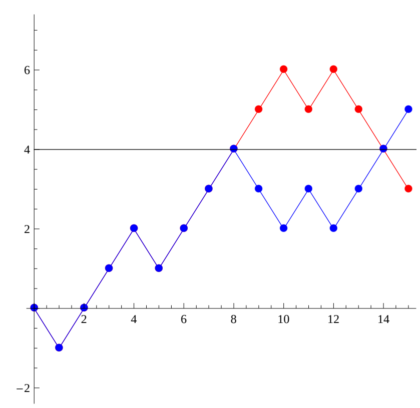
$$W_t = B_{\tau+t} - B_\tau,$$

is a Brownian motion, independent of $\mathcal{F}_\tau^{B+}$.

Proof. We do the proof only in the case of a bounded τ. We'll come back to the general case a bit later. Thanks to the characterization in Proposition 17.4, we need to show that the exponential process $\{e^{irX_{\tau+t}+\frac{1}{2}r^2(\tau+t)}\}_{t\in[0,\infty)}$ is a martingale with respect to $\{\mathcal{F}_{\tau+t}\}_{t\in[0,\infty)}$, for all $r \in \mathbb{R}$. Thanks to boundedness of τ, this is a direct consequence of the optional sampling theorem (specifically, Proposition 16.23). □

The reflection principle

Before we deal with the continuous time, search for "reflection principle" and "ballot problem" for a neat simple idea in enumerative combinatorics. We are going to apply it in continuous time to the Brownian motion to say something about the distribution of its running maximum. As a preparation for the mathematics, do the following problem first:



The idea behing the reflection principle.

Problem 17.1. Let $\{X_t\}_{t\in[0,\infty)}$ be a stochastic process with continuous trajectories, and let τ be an $[0,\infty]$-valued random variable. Define the mapping $\kappa : \Omega \to C[0,\infty)$ by

$$\kappa(\omega) = \{X_{t\wedge\tau(\omega)}(\omega)\}_{t\in[0,\infty)}, \text{ for } \omega \in \Omega.$$

Show that κ is measurable from $(\Omega, \mathcal{F})$ into $(C[0,\infty), \mathcal{B}(C[0,\infty)))$.

Proposition 17.9 (Reflection principle). *Let $\{B_t\}_{t\in[0,\infty)}$ be a Brownian motion, and let τ be a stopping time with $\tau(\omega) < \infty$, for all $\omega \in \Omega$. Then the process $\{\tilde{B}\}_{t\in[0,\infty)}$, given by*

$$\tilde{B}_t(\omega) = \begin{cases} B_t, & t \leq \tau, \\ B_\tau - (B_t - B_\tau), & t > \tau, \end{cases} \tag{17.4}$$

is also a Brownian motion.

Proof. Consider the mapping $K : \Omega \to [0,\infty) \times C[0,\infty) \times C[0,\infty)$, given by

$$K(\omega) = \Big(\tau(\omega), \{B_{t\wedge\tau(\omega)}(\omega)\}_{t\in[0,\infty)}, \{B_{\tau(\omega)+t}(\omega) - B_{\tau(\omega)}(\omega)\}_{t\in[0,\infty)}\Big).$$

It follows from Problem 17.1 (or a very similar argument) that K is $\mathcal{F}$–$\mathcal{B}([0,\infty)) \times \mathcal{B}(C[0,\infty)) \times \mathcal{B}(C[0,\infty))$-measurable, so it induces a measure $\mathbb{P}_K$ - a pushforward of $\mathbb{P}$ - on $[0,\infty) \times C[0,\infty) \times C[0,\infty)$. Moreover, by the strong Markov property of the Brownian motion, the process $\{B_{\tau+t} - B_\tau\}_{t\in[0,\infty)}$ is a Brownian motion, independent of $\mathcal{F}^{B+}_\tau$. Since both τ and $\{B_{\tau\wedge t}\}_{t\in[0,\infty)}$ are measurable with respect to $\mathcal{F}^{B+}_\tau$, the first two components $\tau(\omega), \{B_{t\wedge\tau(\omega)}(\omega)\}_{t\in[0,\infty)}$ of K are independent of the third one. Consequently, the measure $\mathbb{P}_K$ is a product of a probability measure $\mathbb{P}'$ on $[0,\infty) \times C[0,\infty)$ and a probability measure $\mathbb{P}_W$ on $C[0,\infty)$, where, as the notation suggests, the measure $\mathbb{P}_W$ is a Wiener measure on $C[0,\infty)$. The same argument shows that the pushforward of $\mathbb{P}$ under the mapping $L : \Omega \to [0,\infty) \times C[0,\infty) \times C[0,\infty)$, given by

$$L(\omega) = \Big(\tau(\omega), \{B_{t\wedge\tau(\omega)}(\omega)\}_{t\in[0,\infty)}, \{-(B_{\tau(\omega)+t}(\omega) - B_{\tau(\omega)}(\omega))\}_{t\in[0,\infty)}\Big),$$

is also equal to $\mathbb{P}_K = \mathbb{P}' \times \mathbb{P}_W$, because $\{-W_t\}_{t\in[0,\infty)}$ is a Brownian motion, whenever $\{W_t\}_{t\in[0,\infty)}$ is one.

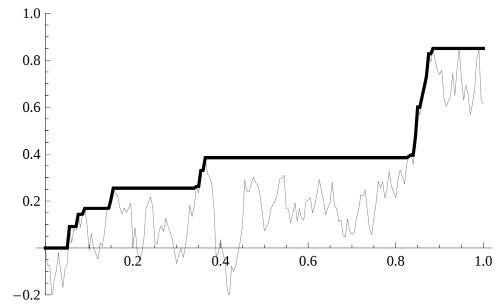
Define the mapping $S : [0,\infty) \times C[0,\infty) \times C[0,\infty) \to C[0,\infty)$, by $S(\tau, x_1, x_2) = x$, where

$$x(t) = \begin{cases} x_1(t), & t \leq \tau \\ x_1(\tau) + x_2(t), & t > \tau. \end{cases}$$

It is left to the reader to prove that S is continuous, and, therefore, measurable. Therefore, the compositions $S \circ K$ and $S \circ L$, both defined on Ω with values in $C[0,\infty)$ are measurable, and the pushforwards of $\mathbb{P}$ under them are the same (beacuse both K and L push $\mathbb{P}$ forward into the same measure on $\mathbb{R} \times C[0,\infty) \times C[0,\infty)$). The composition $S \circ L$ is easy to describe; in fact, we have

$$S(L(\omega))_t = B_t(\omega),$$

so the pushforward of $\mathbb{P}$ under $S \circ L$ equals to the Wiener measure $\mathbb{P}_W$ on $C[0,\infty)$. Therefore, the pushforward of $\mathbb{P}$ under $S \circ K$ is also a Wiener measure. In other words the process $S \circ K$ - which the reader will readily identify with $\{\tilde{B}_t\}_{t\in[0,\infty)}$ - is a Brownian motion. □



Definition 17.10. Let $\{B_t\}_{t\in[0,\infty)}$ be a Brownian motion. The process $\{M_t\}_{t\in[0,\infty)}$, where

$$M_t = \sup_{s\leq t} B_s, \ t \geq 0$$

is called the **running maximum** of the Brownian motion $\{B_t\}_{t\in[0,\infty)}$.

Proposition 17.11. *Let $\{B_t\}_{t\in[0,\infty)}$ be a Brownian motion, and let $\{M_t\}_{t\in[0,\infty)}$ be its running-maximum process. Then the random vector (M_1, B_1) is absolutely continuous with the density*

$$f_{(M_1,B_1)}(m,b) = 2\varphi'(2m-b)\mathbf{1}_{\{m>0,b<m\}},$$

where φ is the density of the unit normal.

Proof. By scaling we can assume that $t = 1$. We pick a level $m > 0$, and let $\tau_m = \inf\{s \geq 0 : B_s = m\}$ be the first hitting time of the level m. In order to be able to work with a finite stopping time, we set $\tau = \tau_m \wedge 1$, so that $\tau(\omega) < \infty$, for all $\omega \in \Omega$. Proposition 17.9 implies that the process $\{\tilde{B}_t\}_{t\in[0,\infty)}$, given by (17.4), is a Brownian motion. Therefore, the random vectors (B_1, M_1) and $(\tilde{B}_1, \tilde{M}_1)$, where $\tilde{M}_t = \sup_{s\leq t} \tilde{B}_s$, $t \geq 0$, are equally distributed. Moreover, $\{\tilde{M}_1 \geq m\} = \{M_1 \geq m\}$, so for $b < m$, $m > 0$ we have

$$\begin{aligned}\mathbb{P}[M_1 \geq m, B_1 \leq b] &= \mathbb{P}[\tilde{M}_1 \geq m, \tilde{B}_1 \geq 2m-b] \\ &= \mathbb{P}[M_1 \geq m, B_1 \geq 2m-b],\end{aligned}$$

i.e., since $\{M_1 \geq m\} = \{\tau_m \leq 1\}$ and $\{M_1 \geq m\} \supset \{B_1 \geq 2m-b\}$,

$$\mathbb{P}[M_1 \geq m, B_1 \leq b] = \mathbb{P}[B_1 \geq 2m-b]. \tag{17.5}$$

Therefore, the left-hand side is continuously differentiable in both m and b in the region $\{(m,b) \in \mathbb{R}^2 : m > 0, m < a\}$ and

$$\begin{aligned}f_{(M_1,B_1)}(m,b) &= \tfrac{\partial^2}{\partial m\,\partial b}\mathbb{P}[M_1 \geq m, B_1 < b] \\ &= \tfrac{\partial^2}{\partial m\,\partial b}\mathbb{P}[B_1 \geq 2m-b] = -2\varphi'(2m-b).\end{aligned}$$ □

Corollary 17.12. *The three random variables M_1, $M_1 - B_1$ and $|B_1|$ (defined in Proposition 17.11 above) have the same distribution.*

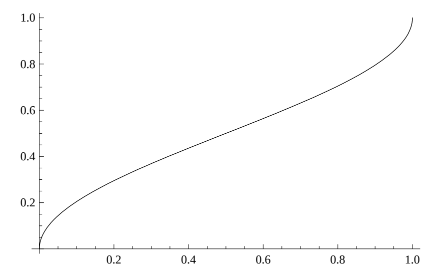
Proof. By (17.5), for $m > 0$, we have

$$\mathbb{P}[M_1 \geq m] = \mathbb{P}[M_1 \geq m, B_1 > m] + \mathbb{P}[M_1 \geq m, B_1 \leq m]$$
$$= \mathbb{P}[B_1 > m] + \mathbb{P}[B_1 \geq 2m - m] = \mathbb{P}[|B_1| \geq m].$$

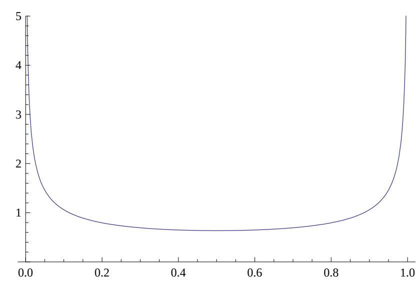
To obtain the second equality in distribution, we use the joint density function: for $a > 0$,

$$\mathbb{P}[B_1 \leq M_1 - a] = \int_0^\infty \int_{-\infty}^{m-a} -2\varphi'(2m - b)\, db\, dm =$$
$$= \int_0^\infty 2\varphi(m + a)\, dm = 2\int_a^\infty \varphi(m)\, dm = \mathbb{P}[|B_1| \geq a]. \square$$

Arcsine laws



The distribution function of the arcsine distribution



The density function of the arcsine distribution

A random variable X is said to have the **arcsine distribution** if it is supported on $[0, 1]$ with the cdf $F(x) = \frac{2}{\pi} \arcsin \sqrt{x}$, for $x \in [0, 1]$. Its density computes to:

$$f(x) = \frac{1}{\pi\sqrt{x(1-x)}},\ x \in (0, 1).$$

Problem 17.2. Show that

1. the random variable $X = \sin^2(\frac{\pi}{2}\alpha)$ has the arcsine distribution, if α is uniformly distributed on $[0, 1]$.
2. the random variable $\frac{\xi^2}{\xi^2+\eta^2}$ has the arcsine distribution, if ξ and η are independent unit ($\mu = 0$, $\sigma = 1$) normals.

Proposition 17.13 (Arcsine laws)**.** *Let $\{B_t\}_{t\in[0,\infty)}$ be a Brownian motion, and let $\{M_t\}_{t\in[0,\infty)}$ be its running maximum. Then the following random variables both have the arcsine distribution:*

$$\tau_1 = \sup\{t \leq 1 : B_t = 0\} \text{ and } \tau_2 = \inf\{t \geq 0 : B_t = M_1\}.$$

Proof. We start with τ_1 and note that $\{\tau_1 < t\} = \{\sup_{t\leq s\leq 1} B_s < 0\} \cup \{\inf_{t\leq s\leq 1} B_s > 0\}$ and that the two sets are disjoint. Then, Proposition 17.11 and the independence of $B_1 - B_t$ and B_t imply that random vectors $(\sup_{t\leq s\leq 1} B_s - B_t, B_t)$ and $(|B_1 - B_t|, B_t)$ have the same joint distribution. Therefore, for $t \in (0,1]$, we have

$$
\begin{aligned}
\mathbb{P}[\tau_1 < t] &= \mathbb{P}\left[\sup_{t\le s\le 1} B_s < 0\right] + \mathbb{P}\left[\inf_{t\le s\le 1} B_s > 0\right] \\
&= 2\mathbb{P}\left[\sup_{t\le s\le 1} B_s < 0\right] = 2\mathbb{P}\left[\sup_{t\le s\le 1} (B_s - B_t) < -B_t\right] \\
&= 2\mathbb{P}\left[|B_1 - B_t| < -B_t\right] = \mathbb{P}\left[|B_1 - B_t| < |B_t|\right] \\
&= \mathbb{P}\left[(B_1 - B_t)^2 < B_t^2\right] = \mathbb{P}\left[(1-t)\eta^2 < t\xi^2\right] \\
&= \mathbb{P}\left[\tfrac{\xi^2}{\xi^2+\eta^2} < t\right],
\end{aligned}
\tag{17.6}
$$

where $\xi = B_t/\sqrt{t}$ and $\eta = (B_1 - B_t)/\sqrt{1-t}$ are independent unit normals. Problem 17.2 implies that the τ_1, indeed, has the arcsine distribution. Moving on to τ_2, we note that

$$
\begin{aligned}
\mathbb{P}[\tau_2 \le t] &= \mathbb{P}\left[\sup_{s\le t}(B_s - B_t) \ge \sup_{s\ge t}(B_s - B_t)\right] \\
&= \mathbb{P}\left[|B_t| \ge |B_1 - B_t|\right] = \mathbb{P}\left[t\xi^2 \ge (1-t)\eta^2\right],
\end{aligned}
\tag{17.7}
$$

and the conclusion follows just like above. □

The zero-set of the Brownian motion

Our final result show that the random set $\{t \ge 0 : B_t = 0\}$ - called the **zero set** of the Brownian motion - looks very much like the Cantor set.

Definition 17.14. A nonempty set $C \subseteq \mathbb{R}$ is said to be **perfect** if it is closed and has no isolated points, i.e., for each $x \in C$ and each $\varepsilon > 0$ there exists $y \in C$ such that $0 < |y - x| < \varepsilon$.

Problem 17.3. Show that perfect sets are uncountable. *Hint:* Assume that a perfect set C is countable, and construct a nested sequence of intervals which "miss" more and more of the elements of C.

Proposition 17.15 (The zero set of the Brownian path)**.** *For $\omega \in \Omega$, let $\mathcal{Z}(\omega) = \{t \ge 0 : B_t(\omega) = 0\}$, where $\{B_t\}_{t\in[0,\infty)}$ is a Brownian motion. Then, for almost all $\omega \in \Omega$,*

1. *$\mathcal{Z}(\omega)$ is perfect,*
2. *$\mathcal{Z}(\omega)$ is uncountable,*
3. *$\mathcal{Z}(\omega)$ is unbounded,*
4. *$\lambda(\mathcal{Z}(\omega)) = 0$, where λ is the Lebesgue measure on $[0,\infty)$.*

Proof.

1. $\mathcal{Z}(\omega)$ is closed because it is a level set of a continuous function. To show that it is perfect, we pick $M > 0$ and for a rational number $q \geq 0$ define the finite stopping time

$$\tau_q = \inf\{t \geq q : B_t = 0\} \wedge M.$$

By the Strong Markov Property, $W_t = B_{\tau_q+t} - B_{\tau_q}$, $t \geq 0$ is a Brownian motion. In particular, there exists a set $A_q \in \mathcal{F}$ with $\mathbb{P}[A_q] = 1$ such that $W(\omega)_t$ takes the value 0 in any (right) neighborhood of 0. We pick $\omega \in A = \cap_q A_q$ and choose a zero t_0 of $B_t(\omega)$ in $[0, M)$ such that either $t_0 = 0$ or t_0 is isolated from the left. Then $t_0 = \tau_q(\omega)$ for q smaller than and close enough to t_0. In particular, $B_{\tau_q(\omega)}(\omega) = 0$ and, since $\omega \in A$, t_0 is not isolated in $\mathcal{Z}(\omega)$ from the right. Consequently, no $t \in \mathcal{Z}(\omega) \cap [0, M)$ can be isolated both from the left and from the right. The statement now follows from the fact that M is arbitrary.

2. Follows directly from (1) and Problem 17.3.

3. It is enough to note that Proposition 15.3 and continuity of the trajectories imply that a path of a Brownian motion will keep changing sign as $t \to \infty$.

4. Define $Z_t = \mathbf{1}_{\{B_t=0\}}$, for $t \geq 0$. Since the map $(t, \omega) \mapsto B_t(\omega)$ is jointly measurable, so is the map $(t, \omega) \mapsto Z_t(\omega)$. By Fubini's theorem, we have

$$\mathbb{E}[\lambda(\mathcal{Z})] = \mathbb{E}[\int_0^\infty Z_t\, dt] = \int_0^\infty \mathbb{E}[Z_t]\, dt = 0,$$

since $\mathbb{E}[Z_t] = \mathbb{P}[B_t = 0] = 0$, for all $t \geq 0$. Thus $\lambda(\mathcal{Z}) = 0$, a.s. □

Additional Problems

Problem 17.4 (The family of Brownian passage times)**.** Let $\{B_t\}_{t\in[0,\infty)}$ be a Brownian motion, and let the stopping times τ_c, $c \geq 0$, be defined as

$$\tau_c = \inf\{t \geq 0 : B_t > c\}.$$

Note: τ_c is also known as the (first) **passage time** of the level c.

1. Show that there exists a function $f : (0, \infty) \to (0, \infty)$ such that the random variables τ_c and $f(c)\tau_1$ have the same distribution.

2. We have derived the expression

$$\mathbb{E}[e^{-\lambda\tau_c}] = e^{-c\sqrt{\lambda}}, \text{ for } c, \lambda > 0,$$

for the Laplace transform of the distribution of τ_c in one of previous problems. Show how one can derive the density of τ_c by using the Brownian running-maximum process $\{M_t\}_{t\in[0,\infty)}$. *Note:* The distributions of τ_c form a family of so-called **Lévy distribution**, and provide one of the few examples of *stable* distributions with an explicit density function. These distributions are also special cases of the wider family of *inverse-Gaussian* distributions.

3. Find all $\alpha > 0$ such that $\mathbb{E}[\tau_c^\alpha] < \infty$. *Hint:* If you are going to use the density from 2. above, compute $\lim_{t\to\infty} \sqrt{t}\,\mathbb{P}[\tau_c > t]$.

4. Show that the process $(\tau_c)_{c\geq 0}$ has independent increments (just like in the definition of the BM) and that the increments over the intervals of the same size have the same distribution (again, just like the BM). Conclude that the distributions of τ_c, $c \geq 0$ form a **convolution semigroup**, i.e., that

$$\mu_{c_1} * \mu_{c_2} = \mu_{c_1+c_2} \text{ for } c_1, c_2 \geq 0,$$

where μ_c denotes the distribution of τ_c and $*$ denotes the convolution of measures. *Hint:* Use the Strong Markov Property or perform a tedious computation using the explicit expression from 2. above.

5. Show that the trajectories of $(\tau_c)_{c\geq 0}$ are a.s. RCLL (like the BM), but *a.s. not left continuous* (unlike the BM). *Hint:* What happens at levels c which correspond to the values of the local maxima of the Brownian path?

Note: $\{\tau_c\}_{c\geq 0}$ is an example of a *Lévy process*, or, more precisely, a *subordinator*.

Problem 17.5 (The planar Brownian motion does not hit points)**.** Let B be a two-dimensional Brownian motion (the coordinate processes are independent Brownian motions), and let $X_t = (0, a_0) + B_t$, for $t \geq 0$ and $a_0 \in \mathbb{R} \setminus \{0\}$. The process X is called the **planar Brownian motion** started at $(0, a_0)$.

The purpose of this problem is to prove that X will never hit $(0,0)$ (with probability 1). In fact, by scaling and rotational invariance, this means that, given any point $(x, y) \neq (0, a_0)$ the planar Brownian motion will never hit (x, y) (with probability 1). In the language of stochastic analysis, we say that points are **polar** for the planar Brownian motion.

1. How can this be, given that each trajectory of X hits at least one point other than $(0, a_0)$? In fact, it hits uncountably many of them.

2. Show that the planar Brownian motion is rotationally invariant. More precisely, if $U : \mathbb{R}^2 \to \mathbb{R}^2$ is an orthogonal linear transformation, then the process $X'_t = UX_t$, $t \geq 0$, is also a planar Brownian motion (started at $U(0, a_0)$.)

3. We start by defining the stopping time

$$T = \inf\{t \geq 0 \,:\, X_t = (0,0)\},$$

so that what we need to prove is that $\mathbb{P}[T < \infty] = 0$. Before it gets the chance to hit $(0,0)$, X will have to hit the x-axis so that $\mathbb{P}[T \geq T_1] = 1$, where

$$T_1 = \inf\left\{t \geq 0 \,:\, X^{(2)}_t = 0\right\},$$

where $X = (X^{(1)}, X^{(2)})$. Let $a_1 = X^{(1)}_{T_1}$ be the x-coordinate of the position at which the x axis is first hit. Identify the distribution of a_1, and conclude that $\mathbb{P}[a_1 = 0] = 0$. *Hint:* Use the formula for the Laplace transform of T_1 and the fact that T_1 is independent of $\{X^{(1)}_t\}_{t\in[0,\infty)}$ (by the Strong Markov Property); also characteristic functions.

4. It follows that $T_1 < \infty$, a.s., and the point at which the process hits the x axis is different from $(0,0)$, with probability 1. Hence, $\mathbb{P}[T_2 > T_1] = 1$, where

$$T_2 = \inf\left\{t \geq T_1 \,:\, X^{(1)}_t = 0\right\}.$$

the first time after T_1 that the y axis is hit. Moreover, it is clear that $T \geq T_2$, a.s. Let $a_2 = X^{(2)}_{T_2}$ be the y-coordinate of the position of the process at time T_2.

The behavior of X from T_1 to T_2 has the same conditional distribution (given $\mathcal{F}_{T_1}$) as the planar Brownian motion started from $(a_1, 0)$ (instead of $(0, a_0)$). Use this idea to compute the conditional distributions of T_2 and a_2, given $\mathcal{F}_{T_1}$, and show that $\zeta_2 = (T_2 - T_1)/a_1^2$ and $\gamma_2 = a_2/a_1$ are independent of $\mathcal{F}_{T_1}$. *Hint:* Strong Markov Property and rotational invariance.

5. Continue the procedure outlined above (alternating the x and y axes) to define the stopping times $T_3, T_4, \dots$ and note that $T \geq T_n$, for each $n \in \mathbb{N}$. Consequently, our main statement will follow if we show that

$$\mathbb{P}[\sum_{n=1}^{\infty} \tau_n = +\infty] = 1,$$

where, $\tau_1 = T_1$ and $\tau_n = T_n - T_{n-1}$, for $n > 1$. Similarly, define the positions $a_3, a_4, \dots$. Show that there exist an iid sequence of random vectors $\{(\gamma_n, \zeta_n)\}_{n\in\mathbb{N}}$ such that

$$a_n = a_{n-1}\gamma_n \text{ and } \tau_n = \zeta_n a_{n-1}^2,$$

for $n \in \mathbb{N}$. What are the distributions of γ_n and ζ_n, for each $n \in \mathbb{N}$? *Note:* Even though it is not used in the proof of the main statement of the problem, the following question is a good test of your intuition: are γ_1 and ζ_1 independent?

6. At this point, forget all about the fact that the sequence $\{(\gamma_n, \zeta_n)\}_{n\in\mathbb{N}}$ comes from a planar Brownian motion, and show, using discrete-time methods, that
$$\sum_{n=1}^{\infty} \tau_n = \sum_{n=1}^{\infty} a_{n-1}^2 \zeta_n = \infty, \text{ a.s.} \tag{17.8}$$
7. Next, show that the event $A = \{\sum_{n=1}^{\infty} a_{n-1}^2 \zeta_n = \infty\}$ is trivial, i.e., that $\mathbb{P}[A] = 0$ or $\mathbb{P}[A] = 1$. Explain why it is enough to show that $\limsup_n \mathbb{P}[a_{n-1}^2 \zeta_n \geq a_0^2] > 0$ to prove that $\mathbb{P}[A] = 1$.
8. For the grand finale, show that the random variable $\ln |\gamma_1|$ is symmetric with respect to the origin and conclude that $\mathbb{P}[a_{n-1}^2 \geq a_0^2] = \frac{1}{2}$. Use that to show that (17.8) holds.

Problem 17.6 (An erroneous conclusion)**.** What is wrong with the following argument which seems to prove that no zero of the Brownian motion is isolated from the right (which would imply that $B_t = 0$ for all $t \geq 0$):

"Define
$$\tau = \inf\{t \geq 0 \,:\, B_t = 0 \text{ and } \exists\, \varepsilon > 0,\ B_s \neq 0 \text{ for } s \in [t, t+\varepsilon]\}$$
and note that τ is an $\{\mathcal{F}_t^{B+}\}_{t\in[0,\infty)}$-stopping time. Therefore the strong Markov property can be applied to the finite stopping time $\tau \wedge t_0$ to conclude that $W_t = B_{\tau\wedge t_0 + t} - B_{\tau \wedge t_0}$, $t \geq 0$, is a Brownian motion with the property that $W_t(\omega)$ does not take the value 0 on some ω-dependent neighborhood of 0 for $\omega \in \{\tau < t_0\}$. By Corollary from the notes, this happens only on a set of probability 0, so $\mathbb{P}[\tau < t_0] = 0$, and, since t_0 is arbitrary, $\tau = \infty$, a.s."

Problem 17.7 (The third arcsine law (*))**.** The **third arcsine law** states that the random variable
$$\tau_3 = \int_0^1 \mathbf{1}_{\{B_u > 0\}}\, du,$$
which is sometimes called the **occupation time of** $(0, \infty)$, has the arcsine distribution. The method of proof outlined in this problem is different from those for the first two arcsine laws. We start with an analogous result for the Brownian motion and use Donsker's invariance principle to apply it to the paths of the Brownian motion.

Let $(X_0, X_1, \dots, X_n)$ be the first n steps of a simple symmetric random walk, i.e., $X_0 = 0$ and the increments $\xi_1, \dots, \xi_n$, where $\xi_k = X_k - X_{k-1}$ are independent coin tosses. As in the proof of the reflection principle for random walks, we think of X as a random variable which takes values in the path space

$$C(0,\dots,n) = \{(x_0, x_1, \dots, x_n) \in \mathbb{Z}^{n+1} : x_0 = 0, x_i - x_{i-1} \in \{0,1\}, \text{ for } i = 1,\dots,n\}.$$

and define the functionals $I_n, J_n : C(0,\dots,n) \to \mathbb{N}_0$ by

$$I_n(x) = \#\{1 \le k \le n : x_k > 0\}$$

and

$$J_n(x) = \min\{0 \le k \le n : x_k = \max_{0\le j\le n} x_j\},$$

for $x \in C(0,\dots,n)$. The first order of business is to show that $I_n(X)$ and $J_n(X)$ have the same distribution for $n \in \mathbb{N}$ when X is a simple random walk (this is known as **Richard's lemma**).

1. Let $P : \{-1,1\}^n$ be the partial sum mapping, i.e., for $c = (c_1,\dots,c_n) \in \{-1,1\}^n$, $P(c) = x$, where $x_0 = 0$, $x_k = \sum_{j=1}^k c_k$, $k = 1,\dots,n$. We define a mapping $T_c : \{-1,1\}^n \to \{-1,1\}^n$ by the following procedure:

- Place in *decreasing* order of k those c_k for which $P(c)_k > 0$.
- Then, place the remaining increments in the *increasing* order of k.

For example,

$$T_c\Big((1,1,-1,-1,-1,1,1)\Big) = (1,-1,1,1,-1,-1,1).$$

With T_c defined, let $T : C(0,\dots,n) \to C(0,\dots,n)$ be the transformation whose value on $x \in C(0,\dots,n)$ you obtain by applying T_c to the increments of x and returning the partial sums, i.e. $T = P^{-1}T_cP$. For example

$$T\Big((0,1,2,1,0,-1,0,1)\Big) = (0,1,0,1,2,1,0,1).$$

Prove that T is a bijection and that if $(Y_0,\dots,Y_n) = T\Big((X_0,\dots,X_n)\Big)$, then $Y_0,\dots,Y_n$ is a simple symmetric random walk. *Hint:* Induction.

2. Show that $I_n(x) = J_n(T(x))$, for all $n \in \mathbb{N}$, $x \in C(0,\dots,n)$. Deduce Richard's lemma. *Hint:* Draw pictures and use induction.

3. Prove that functional

$$g(f) = \inf\{t \in [0,1] : f(t) = \sup_{s\in[0,1]} f(s)\}$$

is not continuous on $C[0,1]$, but it is continuous at every $f \in C[0,1]$ which admits a unique maximum (hence a.s. with respect to the Wiener measure on $C[0,1]$.) *Note:* We are assuming the reader will easily adapt the notions and results from the canonical space $C[0,\infty)$ to its finite-horizon counterpart $C[0,1]$.

4. Show that the functional h, defined by

$$h(f) = \int_0^1 \mathbf{1}_{\{f(t)>0\}}\, dt,$$

is not continuous on $C[0,1]$, but is continuous at each $f \in C[0,1]$ which has the property that $\lim_{\varepsilon\searrow 0} \int_0^1 \mathbf{1}_{\{0\leq f(t)\leq \varepsilon\}}\, dt = 0$, and, hence, a.s. with respect to the Wiener measure on $C[0,1]$.

5. Let $\{X^n_t\}_{t\in[0,1]}$ be the n-scaled interpolated random walk constructed from X as in Definition 14.37. Show that the difference

$$\tfrac{1}{n} I_n(X) - \int_0^1 \mathbf{1}_{\{X^n_t>0\}}\, dt$$

converges to 0 in probability.

6. Use Donsker's invariance principle (Theorem 14.38) and the Portmanteau theorem to identify the distribution of τ_3.