Course: Theory of Probability II
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Lecture 15
First Properties of the Brownian Motion

This lecture deals with some of the more immediate properties of the Brownian motion and its trajectories. Many other properties which require various tools from stochastic analysis will be scattered throughout the remainder of the notes.

Proposition 15.1 (Symmetries of the Brownian motion)**.** *Let $\{B_t\}_{t\in[0,\infty)}$ be a Brownian motion. Then the following processes are also Brownian motions (only on $[0,1]$ in (5)):*

1. $\{-B_t\}_{t\in[0,\infty)}$, (reflection),
2. $\{\frac{1}{\sqrt{\alpha}}B_{\alpha t}\}_{t\in[0,\infty)}$, *for* $\alpha > 0$, (scaling),
3. $\{B_{t_0+t} - B_{t_0}\}_{t\in[0,\infty)}$, *for all* $t_0 \geq 0$, (shifting),
4. $\{X_t\}_{t\in[0,\infty)}$, *where* $X_0 = 0$ *and* $X_t = tB_{1/t}$, *for* $t > 0$, (inversion),
5. $\{B_1 - B_{1-t}\}_{t\in[0,1]}$, (time reversal).

Proof. It is easy to see that all of the above are centered Gaussian processes with the Brownian covariance strucure. Continuity of the paths is clear in 2., 3. and 5., and everywhere except at $t = 0$ in 4. To deal with that case, we may use the Kolmogorov-Čentsov theorem (how?), or use our next result. □

Proposition 15.2 (The Law of Large Numbers for Brownian Motion)**.** *Let $\{B_t\}_{t\in[0,\infty)}$ be a Brownian motion. Then*

$$\lim_{t\to\infty} \frac{B_t}{t} = 0, \text{ a.s.}$$

Proof. Using the (discrete-time) Law of Large numbers, we have

$$\lim_{n\to\infty} \frac{B_n}{n} = \lim_n \frac{1}{n}\sum_{k=1}^n (B_k - B_{k-1}) = 0, \text{ a.s.}$$

The idea is to show that the trajectory of B cannot deviate too much from B_n on $[n, n+1]$. Indeed, if we can prove that

$$\sum_{n=0}^{\infty} \mathbb{P}\left[\sup_{t\in[n,n+1]} |B_t - B_n| \geq n^{2/3}\right] < \infty, \qquad (15.1)$$

the Borel-Cantelli theorem will finish the job for us (why?). Therefore, all we need to do is estimate the probability in (15.1). We use Kolmogorov's inequality (a specialization of the maximal inequality for submartingales) applied to the discrete random walk $(B_{n+k2^{-m}} - B_n)_{k\in\mathbb{N}_0}$ to conclude that

$$\mathbb{P}[\sup_{0<k\leq 2^m} |B_{n+k2^{-m}} - B_n| \geq n^{2/3}] \leq \tfrac{1}{n^{4/3}} \mathbb{E}[(B_{n+1} - B_n)^2] = \tfrac{1}{n^{4/3}}.$$

However, by the continuity of trajectories of B, we have

$$\Big\{ \sup_{t\in[n,n+1]} |B_t - B_n| \geq n^{2/3} \Big\} = \bigcup_{m\in\mathbb{N}} \Big\{ \sup_{0<k\leq 2^m} |B_{n+k2^{-m}} - B_n| \geq n^{2/3} \Big\},$$

and the union is increasing. Therefore,

$$\mathbb{P}\left[\sup_{t\in[n,n+1]} |B_t - B_n| \geq n^{2/3} \right] \leq \tfrac{1}{n^{4/3}},$$

and (15.1) follows. □

Proposition 15.3 (Long-term behavior of trajectories)**.** *Let* $\{B_t\}_{t\in[0,\infty)}$ *be a Brownian motion. Then,*

$$\limsup_{t\to\infty} \tfrac{B_t}{\sqrt{t}} = \infty, \text{ and } \liminf_{t\to\infty} \tfrac{B_t}{\sqrt{t}} = -\infty, \text{ a.s.}$$

Proof. For $K \in (0, \infty)$, we have

$$\begin{aligned} \mathbb{P}[B_n > K\sqrt{n} \text{ i.o.}] &= \mathbb{P}[\cap_{n\in\mathbb{N}} \cup_{m\geq n} \{B_m > K\sqrt{m}\}] \\ &= \lim_{n\to\infty} \mathbb{P}[\cup_{m\geq n}\{B_m > K\sqrt{m}\}] \\ &\geq \lim_{n\to\infty} \mathbb{P}[B_n > K\sqrt{n}] = \mathbb{P}[B_1 > K] > 0. \end{aligned} \tag{15.2}$$

We can view B_n as a sum $B_n = \sum_{k=1}^n \xi_k$, where $\xi_k = B_k - B_{k-1}$ of independent random variables and note that for any $n_0 \in \mathbb{N}$,

$$\limsup_n B_n/\sqrt{n} > K \text{ if and only if } \limsup_n (B_n - B_{n_0})/\sqrt{n} > K.$$

By Kolmogorov's 0-1 law[1] we have

$$\mathbb{P}[B_n > K\sqrt{n} \text{ i.o.}] = 1, \text{ so that } \limsup \tfrac{B_n}{\sqrt{n}} \geq K, \text{ a.s., for all } K > 0.$$

Therefore,

$$\limsup_{t\to\infty} \tfrac{B_t}{\sqrt{t}} \geq \limsup \tfrac{B_n}{\sqrt{n}} = +\infty, \text{ a.s.},$$

The statement about the lim inf follows from the fact that $\{-B_t\}_{t\in[0,\infty)}$ is also a Brownian motion. □

[1] that is the one that says that the σ-algebra $\cap_{n\in\mathbb{N}}\sigma(\xi_k; k\geq n)$ is trivial when $\{\xi_n\}_{n\in\mathbb{N}}$ are independent.

A nice corollary of the above result is that the Brownian motion visits each point $a \in \mathbb{R}$ in each time-interval $[t, \infty)$, $t \geq 0$.

Corollary 15.4 (Recurrence of the Brownian motion)**.** *The Brownian motion is recurrent, i.e., for each $a \in \mathbb{R}$,*

the (random) set $\mathcal{L}^a(\omega) = \{t \in [0, \infty) : B_t(\omega) = a\}$ is unbounded, a.s.

The quadratic variation of the Brownian motion

We start by introducing some space-saving notation related to partitions. Given $t > 0$, a sequence

$$0 = t_0 < t_1 < \cdots < t_k = t$$

is called a **partition of** $[0, t]$ and the set of all partitions of $[0, t]$ is denoted by $P_{[0,t]}$. The elements $t_0, t_1, \ldots$ of a partition are referred to as its **nodes**. For $\Delta = \{t_0, \ldots, t_k\} \in P_{[0,t]}$, the **mesh** $|\Delta|_{[0,t]}$ of Δ is defined by

$$|\Delta|_{[0,t]} = \sup_{i \in \{1,\ldots,k\}} |t_i - t_{i-1}| .$$

A sequence $\{\Delta_n\}_{n\in\mathbb{N}}$ in $P_{[0,t]}$ is said to **converge to identity**, denoted by $\Delta_n \to \mathrm{Id}$, if $|\Delta_n|_{[0,t]} \to 0$, for each $t \geq 0$. Additionally, we say that the convergence is **fast**, denoted by $\Delta_n \overset{sum}{\to} \mathrm{Id}$, if $\sum_n |\Delta_n|_{[0,t]} < \infty$.

For $\Delta = \{t_0, \ldots, t_k\} \in P_{[0,t]}$, a real function $f : [0, t] \to \mathbb{R}$ (or, more generally, $f : [0, \infty) \to \mathbb{R}$) and $p \geq 1$, we define the ***p*-variation** $\mathrm{Var}_p(f; \Delta)$ **of** f **along** Δ by

$$\mathrm{Var}_p(f; \Delta) = |f(0)|^p + \textstyle\sum_{i=1}^k |f(t_i) - f(t_{i-1})|^p .$$

The **total *p*-variation** $\mathrm{Var}_p(f; [0, t])$ **of** f **is given by**

$$\mathrm{Var}_p(f; [0, t]) = \sup_{\Delta \in P_{[0,t]}} \mathrm{Var}_p(f; \Delta),$$

and the function f is said to be **of finite *p*-variation** if $\mathrm{Var}_p(f; [0, t]) < \infty$, for all $t \geq 0$. When $p = 1$ we simplify the notation by writing Var for Var_1 and refer to the functions of finite 1-variation simply as **functions of finite variation** or **rectifiable functions**.

For a stochastic process $\{X\}_{t\in[0,\infty)}$, and $\Delta \in P_{[0,t]}$, the expression $\mathrm{Var}_p(X; \Delta)$ refers to the random variable, whose value on $\omega \in \Omega$ is $\mathrm{Var}_p(X_\cdot(\omega); \Delta)$. A similar interpretation can be applied for the total variation $\mathrm{Var}_p(X; [0, t])$.

Proposition 15.5 (The quadratic variation of the Brownian Motion)**.** *Let $\{B_t\}_{t\in[0,\infty)}$ be a Brownian motion, and, for $t \geq 0$, let $\{\Delta_n\}_{n\in\mathbb{N}}$ be a sequence in $P_{[0,t]}$ with $|\Delta_n|_{[0,t]} \to 0$. Then*

$$\lim_n \mathrm{Var}_2(B; \Delta_n) = t, \text{ in } \mathbb{L}^2. \tag{15.3}$$

If $\Delta_n \overset{sum}{\to} \mathrm{Id}$, then the convergence in (15.3) also holds in the a.s.-sense.

Proof. We start with two very simple identities, valid for all $0 \leq r \leq s \leq u \leq v$:

$$\mathbb{E}\left[\left((B_s - B_r)^2 - (s-r)\right)^2\right] = 2(s-r)^2,$$

and

$$\mathbb{E}\left[\left((B_s - B_r)^2 - (s-r)\right)\left((B_v - B_u)^2 - (v-u)\right)\right] = 0.$$

It follows that, for any partition $\Delta \in P_{[0,t]}$, we have

$$\mathbb{E}\left[\left(\operatorname{Var}_2(B;\Delta) - t\right)^2\right] = 2\operatorname{Var}_2(\mathrm{Id};\Delta),$$

where Id denotes the identity function $s \mapsto s$. The first claim is now a consequence of the estimate

$$\operatorname{Var}_2(\mathrm{Id},\Delta) = \sum_{i=1}^{k} |t_i - t_{i-1}|^2 \leq t\,|\Delta|_{[0,t]}\,.$$

The second one follows from the first one and an application of the Borel-Cantelli lemma. □

Remark 15.6. Note that Proposition 15.5 *does not* imply that the paths of Brownian motion have finite 2-variation, a.s. In fact, it can be proved that, for each $t > 0$, $\operatorname{Var}_2(B;[0,t]) = \infty$, a.s.

Corollary 15.7 (Non-rectifiability of Brownian paths)**.** *Paths of the Brownian Motion have infinite variation on $[0,t]$, for all $t \geq 0$, a.s., i.e.*

$$\forall\, t > 0,\ \operatorname{Var}_1(B;[0,t]) = \infty,\ \textit{a.s.}$$

Proof. For a partition $\Delta = \{t_0, \dots, t_k\} \in P_{[0,t]}$, we clearly have

$$\operatorname{Var}_2(B;\Delta) \leq \delta(B;\Delta)\operatorname{Var}_1(B,\Delta) \text{ where } \delta(B,\Delta) = \sup_{1\leq i\leq k} \left|B_{t_i} - B_{t_{i-1}}\right|.$$

If $\{\Delta_n\}_{n\in\mathbb{N}}$ is a sequence of partitions in $P_{[0,t]}$ with $\Delta_n \stackrel{sum}{\to} \mathrm{Id}$, Proposition 15.5 implies that $\operatorname{Var}_2(B,\Delta_n) \to t$, a.s. On the other hand, the continuity (and therefore uniform continuity on compacts) of the paths of the Brownian motion allows us to conclude that $\delta(B,\Delta_n) \to 0$, a.s. It follows that, necessarily, $\operatorname{Var}_1(B,\Delta_n) \to \infty$, and, a fortiori, that

$$\operatorname{Var}_1(B,[0,t]) = \sup_{\Delta\in P_{[0,t]}} \operatorname{Var}_1(B;\Delta) \geq \sup_n \operatorname{Var}_1(B;\Delta_n) = \infty, \text{ a.s.} \quad \square$$

Logarithmic laws (*)

We finish with some fine regularity properties of Brownian paths.

Definition 15.8 (Uniform local modulus of continuity)**.** The map δ, defined in a right neighborhood of 0, is called the **uniform local modulus of continuity** for the function $f : [0,1] \to \mathbb{R}$, if there exists $h_0 > 0$ such that, for each $t \in [0,1]$,

$$|f(t+h) - f(t)| \leq \delta(h), \text{ for all } 0 < h < h_0 \text{ with } t+h \leq 1.$$

The estimates from the Lévy-Ciesielski construction above, lead relatively directly to the following result:

Proposition 15.9 (A uniform local modulus of continuity for the Brownian motion)**.** *There exists a constant C such that, for almost all* ω

$$\delta(h) = C\sqrt{h \log(1/h)}$$

is the local uniform modulus of continuity of the trajectory $t \mapsto B_t(\omega)$*.*

Proof. We use the notation from the Lévy-Ciesielski construction and recall that

$$B_t^{(n)} = \sum_{k=1}^{n} \Delta_t^{(n)} \to B_t, \text{ uniformly over } t \in [0,1], \text{ a.s.}$$

In the course of the proof of Proposition 14.19, we established that the following bound

$$\mathbb{P}\left[\sup_{t\in[0,1]} \left|\Delta_t^{(n)}\right| \leq C_1\sqrt{n2^{-n}} \text{ ev.}\right] = 1, \tag{15.4}$$

holds for all C_1 large enough, say $C_1 = \sqrt{2}$. Thanks to the piecewise-linear structure of $\Delta^{(n)}$, and the fact that the intervals of linearity are of the size 2^{-n}, we clearly have

$$\mathbb{P}\left[\sup_{t\in[0,1]} \left|\tfrac{d}{dt}\Delta_t^{(n)}\right| \leq C_1\sqrt{n2^{n}} \text{ ev.}\right] = 1. \tag{15.5}$$

Therefore, there exists a random variable L with values in $\mathbb{N}$ such that

$$\sup_{t\in[0,1]} \left|\Delta_t^{(n)}\right| \leq C_1\sqrt{n2^{-n}} \text{ and } \sup_{t\in[0,1]} \left|\tfrac{d}{dt}\Delta_t^{(n)}\right| \leq C_1\sqrt{n2^{n}}, \text{ for all } n \geq L, \text{ a.s.}$$

For any random variable $K \in \mathbb{N}$ with $K \geq L$, $t \in [0,1]$, and $h > 0$, we have

$$|B_{t+h} - B_t| \leq \sum_{n=0}^{\infty} \left|\Delta^{(n)}(t+h) - \Delta^{(n)}(t)\right| \leq (I) + (II) + (III),$$

where

$$(I) = h\sum_{n=0}^{L} \sup_{t\in[0,1]} \left|\tfrac{d}{dt}\Delta_t^{(n)}\right|, \; (II) = h\sum_{n=L+1}^{K} C_1\sqrt{n2^{n}},$$

and

$$(III) = 2 \sum_{n=K+1}^{\infty} C_1 \sqrt{n2^{-n}}.$$

First, we pick (a random) $h_0 > 0$ so small that

$$(I) \leq \sqrt{h \log(1/h)}, \text{ for } 0 < h \leq h_0.$$

Then, given $0 < h < h_0$, we construct an $\mathbb{N}$-valued random variable K' such that $2^{-K'-1} \leq h < 2^{-K'}$, and set $K = \max(K', L+1)$. Using the estimate

$$\begin{aligned}\sum_{n=N}^{\infty} \sqrt{n2^{-n}} &= N2^{-N} \sum_{n=N}^{\infty} \sqrt{\tfrac{n}{N} 2^{-(n-N)}} \\ &= N2^{-N} \sum_{k=0}^{\infty} \sqrt{(1+\tfrac{k}{N})2^{-k}} \leq C_2 N 2^{-N},\end{aligned}$$

where $C_2 = \sum_{k\geq 0} \sqrt{(1+k)2^{-k}} < \infty$, we conclude that

$$\begin{aligned}(III) &\leq 2C_1 \sum_{n=K'+1}^{\infty} \sqrt{n2^{-n}} \leq 2C_1C_2 \sqrt{(1+K')2^{-K'-1}} \\ &\leq 4C_1C_2 \sqrt{h \log(1/h)}.\end{aligned}$$

Finally,

$$\begin{aligned}(II) &\leq hC_1 \sum_{n=1}^{K'} \sqrt{n2^n} \leq C_1 \sum_{n=1}^{K'} \sqrt{n2^{(n-2K')}} = C_1 \sum_{k=K'}^{2K'} \sqrt{(k-K')2^{-k}} \\ &\leq C_1 \sum_{k=K'}^{\infty} \sqrt{k2^{-k}} \leq C_1C_2\sqrt{K'2^{-K'}} \leq 2C_1C_2\sqrt{h \log(1/h)}.\end{aligned}$$

All in all, $|B_{t+h} - B_t| \leq (1 + 6C_1C_2)\sqrt{h \log(1/h)}$, for $h \leq h_0$. □

The full result of Lévy (whose proof we omit) is that the function δ of Proposition 15.9 is optimal, and that the the constant C can be chosen to be equal to $\sqrt{2}$.

Theorem 15.10 (Lévy's modulus of continuity)**.**

$$\sup_{t\in[0,1]} \limsup_{h\searrow 0} \frac{|B_{t+h}-B_t|}{\sqrt{2h\log(1/h)}} = 1, \text{ a.s.}$$

If one focuses on the fluctuations of the Brownian motion around a single, fixed, point $t \geq 0$, one gets a slightly tighter estimate.

Theorem 15.11 (Law of iterated logarithm)**.** *For each $t \geq 0$, we have*

$$\limsup_{h\searrow 0} \frac{|B_{t+h}-B_t|}{\sqrt{2h\log(\log(1/h))}} = 1, \text{ a.s.}$$

Additional Problems

Problem 15.1 (p-variations of functions)**.**

1. Show that, for a function $f : [0,t] \to \mathbb{R}$, $\text{Var}_p(f;[0,t]) < \infty$ implies $\text{Var}_q(f;[0,t]) < \infty$ for $q > p > 0$.

2. For each $q > 1$, find an example of a function $f : [0,1] \to \mathbb{R}$ with the property that

$$\begin{cases} \text{Var}_p(f;[0,1]) < \infty, & p > q, \\ \text{Var}_p(f;[0,1]) = \infty, & p \leq q, \end{cases}$$

3. For each $q \geq 1$, find an example of a function $f : [0,1] \to \mathbb{R}$ with the property that

$$\begin{cases} \text{Var}_p(f;[0,1]) < \infty, & p \geq q, \\ \text{Var}_p(f;[0,1]) = \infty, & p < q. \end{cases}$$

4. For a function $f : [0,\infty) \to \mathbb{R}$ of finite variation (remember, that means that $\text{Var}_1(f;[0,t]) < \infty$ for all $t > 0$), define $F : [0,\infty) \to \mathbb{R}$ by $F(t) = \text{Var}(f;[0,t])$, $t \geq 0$. Show that $|f(t) - f(s)| \leq |F(t) - F(s)|$, for all $t, s \in [0,\infty)$. Deduce that f can be written as a difference of two monotone functions, and, more generally, the f is of finite variation if and only if it can be written as a difference of two monotone functions.

Problem 15.2 (Quadratic covariation of independent Brownian motions)**.** Let the stochastic processes $\{X_t\}_{t\in[0,\infty)}$ and $\{Y_t\}_{t\in[0,\infty)}$ be defined on the same probability space. For a partition $\Delta \in P_{[0,t]}$, we define the **quadratic covariation of** X **and** Y **along** Δ by

$$\text{Var}_2(X,Y;\Delta) = \sum_{i=1}^{k} (X_{t_i} - X_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}}),$$

where $\Delta = \{0 = t_0, \dots, t_k = t\}$. If X and Y are two independent Brownian motions, i.e., such that the σ-algebras $\sigma(X_t; t \geq 0)$ and $\sigma(Y_t; t \geq 0)$ are independent, show that $\text{Var}_2(X,Y;\Delta^{(n)}) \to 0$ in $\mathbb{L}^2$, for each sequence $\{\Delta^{(n)}\}_{n\in\mathbb{N}}$ in $P_{[0,t]}$ with $\Delta^{(n)} \to \text{Id}$.

Problem 15.3 (Higher-dimensional Brownian motion)**.** For $d \in \mathbb{N}$, a vector-valued stochastic process $(B_t^{(1)}, \dots, B_t^{(d)})_{t\geq 0}$, with values in $\mathbb{R}^d$, is said to be a **d-dimensional Brownian motion** if its components $B^{(1)}$, $\dots$, $B^{(d)}$ are independent Brownian motions. Given $n \in \mathbb{N}$, find necessary and sufficient conditions on the $\mathbb{R}^{n\times d}$-matrix H such that the process

$$W_t = \begin{bmatrix} W_t^{(1)} \\ \vdots \\ W_t^{(n)} \end{bmatrix} \text{ where } W_t = HB_t \text{ and } B_t = \begin{bmatrix} B_t^{(1)} \\ \vdots \\ B_t^{(d)} \end{bmatrix} \quad t \geq 0$$

is an n-dimensional Brownian motion.

Problem 15.4 (Monotonicity and maxima of the Brownian path)**.** Prove the following statements for a Brownian motion B:

1. B is monotone on no interval of the form $[r, s]$, $0 \leq r < s$, a.s.
2. For each $p > 0$, the distribution of the random variable $M_t = \sup_{s \leq t} B_s$ is diffuse, i.e. $\mathbb{P}[M_t = a] = 0$, for all $a \in \mathbb{R}$. *Hint:* Argue, first, that it is enough to assume that $t = 1$. Let $\tilde{M}_1$ be an independent random variable with the same distribution as M_1. Show that $\sqrt{2}M_1$ and $\max(M_1, W_1 + \tilde{M}_1)$ have the same distribution. Deduce that the only possible atom for M_1 is 0. Then show that $\mathbb{P}[M_1 > 0] = 1$.
3. B attains different maxima on any two non-overlapping intervals (r_1, s_1) and (r_2, s_2), a.s.
4. Each local maximum of B is a strict local maximum, a.s.
5. B achieves its global maximum on $[0, 1]$ in exactly one point, a.s.

Problem 15.5 (Non-differentiability of Brownian paths)**.**

1. Show that if $f : [0, 1) \to \mathbb{R}$ is differentiable at $t \in [0, 1)$ (the right derivative is considered at the $t = 0$), then there exists $l, n_0 \in \mathbb{N}$ such that
$$\left| f(\tfrac{j}{n}) - f(\tfrac{j-1}{n}) \right| \leq \tfrac{l}{n},$$
for all $n \geq n_0$ and all $i < j \leq i + 3$, where $i = \lfloor nt \rfloor + 1$ and $\lfloor x \rfloor$ denotes the largest integer not larger than x.
2. Let $\{B_t\}_{t \in [0,\infty)}$ be a Brownian motion. For $s \in [0, 1)$ we define D_s as the set of all $\omega \in \Omega$ such that the trajectory $t \mapsto B_t(\omega)$ is differentiable at s. Show that
$$\bigcup_{s \in [0,1)} D_s \subseteq \Gamma, \text{ where } \Gamma = \bigcup_{l \geq 1} \liminf_{n \to \infty} \bigcup_{i=1}^{n+1} \bigcap_{j=i+1}^{i+3} \left\{ \left| B_{j/n} - B_{(j-1)/n} \right| \leq \tfrac{l}{n} \right\}.$$
3. (*) Show that $\mathbb{P}[\Gamma] = 0$.