**Course:** Theory of Probability II
**Term:** Spring 2015
**Instructor:** Gordan Zitkovic

# Lecture 22
## Girsanov's Theorem

### *An example*

Consider a finite Gaussian random walk

$$X_n = \sum_{k=1}^{n} \xi_k,\ n = 0, \dots, N,$$

where $\xi_k$ are independent $N(0,1)$ random variables. The random vector $(X_1, \dots, X_N)$ is then, itself, Gaussian, and admits the density

$$f(x_1, \dots, x_N) = C_N e^{-\frac{1}{2}\left(x_1^2 + (x_2-x_1)^2 + \dots + (x_N - x_{N-1})^2\right)}$$

with respect to the Lebesgue measure on $\mathbb{R}^N$, for some $C_N > 0$.

Let us now repeat the whole construction, with the $n$-th step having the $N(\mu_n, 1)$-distribution, for some $\mu_1, \dots, \mu_N \in \mathbb{R}$. The resulting, Gaussian, distribution still admits a density with respect to the Lebesgue measure, and it is given by

$$\tilde{f}(x_1, \dots, x_N) = C_N e^{-\frac{1}{2}\left((x_1-\mu_1)^2 + (x_2-x_1-\mu_2)^2 + \dots + (x_N - x_{N-1}-\mu_N)^2\right)}.$$

The two densities are everywhere positive, so the two Gaussian measures are equivalent to each other and the Radon-Nikodym derivative turns out to be

$$\begin{aligned}\frac{d\mathbb{Q}}{d\mathbb{P}} &= \frac{\tilde{f}(X_1,\dots,X_N)}{f(X_1,\dots,X_N)} \\ &= e^{-\left(\mu_1 X_1 - \mu_2(X_2-X_1) - \dots - \mu_N(X_N - X_{N-1})\right) + \frac{1}{2}\left(\mu_1^2 + \mu^2 + \dots + \mu_N^2\right)} \\ &= e^{\sum_{k=1}^N \mu_k (X_k - X_{k-1}) - \frac{1}{2}\sum_{k=1}^N \mu_k^2}.\end{aligned}$$

### *Equivalent measure changes*

Let $\mathbb{Q}$ be a probability measure on $\mathcal{F}$, equivalent to $\mathbb{P}$, i.e., $\forall A \in \mathcal{F}$, $\mathbb{P}[A] = 0$ if and only if $\mathbb{Q}[A] = 0$. Its Radon-Nikodym derivative

$Z = \frac{d\mathbb{Q}}{d\mathbb{P}}$ is a non-negative random variable in $\mathbb{L}^1$ with $\mathbb{E}[Z] = 1$. The uniformly-integrable martingale

$$Z_t = \mathbb{E}[Z|\mathcal{F}_t],\ t \geq 0,$$

is called the **density** of $\mathbb{Q}$ with respect to $\mathbb{P}$ (note that we can - and do - assume that $\{Z_t\}_{t\in[0,\infty)}$ is càdlàg). We will often use the shortcut **$\mathbb{Q}$-(local, semi-, etc.) martingale** for a process which is a (local, semi-, etc.) martingale for $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$.

**Proposition 22.1.** *Let $\{X_t\}_{t\in[0,\infty)}$ be a càdlàg and adapted process. Then $X$ is a $\mathbb{Q}$-local martingale if and only if the product $\{Z_t X_t\}_{t\in[0,\infty)}$ is a càdlàg $\mathbb{P}$-local martingale.*

Before we give a proof, here is a simple and useful lemma. Since the measures involved are equivalent, we are free to use the phrase "almost surely" without explicit mention of the probability.

**Lemma 22.2.** *Let $(\Omega, \mathcal{H}, \mathbb{P})$ be a probability space, and let $\mathcal{G} \subseteq \mathcal{H}$ be a sub-$\sigma$-algebra of $\mathcal{H}$. Givan a probability measure $\mathbb{Q}$ on $\mathcal{H}$, equivalent to $\mathbb{P}$, let $Z = \frac{d\mathbb{Q}}{d\mathbb{P}}$ be its Radon-Nikodym derivative with respect to $\mathbb{P}$. For a random variable $X \in \mathbb{L}^1(\mathcal{F}, \mathbb{Q})$ we have $XZ \in \mathbb{L}^1(\mathbb{P})$ and*

$$\mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}] = \frac{1}{\mathbb{E}[Z|\mathcal{G}]}\mathbb{E}[XZ|\mathcal{G}],\ a.s.$$

*where $\mathbb{E}^{\mathbb{Q}}[\cdot|\mathcal{G}]$ denotes the conditional expectation on $(\Omega, \mathcal{H}, \mathbb{Q})$.*

*Proof.* First of all, note that the Radon-Nikodym theorem implies that $XZ \in \mathbb{L}^1(\mathbb{P})$ and that the set $\{\mathbb{E}[Z|\mathcal{G}] = 0\}$ has $\mathbb{Q}$-probability (and, therefore $\mathbb{P}$-probability) 0. Indeed,

$$\begin{aligned}\mathbb{Q}[\mathbb{E}[Z|\mathcal{G}] = 0] &= \mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{\{\mathbb{E}[Z|\mathcal{G}]=0\}}] = \mathbb{E}[Z\mathbf{1}_{\{\mathbb{E}[Z|\mathcal{G}]=0\}}] \\ &= \mathbb{E}[\mathbb{E}[Z\mathbf{1}_{\{\mathbb{E}[Z|\mathcal{G}]=0\}}|\mathcal{G}]]\end{aligned}$$

Therefore, the expression on the right-hand side is well-defined almost surely, and is clearly $\mathcal{G}$-measurable. Next, we pick $A \in \mathcal{G}$, observe that

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[\mathbf{1}_A \tfrac{1}{\mathbb{E}[Z|\mathcal{G}]}\mathbb{E}[XZ|\mathcal{G}]] &= \mathbb{E}[Z\mathbf{1}_A \tfrac{1}{\mathbb{E}[Z|\mathcal{G}]}\mathbb{E}[XZ|\mathcal{G}]] \\ &= \mathbb{E}[\mathbb{E}[Z|\mathcal{G}]\mathbf{1}_A \tfrac{1}{\mathbb{E}[Z|\mathcal{G}]}\mathbb{E}[XZ|\mathcal{G}]] \\ &= \mathbb{E}[\mathbb{E}[ZX\mathbf{1}_A|\mathcal{G}]] = \mathbb{E}^{\mathbb{Q}}[X\mathbf{1}_A],\end{aligned}$$

and remember the definition of conditional expectation. □

*Proof of Proposition 22.1.* Suppose, first, that $X$ is a $\mathbb{Q}$-martingale. Then $\mathbb{E}^{\mathbb{Q}}[X_t|\mathcal{F}_s] = X_s$, $\mathbb{Q}$-a.s. By the tower property of conditional expectation, the random variable $Z_t$ is the Radon-Nikodym derivative of (the restriction of) $\mathbb{Q}$ with respect to (the restriction of) $\mathbb{P}$ on the probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$ (prove this yourself!). Therefore, we can use Lemma 22.2 with $\mathcal{F}_t$ playing the role of $\mathcal{H}$ and $\mathcal{F}_s$ the role $\mathcal{G}$, and rewrite the $\mathbb{Q}$-martingale property of $X$ as

(22.1) $\frac{1}{Z_s}\mathbb{E}[X_t Z_t|\mathcal{F}_s] = X_s,\ \mathbb{Q}-\text{a.s., i.e. } \mathbb{E}[X_t Z_t|\mathcal{F}_s] = Z_s X_s,\ \mathbb{P}-\text{a.s.}$

We leave the other direction, as well as the case of a local martingale to the reader. □

**Proposition 22.3.** *Suppose that the density process $\{Z_t\}_{t\in[0,\infty)}$ is continuous. Let $X$ be a continuous semimartingale under $\mathbb{P}$ with decomposition $X = X_0 + M + A$. Then $X$ is also a $\mathbb{Q}$-semimartingale, and its $\mathbb{Q}$-semimartingale decomposition is given by $X = X_0 + N + B$, where*

$$N = M - F,\ B = A + F \text{ where } F_t = \int_0^t \tfrac{1}{Z_t}\, d\langle M, Z\rangle.$$

*Proof.* The process $F$ is clearly well-defined, continuous, adapted and of finite variation, so it will be enough to show that $M - F$ is a $\mathbb{Q}$-local martingale. Using Proposition 22.1, we only need to show that $Y = Z(M - F)$ is a $\mathbb{P}$-local martingale. By Itô's formula (integration-by-parts), the finite-variation part of $Y$ is given by

$$-\int_0^t Z_u\, dF_u + \langle Z, M\rangle_t,$$

and it is easily seen to vanish using the associative property of Stieltjes integration. □

One of the most important applications of the above result is to the case of a Brownian motion.

**Theorem 22.4** (Girsanov; Cameron and Martin)**.** *Suppose that the filtration $\{\mathcal{F}_t\}_{t\in[0,\infty)}$ is the usual augmentation of the natural filtration generated by a Brownian motion $\{B_t\}_{t\in[0,\infty)}$.*

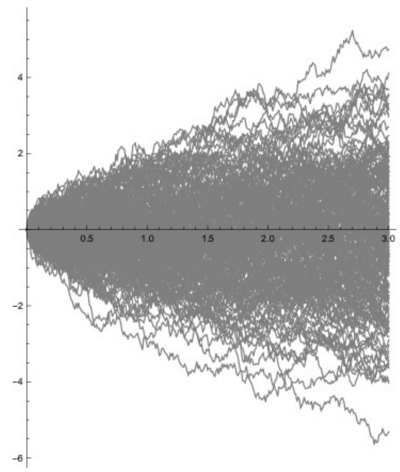
1. *Let $\mathbb{Q} \sim \mathbb{P}$ be a probability measure on $\mathcal{F}$ and let $\{Z_t\}_{t\in[0,\infty)}$ be the corresponding density process, i.e., $Z_t = \mathbb{E}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t]$. Then, here exists a predictable process $\{\theta_t\}_{t\in[0,\infty)}$ in $L(B)$ such that $Z = \mathcal{E}(\int_0^\cdot \theta_u\, dB_u)$ and*

$$B_t - \int_0^t \theta_u\, du \text{ is a } \mathbb{Q}\text{-Brownian motion.}$$

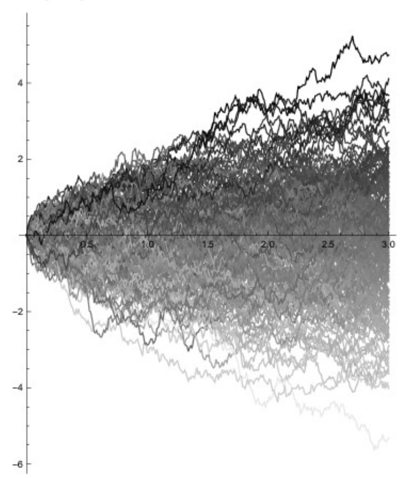
2. *Conversely, let $\{\theta_t\}_{t\in[0,\infty)} \in L(B)$ have the property that the process $Z = \mathcal{E}(\int_0^\cdot \theta_u\, dB_u)$ is a uniformly-integrable martingale with $Z_\infty > 0$, a.s. For any probability measure $\mathbb{Q} \sim \mathbb{P}$ such that $\mathbb{E}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_\infty] = Z_\infty$,*

$$B_t - \int_0^t \theta_u\, du,\ t \geq 0,$$

*is a $\mathbb{Q}$-Brownian motion.*



A cloud of simulated Brownian paths on $[0,3]$



The same cloud with darker-colored paths corresponding to higher values of the Radon-Nikodym derivative $Z_3$.

*Proof.*

1. We start with an application of the martingale representation theorem (Proposition 21.16). It implies that there exists a process $\rho \in L(B)$ such that
$$Z_t = 1 + \int_0^t \rho_u \, dB_u.$$
Since $Z$ is continuous and bounded away from zero on each segment, the process $\{\theta_t\}_{t\in[0,\infty)}$, given by $\theta_t = \rho_t / Z_t$ is in $L(B)$ and we have
$$Z_t = 1 + \int_0^t Z_u \theta_u dB_u.$$
Hence, $Z = \mathcal{E}(\int_0^\cdot \theta_u \, dB_u)$. Proposition 22.3 states that $B$ is a $\mathbb{Q}$-semimartingale with decomposition $B = (B - F) + F$, where the continuous FV-process $F$ is given by
$$F_t = \int_0^t \tfrac{1}{Z_u} \langle B, Z \rangle_u = \int_0^t \tfrac{1}{Z_u} Z_u \theta_u \, du = \int_0^t \theta_u \, du.$$
In particular, $B - F$ is a $\mathbb{Q}$-local martingale. On the other hand, its quadratic variation (as a limit in $\mathbb{P}$-, and therefore in $\mathbb{Q}$-probability) is that of $B$, so, by Lévy's characterization, $B - F$ is a $\mathbb{Q}$-Brownian motion.

2. We only need to realize that any measure $\mathbb{Q} \sim \mathbb{P}$ with $\mathbb{E}[\frac{d\mathbb{Q}}{d\mathbb{P}} | \mathcal{F}_\infty] = Z_\infty$ will have $Z$ as its density process. The rest follows from (1). □

Even though we stated it on $[0, \infty)$, most of applications of the Girsanov's theorem are on finite intervals $[0, T]$, with $T > 0$. The reason is that the condition that $\mathcal{E}(\int_0^\cdot \theta_u \, dB_u)$ be uniformly integrable *on the entire* $[0, \infty)$ is either hard to check or even not satisfied for most practically relevant $\theta$. The simplest conceivable example $\theta_t = \mu$, for all $t \geq 0$ and $\mu \in \mathbb{R} \setminus \{0\}$ gives rise to the exponential martingale $Z_t = e^{\mu B_t - \frac{1}{2}\mu^2 t}$, which is not uniformly integrable on $[0, \infty)$ (why?). On any finite horizon $[0, T]$, the (deterministic) process $\mu \mathbf{1}_{\{t \leq T\}}$ satisfies the conditions of Girsanov's theorem, and there exists a probability measure $\mathbb{P}^{\mu,T}$ on $\mathcal{F}_T$ with the property that $\hat{B}_t = B_t - \mu t$ is a $\mathbb{P}^{\mu,T}$ Brownian motion on $[0, T]$. It is clear, furthermore, that for $T_1 < T_2$, $\mathbb{P}^{\mu,T_1}$ coincides with the restriction of $\mathbb{P}^{\mu,T_2}$ onto $\mathcal{F}_{T_1}$. Our life would be easier if this consistency property could be extended all the way up to $\mathcal{F}_\infty$. It can be shown that this can, indeed, be done in the canonical setting, but not in same equivalence class. Indeed, suppose that there exists a probability measure $\mathbb{P}^\mu$ on $\mathcal{F}_\infty$, equivalent to $\mathbb{P}$, such that $\mathbb{P}^\mu$, restricted to $\mathcal{F}_T$, coincides with $\mathbb{P}^{\mu,T}$, for each $T > 0$. Let $\{Z_t\}_{t\in[0,\infty)}$ be the density process of $\mathbb{P}^\mu$ with respect to $\mathbb{P}$. It follows that
$$Z_T = \exp(\mu B_T - \tfrac{1}{2}\mu^2 T), \text{ for all } T > 0.$$

Since $\mu \neq 0$, we have $B_t - \frac{1}{2}\mu T \to -\infty$, a.s., as $T \to \infty$ and, so, $Z_\infty = \lim_{T\to\infty} Z_T = 0$, a.s. On the other hand, $Z_\infty$ is the Radon-Nikodym derivative of $\mathbb{P}^\mu$ with respect to $\mathbb{P}$ on $\mathcal{F}_\infty$, and we conclude that $\mathbb{P}^\mu$ must be singular with respect to $\mathbb{P}$. Here is slightly different perspective on the fact that $\mathbb{P}$ and $\mathbb{P}^\mu$ must be mutually singular: for the event $A \in \mathcal{F}_\infty$, given by

$$A = \Big\{ \lim_{t\to\infty} \tfrac{B_t}{t} = 0 \Big\},$$

we have $\mathbb{P}[A] = 1$, by the Law of Large Numbers for the Brownian motion. On the other hand, with $\hat{B}_t$ being a $\mathbb{P}^\mu$ Brownian motion, we have

$$\mathbb{P}^\mu[A] = \mathbb{P}^\mu[\lim_{t\to\infty} \tfrac{B_t}{t} = 0] = \mathbb{P}^\mu[\lim_{t\to\infty} \tfrac{\hat{B}_t}{t} = -\mu] = 0,$$

because $\hat{B}_t/t \to 0$, $\mathbb{P}^\mu$-a.s. Not everything is lost, though, as we can still use employ Girsanov's theorem in many practical situations. Here is one (where we take $\mathbb{P}[X \in dx] = f(x)\,dx$ to mean that $f(x)$ is the density of the distribution of $X$.)

**Example 22.5** (Hitting times of the Brownian motion with drift)**.** Define $\tau_a = \inf\{t \geq 0 : B_t = a\}$ for $a > 0$. By the formula derived in a homework, we have

$$\mathbb{P}[\tau_a \in dt] = \tfrac{|a|}{\sqrt{2\pi t^3}} e^{-\frac{a^2}{2t}}\,dt.$$

For $T \geq 0$, (the restriction of) $\mathbb{P}$ and $\mathbb{P}^{\mu,T}$ are equivalent on $\mathcal{F}_T$ with the Radon-Nikodym derivative

$$\tfrac{d\mathbb{P}^{\mu,T}}{d\mathbb{P}} = \exp(\mu B_T - \tfrac{1}{2}\mu^2 T).$$

The optional sampling theorem (justified by the uniform integrability of the martingale $\exp(\mu B_t - \frac{1}{2}\mu^2 t)$ on $[0,T]$) and the fact that $\{\tau_a \leq T\} \in \mathcal{F}_{\tau_a \wedge T} \subseteq \mathcal{F}_T$ imply that

$$\mathbb{E}[\exp(\mu B_T - \tfrac{1}{2}\mu^2 T)|\mathcal{F}_{\tau_a\wedge T}] = \exp(\mu B_{\tau_a\wedge T} - \tfrac{1}{2}\mu^2(\tau_a \wedge T)).$$

Therefore,

$$\begin{aligned}
\mathbb{P}^{\mu,T}[\tau_a \leq T] &= \mathbb{E}^{\mu,T}[\mathbf{1}_{\{\tau_a\leq T\}}] = \mathbb{E}[\exp(\mu B_T - \tfrac{1}{2}\mu^2 T)\mathbf{1}_{\{\tau_a\leq T\}}] \\
&= \mathbb{E}[\exp(\mu B_{\tau_a\wedge T} - \tfrac{1}{2}\mu^2(\tau_a\wedge T))\mathbf{1}_{\{\tau_a\leq T\}}] \\
&= \mathbb{E}[\exp(\mu B_{\tau_a} - \tfrac{1}{2}\mu^2\tau_a)\mathbf{1}_{\{\tau_a\leq T\}}] \\
&= \mathbb{E}[\exp(\mu a - \tfrac{1}{2}\mu^2\tau_a)\mathbf{1}_{\{\tau_a\leq T\}}] = \int_0^T e^{\mu a - \frac{1}{2}\mu^2 t}\,\mathbb{P}[\tau_a \in dt] \\
&= \int_0^T e^{\mu a - \frac{1}{2}\mu^2 t} \tfrac{|a|}{\sqrt{2\pi t^3}} e^{-\frac{a^2}{2t}}\,dt.
\end{aligned}$$

On the other hand, $\{B_t - \mu t\}_{t\in[0,T]}$ is a Brownian motion under $\mathbb{P}^{\mu,T}$, so

$$\mathbb{P}^{\mu,T}[\tau_a \leq T] = \mathbb{P}[\hat{\tau}_a \leq T],$$

where $\hat{\tau}_a$ is the first hitting time of the level $a$ of the Brownian motion with drift $\mu$. It follows immediately that the "density" of $\hat{\tau}_a$ is given by

$$\mathbb{P}[\hat{\tau}_a \in dt] = \tfrac{|a|}{\sqrt{2\pi t^3}} e^{-\frac{(a-\mu t)^2}{2t}}\, dt.$$

We quote the word "density" because, if one tries to integrate it over all $t \geq 0$, one gets

$$\mathbb{P}[\hat{\tau}_a < \infty] = \int_0^\infty \tfrac{|a|}{\sqrt{2\pi t^3}} e^{-\frac{(a-\mu t)^2}{2t}}\, dt = \exp(\mu a - |\mu a|).$$

In words, if $\mu$ and $a$ have the same sign, the Brownian motion with drift $\mu$ will hit $a$ sooner or later. On the other hand, if they differ in sign, the probability that it will never get there is strictly positive and equal to $e^{2\mu a}$.

## *Kazamaki's and Novikov's criteria*

The message of the second part of Theorem 22.4 is that, given a "drift" process $\{\theta_t\}_{t\in[0,\infty)}$, we can turn a Brownian motion into a Brownian motion with drift $\theta$, provided, essentially, that a certain exponential martingale is a UI martingale. Even though useful sufficient conditions for martingality of stochastic integrals are known, the situation is much less pleasant in the case of stochastic exponentials. The most well-known criterion is the one of Novikov. Novikov's criterion is, in turn, implied by a slightly stronger criterion of Kazamaki. We start with an auxiliary integrability result. In addition to the role it plays in the proof of Kazamaki's criterion, it is useful when one needs $\mathcal{E}(M)$ to be a little more than just a martingale.

**Lemma 22.6.** *Let $\mathcal{E}(M)$ be the stochastic exponential of $M \in \mathcal{M}_0^{loc,c}$. If*

$$\sup_{\tau\in\mathcal{S}_b} \mathbb{E}[e^{aM_\tau}] < \infty,$$

*for some constant $a > \frac{1}{2}$, where the supremum is taken over the set $\mathcal{S}_b$ of all finite-valued stopping times $\tau$, then $\mathcal{E}(M)$ is an $\mathbb{L}^p$-bounded martingale for $p = \frac{4a^2}{4a-1} \in (1,\infty)$.*

*Proof.* We pick a finite stopping time $\tau$ and start from the following identity, which is valid for all constants $p, s > 0$:

$$\mathcal{E}(M)_\tau^p = \mathcal{E}(\sqrt{p/s}M)_\tau^s e^{(p-\sqrt{ps})M_\tau}.$$

For $1 > s > 0$, we can use Hölder's inequality (note that $1/s$ and $1/(1-s)$ are conjugate exponents), to obtain

$$\mathbb{E}[\mathcal{E}(M)^p] \leq (\mathbb{E}[\mathcal{E}(\sqrt{p/s}M_\tau)])^s (\mathbb{E}[\exp(\tfrac{p-\sqrt{ps}}{1-s}M_\tau)])^{1-s}. \tag{22.2}$$

The first term of the product is the $s$-th power of the expectation a positive local martingale (and, therefore, supermartingale) sampled at a finite stopping time. By the optional sampling theorem it is always finite (actually, it is less then 1). As for the second term, one can easily check that the expression $\frac{p-\sqrt{ps}}{1-s}$ attains its minimum in $s$ over $(0,1)$ for $s = 2p-1-2\sqrt{p^2-p}$, and that this minimum value equals to $f(p)$, where $f(p) = \frac{1}{2}\frac{\sqrt{p}}{\sqrt{2p-1-2\sqrt{p^2-p}}}$. If we pick $p = \frac{4a^2}{4a-1}$, then $f(p) = a$ and both terms on the right hand side of (22.2) are bounded, uniformly in $\tau$, so that $\mathcal{E}(M)$ is in fact a martingale and bounded in $\mathbb{L}^p$ (why did we have to consider all stopping times $\tau$, and only deterministic times?). □

**Proposition 22.7** (Kazamaki's criterion)**.** *Suppose that for $M \in \mathcal{M}_0^{loc,c}$ we have*

$$\sup_{\tau \in \mathcal{S}_b} \mathbb{E}[e^{\frac{1}{2}M_\tau}] < \infty,$$

*where the supremum is taken over the set $\mathcal{S}_b$ of all finite-valued stopping times, then $\mathcal{E}(M)$ is a uniformly integrable martingale.*

*Proof.* Note, first, that the function $x \mapsto \exp(\frac{1}{2}x)$ is a test function of uniform integrability, so that the local martingale $M$ is a uniformly integrable martingale and admits the last element $M_\infty$. For the continuous martingale $cM$, where $0 < c < 1$ is an arbitrary constant, Lemma 22.6 and the assumption imply that the local martingale $\mathcal{E}(cM)$ is, in fact, a martingale bounded in $\mathbb{L}^p$, for $p = \frac{1}{2c-c^2}$. In particular, it is uniformly integrable. Therefore,

$$\mathcal{E}(cM)_t = \exp(cM_t - \tfrac{1}{2}c^2\langle M\rangle_t) = \mathcal{E}(M)_t^{c^2} e^{c(1-c)M_t}. \tag{22.3}$$

By letting $t \to \infty$ in (22.3), we conclude that $\mathcal{E}(M)$ has the last element $\mathcal{E}(M)_\infty$, and that the equality in (22.3) holds at $t = \infty$, as well. By Hölder's inequality with conjugate exponents $1/c^2$ and $1/(1-c^2)$, we have

$$1 = \mathbb{E}[\mathcal{E}(cM)_\infty] \leq \mathbb{E}[\mathcal{E}(M)_\infty]^{c^2}\mathbb{E}[\exp(\tfrac{c}{1+c}M_\infty)]^{1-c^2}.$$

Jensen's inequality implies that $\mathbb{E}[\exp(\frac{c}{1+c}M_\infty)] \leq \mathbb{E}[\exp(\frac{1}{2}M_\infty)]^{\frac{2c}{1+c}}$, and so

$$1 \leq \mathbb{E}[\mathcal{E}(M)_\infty]^{c^2}\mathbb{E}[\exp(\tfrac{1}{2}M_\infty)]^{2c(1-c)}.$$

We let $c \to 1$ to get $\mathbb{E}[\mathcal{E}(M)_\infty] \geq 1$, which, together with the non-negative supermartingale property of $\mathcal{E}(M)$ implies that $\mathcal{E}(M)$ is a uniformly-integrable martingale. □

**Theorem 22.8** (Novikov's criterion)**.** *If $M \in \mathcal{M}_0^{loc,c}$ is such that*

$$\mathbb{E}[e^{\frac{1}{2}\langle M\rangle_\infty}] < \infty,$$

*then $\mathcal{E}(M)$ is a uniformly integrable martingale.*

*Proof.* Since $e^{\frac{1}{2}M_\tau} = \mathcal{E}(M)^{\frac{1}{2}} e^{\frac{1}{4}\langle M\rangle_\tau}$, the Cauchy-Schwarz inequality implies that

$$\mathbb{E}[e^{\frac{1}{2}M_\tau}] = \mathbb{E}[\mathcal{E}(M)_\tau]^{1/2}\mathbb{E}[e^{\frac{1}{2}\langle M\rangle_\tau}]^{1/2} \leq \mathbb{E}[e^{\frac{1}{2}\langle M\rangle_\infty}],$$

and Kazamaki's criterion can be applied. □