Course: M362K Intro to Stochastic Processes
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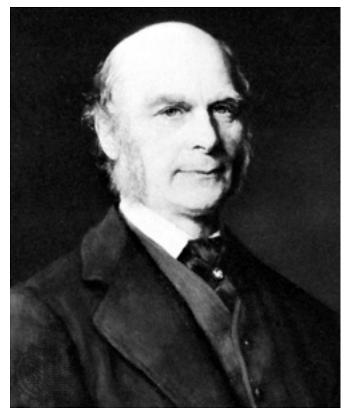
Lecture 7
Branching processes

A BIT OF HISTORY

In the mid 19th century several aristocratic families in Victorian England realized that their family names could become extinct. Was it just unfounded paranoia, or did something real prompt them to come to this conclusion? They decided to ask around, and Sir Francis Galton (a "polymath, anthropologist, eugenicist, tropical explorer, geographer, inventor, meteorologist, proto-geneticist, psychometrician and statistician" and half-cousin of Charles Darwin) posed the following question (1873, *Educational Times*):

> *How many male children (on average) must each generation of a family have in order for the family name to continue in perpetuity?*



Sir Francis Galton

An answer came from Reverend Henry William Watson soon after, and the two wrote a joint paper entitled *One the probability of extinction of families* in 1874. By the end of this lecture, you will be able to give a precise answer to Galton's question.

A MATHEMATICAL MODEL

The model proposed by Watson[1] was the following:

1. A population starts with one individual at time $n = 0$: $Z_0 = 1$.
2. After one unit of time (at time $n = 1$) the sole individual produces Z_1 identical clones of itself and dies. Z_1 is an $\mathbb{N}_0$-valued random variable.
3. (a) If Z_1 happens to be equal to 0 the population is dead and nothing happens at any future time $n \geq 2$.
   (b) If $Z_1 > 0$, a unit of time later, each of Z_1 individuals gives birth to a random number of children and dies. The first one has $Z_{1,1}$ children, the second one $Z_{1,2}$ children, etc. The last, Z_1^{th} one, gives birth to Z_{1,Z_1} children. We assume that the distribution of

[1] A similar model was proposed and analyzed earlier, in 1845, independently, by Irénée-Jules Bienaymé.

Irénée-Jules Bienaymé

the number of children is the same for each individual in every generation and independent of either the number of individuals in the generation and of the number of children the others have. This distribution, shared by all $Z_{n,i}$ and Z_1, is called the *offspring distribution*. The total number of individuals in the second generation is now

$$Z_2 = \sum_{k=1}^{Z_1} Z_{1,k}.$$

(c) The third, fourth, etc. generations are produced in the same way. If it ever happens that $Z_n = 0$, for some n, then $Z_m = 0$ for all $m \geq n$ - the population is extinct. Otherwise,

$$Z_{n+1} = \sum_{k=1}^{Z_n} Z_{n,k}.$$

Definition 7.1. A stochastic process with the properties described in (1), (2) and (3) above is called a **(simple) branching process**.

The mechanism that produces the next generation from the present one can differ from application to application. It is the offspring distribution alone that determines the evolution of a branching process. With this new formalism, we can pose Galton's question more precisely:

Under what conditions on the offspring distribution will the process $\{Z_n\}_{n\in\mathbb{N}_0}$ never go extinct, i.e., when does

$$\mathbb{P}[Z_n \geq 1 \text{ for all } n \in \mathbb{N}_0] = 1 \tag{7.1}$$

hold?

CONSTRUCTION AND SIMULATION OF BRANCHING PROCESSES

Before we answer Galton's question, let us figure out how to simulate a branching process, for a given offspring distribution $\{p_n\}_{n\in\mathbb{N}_0}$ ($p_k = \mathbb{P}[Z_1 = k]$). The distribution $\{p_n\}_{n\in\mathbb{N}_0}$ is $\mathbb{N}_0$-valued - we have learned how to simulate such distributions in Lecture 3. We can, therefore, assume that a transformation function F is known, i.e., that the random variable $\eta = F(\gamma)$ is $\mathbb{N}_0$-valued with pmf $\{p_n\}_{n\in\mathbb{N}_0}$, where $\gamma \sim U[0,1]$.

Some time ago we assumed that a probability space with a sequence $\{\gamma_n\}_{n\in\mathbb{N}_0}$ of independent $U[0,1]$ random variables is given. We think of $\{\gamma_n\}_{n\in\mathbb{N}_0}$ as a sequence of random numbers produced by a computer. Let us first apply the function F to each member of $\{\gamma_n\}_{n\in\mathbb{N}_0}$ to obtain an independent sequence $\{\eta_n\}_{n\in\mathbb{N}_0}$ of $\mathbb{N}_0$-valued random variables with pmf $\{p_n\}_{n\in\mathbb{N}_0}$. In the case of a simple random walk, we would be done at this point - an accumulation of the first n elements of $\{\eta_n\}_{n\in\mathbb{N}_0}$ would give you the value X_n of the random walk at time n. Branching processes are a bit more complicated; the increment $Z_{n+1} - Z_n$ depends on Z_n: the more individuals in a generation, the more offspring they will produce. In other words, we need a black box with two inputs - "randomness" and Z_n - which will produce Z_{n+1}. What do we mean by "randomness"? Ideally, we would need exactly Z_n (unused) elements of $\{\eta_n\}_{n\in\mathbb{N}_0}$ to simulate the number of children for each of Z_n members of generation n. This is exactly how one would do it in practice: given the size Z_n of generation n, one would draw Z_n simulations from the distribution $\{p_n\}_{n\in\mathbb{N}_0}$, and sum up the results to get Z_{n+1}. Mathematically, it is easier to be more wasteful. The sequence $\{\eta_n\}_{n\in\mathbb{N}_0}$ can be rearranged into a double sequence[2] $\{Z_{n,i}\}_{n\in\mathbb{N}_0,i\in\mathbb{N}}$. In words, instead of one sequence of independent random variables with pmf $\{p_n\}_{n\in\mathbb{N}_0}$, we have a sequence of sequences. Such an abundance allows us to feed the whole "row" $\{Z_{n,i}\}_{i\in\mathbb{N}}$ into the black box which produces Z_{n+1} from Z_n. You can think of $Z_{n,i}$ as the number of children the i^{th} individual in the n^{th} generation would have had he been born. The black box uses only the first Z_n elements of $\{Z_{n,i}\}_{i\in\mathbb{N}}$ and discards the rest:

[2] Can you find a one-to-one and onto mapping from $\mathbb{N}$ into $\mathbb{N}\times\mathbb{N}$?

$$Z_0 = 1, Z_{n+1} = \sum_{i=1}^{Z_n} Z_{n,i},$$

where all $\{Z_{n,i}\}_{n\in\mathbb{N}_0,i\in\mathbb{N}}$ are independent of each other and have the same distribution with pmf $\{p_n\}_{n\in\mathbb{N}_0}$. Once we learn a bit more about the probabilistic structure of $\{Z_n\}_{n\in\mathbb{N}_0}$, we will describe another way to simulate it.

A GENERATING-FUNCTION APPROACH

Having defined and constructed a branching process with offspring distribution $\{Z_n\}_{n\in\mathbb{N}_0}$, let us analyze its probabilistic structure. The first question the needs to be answered is the following: *What is the distribution of Z_n, for $n \in \mathbb{N}_0$?* It is clear that Z_n must be $\mathbb{N}_0$-valued, so its distribution is completely described by its pmf, which is, in turn, completely determined by its generating function. While an explicit expression for the pmf of Z_n may not be available, its generating function can always be computed:

Proposition 7.2. *Let $\{Z_n\}_{n\in\mathbb{N}_0}$ be a branching process, and let the generating function of its offspring distribution $\{p_n\}_{n\in\mathbb{N}_0}$ be given by $P(s)$. Then the generating function of Z_n is the n-fold composition of P with itself, i.e.,*

$$P_{Z_n}(s) = \underbrace{P(P(\dots P(s)\dots))}_{n\ P\text{'s}}, \text{ for } n \geq 1.$$

Proof. For $n = 1$, the distribution of Z_1 is exactly $\{p_n\}_{n\in\mathbb{N}_0}$, so $P_{Z_1} = P(s)$. Suppose that the statement of the proposition holds for some $n \in \mathbb{N}$. Then

$$Z_{n+1} = \sum_{i=1}^{Z_n} Z_{i,n},$$

can be viewed as a random sum of Z_n independent random variables with pmf $\{p_n\}_{n\in\mathbb{N}_0}$, where the number of summands Z_n is independent of $\{Z_{n,i}\}_{i\in\mathbb{N}}$. By Proposition 5.16 in the lecture on generating functions, we have seen that the generating function $P_{Z_{n+1}}$ of Z_{n+1} is a composition of the generating function $P(s)$ of each of the summands and the generating function P_{Z_n} of the random time Z_n. Therefore,

$$P_{Z_{n+1}}(s) = P_{Z_n}(P(s)) = \underbrace{P(P(\dots P(P(s))\dots)))}_{n+1 \text{ P's}},$$

and the full statement of the Proposition follows by induction. □

Let us use Proposition 7.2 in some simple examples.

Example 7.3. Let $\{Z_n\}_{n\in\mathbb{N}_0}$ be a branching process with offspring distribution $\{p_n\}_{n\in\mathbb{N}_0}$. In the first three examples no randomness occurs and the population growth can be described exactly. In the other examples, more interesting things happen.

1. $p_0 = 1$, $p_n = 0$, $n \in \mathbb{N}$:

   In this case $Z_0 = 1$ and $Z_n = 0$ for all $n \in \mathbb{N}$. This infertile population dies after the first generation.

2. $p_0 = 0, p_1 = 1, p_n = 0, n \geq 2$:

   Each individual produces exactly one child before he/she dies. The population size is always 1: $Z_n = 1$, $n \in \mathbb{N}_0$.

3. $p_0 = 0, p_1 = 0, \dots, p_k = 1, p_n = 0, n \geq k$, for some $k \geq 2$:

   Here, there are k kids per individual, so the population grows exponentially: $P(s) = s^k$, so $P_{Z_n}(s) = ((\dots(s^k)^k \dots)^k)^k = s^{k^n}$. Therefore, $Z_n = k^n$, for $n \in \mathbb{N}$.

4. $p_0 = p, p_1 = q = 1 - p, p_n = 0, n \geq 2$:

   Each individual tosses a (a biased) coin and has one child of the outcome is *heads* or dies childless if the outcome is *tails*. The generating function of the offspring distribution is $P(s) = p + qs$. Therefore,

   $$P_{Z_n}(s) = \underbrace{(p + q(p + q(p + q(\dots(p + qs)))))}_{n \text{ pairs of parentheses}}.$$

   The expression above can be simplified considerably. One needs to realize two things:

(a) After all the products above are expanded, the resulting expression must be of the form $A + Bs$, for some A, B. If you inspect the expression for P_{Z_n} even more closely, you will see that the coefficient B next to s is just q^n.

(b) P_{Z_n} is a generating function of a probability distribution, so $A + B = 1$.

Therefore,

$$P_{Z_n}(s) = (1 - q^n) + q^n s.$$

Of course, the value of Z_n will be equal to 1 if and only if all of the coin-tosses of its ancestors turned out to be *heads*. The probability of that event is q^n. So we didn't need Proposition 7.2 after all.

This example can be interpreted alternatively as follows. Each individual has exactly one child, but its gender is determined at random - male with probability q and female with probability p. Assuming that all females change their last name when they marry, and assuming that all of them marry, Z_n is just the number of individuals carrying the family name after n generations.

5. $p_0 = p^2, p_1 = 2pq, p_2 = q^2, p_n = 0, n \geq 3$:

   In this case each individual has exactly two children and their gender is female with probability p and male with probability q, independently of each other. The generating function P of the offspring distribution $\{p_n\}_{n\in\mathbb{N}}$ is given by $P(s) = (p + qs)^2$. Then

$$P_{Z_n} = \underbrace{(p + q(p + q(\dots p + qs)^2 \dots)^2)^2}_{n \text{ pairs of parentheses}}.$$

   Unlike the example above, it is not so easy to simplify the above expression.

Proposition 7.2 can be used to compute the mean and variance of the population size Z_n, for $n \in \mathbb{N}$.

Proposition 7.4. *Let $\{p_n\}_{n\in\mathbb{N}_0}$ be a pmf of the offpsring distribution of a branching process $\{Z_n\}_{n\in\mathbb{N}_0}$. If $\{p_n\}_{n\in\mathbb{N}_0}$ admits an expectation, i.e., if*

$$\mu = \sum_{k=0}^{\infty} k p_k < \infty,$$

then

$$\mathbb{E}[Z_n] = \mu^n. \tag{7.2}$$

If the variance of $\{p_n\}_{n\in\mathbb{N}_0}$ is also finite, i.e., if

$$\sigma^2 = \sum_{k=0}^{\infty} (k - \mu)^2 p_k < \infty,$$

then

$$\begin{aligned}\mathrm{Var}[Z_n] &= \sigma^2 \mu^n (1+\mu+\mu^2+\dots+\mu^n) \\ &= \begin{cases} \sigma^2\mu^n \frac{1-\mu^{n+1}}{1-\mu}, & \mu \neq 1, \\ \sigma^2(n+1), & \mu = 1 \end{cases}\end{aligned} \tag{7.3}$$

Proof. Since the distribution of Z_1 is just $\{p_n\}_{n\in\mathbb{N}_0}$, it is clear that $\mathbb{E}[Z_1] = \mu$ and $\mathrm{Var}[Z_1] = \sigma^2$. We proceed by induction and assume that the formulas (7.2) and (7.3) hold for $n \in \mathbb{N}$. By Proposition 7.2, the generating function $P_{Z_{n+1}}$ is given as a composition $P_{Z_{n+1}}(s) = P_{Z_n}(P(s))$. Therefore, if we use the identity $\mathbb{E}[Z_{n+1}] = P'_{Z_{n+1}}(1)$, we get

$$\begin{aligned}P'_{Z_{n+1}}(1) &= P'_{Z_n}(P(1))P'(1) = P'_{Z_n}(1)P'(1) = \mathbb{E}[Z_n]\mathbb{E}[Z_1] = \mu^n\mu \\ &= \mu^{n+1}.\end{aligned}$$

A similar (but more complicated and less illuminating) argument can be used to establish (7.3). □

EXTINCTION PROBABILITY

We now turn to the central question (the one posed by Galton). We define **extinction** to be the following event:

$$E = \{\omega \in \Omega : Z_n(\omega) = 0 \text{ for some } n \in \mathbb{N}\}.$$

It is the property of the branching process that $Z_m = 0$ for all $m \geq n$ whenever $Z_n = 0$. Therefore, we can write E as an *increasing* union of sets E_n, where

$$E_n = \{\omega \in \Omega : Z_n(\omega) = 0\}.$$

Therefore, the sequence $\{\mathbb{P}[E_n]\}_{n\in\mathbb{N}}$ is non-decreasing and "continuity of probability" (see the very first lecture) implies that

$$\mathbb{P}[E] = \lim_{n\in\mathbb{N}} \mathbb{P}[E_n].$$

The number $p_E = \mathbb{P}[E]$ is called the **extinction probability**. Using generating functions, and, in particular, the fact that $\mathbb{P}[E_n] = \mathbb{P}[Z_n = 0] = P_{Z_n}(0)$ we get

$$p_E = \mathbb{P}[E] = \lim_{n\in\mathbb{N}} P_{Z_n}(0) = \lim_{n\in\mathbb{N}} \underbrace{P(P(\dots P(0)\dots))}_{n \text{ P's}}.$$

It is amazing that this probability can be computed, even if the explicit form of the generating function P_{Z_n} is not known.

Proposition 7.5. *The extinction probability p_E is the smallest non-negative solution of the equation*

$$x = P(x), \text{ called the } \textbf{\textit{extinction equation}},$$

where P is the generating function of the offspring distribution.

Proof. Let us show first that p_E is *a* solution of the equation $x = P(x)$. Indeed, P is a continuous function, so $P(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} P(x_n)$ for every convergent sequence $\{x_n\}_{n\in\mathbb{N}_0}$ in $[0,1]$ with $x_n \to x_\infty$. Let us take a particular sequence given by

$$x_n = \underbrace{P(P(\dots P(0)\dots))}_{n \text{ P's}}.$$

Then

1. $p_E = \mathbb{P}[E] = \lim_{n\in\mathbb{N}} x_n$, and
2. $P(x_n) = x_{n+1}$.

Therefore,

$$p_E = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} P(x_n) = P(\lim_{n\to\infty} x_n) = P(p_E),$$

and so p_E solves the equation $P(x) = x$.

The fact that p_E is the *smallest* solution of $x = P(x)$ on $[0,1]$ is a bit trickier to get. Let p' be another solution of $x = P(x)$ on $[0,1]$. Since $0 \leq p'$ and P is a non-decreasing function, we have

$$P(0) \leq P(p') = p'.$$

We can apply the function P to both sides of the inequality above to get

$$P(P(0)) \leq P(P(p')) = P(p') = p'.$$

Continuing in the same way we get

$$P[E_n] = \underbrace{P(P(\dots P(0)\dots))}_{n \text{ P}'} \leq p',$$

we get $p_E = \lim_{n\in\mathbb{N}} \mathbb{P}[E_n] \leq \lim_{n\in\mathbb{N}} p' = p'$, so p_E is not larger then any other solution p' of $x = P(x)$. □

Example 7.6. Let us compute extinction probabilities in the cases from Example 7.3.

1. $p_0 = 1$, $p_n = 0$, $n \in \mathbb{N}$:

   No need to use any theorems. $p_E = 1$ in this case.

2. $p_0 = 0, p_1 = 1, p_n = 0, n \geq 2$:

   Like above, the situation is clear - $p_E = 0$.

3. $p_0 = 0, p_1 = 0, \dots, p_k = 1, p_n = 0, n \geq k$, for some $k \geq 2$:

   No extinction here - $p_E = 0$.

4. $p_0 = p, p_1 = q = 1 - p, p_n = 0, n \geq 2$:
   Since $P(s) = p + qs$, the extinction equation is $s = p + qs$. If $p = 0$, the only solution is $s = 0$, so no extinction occurs. If $p > 0$, the only solution is $s = 1$ - the extinction is guaranteed. It is interesting to note the jump in the extinction probability as p changes from 0 to a positive number.

5. $p_0 = p^2, p_1 = 2pq, p_2 = q^2, p_n = 0, n \geq 3$:

   Here $P(s) = (p + qs)^2$, so the extinction equation reads

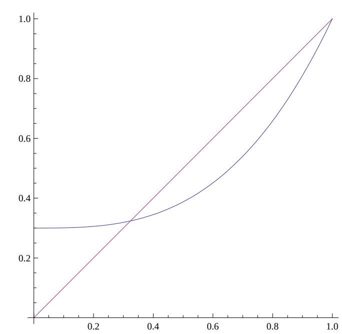
   $$s = (p + qs)^2.$$

   This is a quadratic in s and its solutions are $s_1 = 1$ and $s_2 = \frac{p^2}{q^2}$, if we assume that $q > 0$. When $p < q$, the smaller of the two is s_2. When $p \geq q$, $s = 1$ is the smallest solution. Therefore

   $$p_E = \min(1, \frac{p^2}{q^2}).$$

PROBLEMS

Problem 7.1. Let $\{Z_n\}_{n \in \mathbb{N}_0}$ be a simple branching process which starts from one individual. Each individual has exactly three children, each of whom survives until reproductive age with probability $0 < p < 1$, and dies before he/she is able to reproduce with probability $q = 1 - p$, independently of his/her siblings. The children that reach reproductive age reproduce according to the same rule.



1. Write down the generating function for the offspring distribution.

2. For what values of p will the population go extinct with certainty (probability 1). *Hint:* You don't need to compute much. Just find the derivative $P'(1)$ and remember the picture from class one the right

Solution:

1. The number of children who reach reproductive age is binomially distributed with parameters 3 and p. Therefore, $P(s) = (ps + q)^3$.

2. We know that the extinction happens with probability 1 if and only if the graphs of functions s and $P(s)$ meet only at $s = 1$, for $s \in$

$[0,1]$. They will have met once before somewhere on $[0,1)$ if and only if $P'(1) > 1$. Therefore, the extinction happens with certainty if $P'(1) = 3p(p\,1+q) = 3p \leq 1$, i.e., if $p \leq 1/3$.

Problem 7.2. Let $\{Z_n\}_{n\in\mathbb{N}_0}$ be a branching process with offspring distribution $\{p_n\}_{n\in\mathbb{N}_0}$ and generating function $P(s) = \sum_{n=0}^{\infty} p_n s^n$. The extinction is inevitable if

(a) $p_0 > 0$

(b) $P(\frac{1}{2}) > \frac{1}{4}$

(c) $P'(1) < 1$

(d) $P'(1) \geq 1$

(e) None of the above

Solution: The correct answer is (c).

(a) False. Take $p_0 = 1/3, p_1 = 0, p_2 = 2/3$, and $p_n = 0$, $n > 2$. Then the equation $P(s) = s$ reads $1/3 + 2/3s^2 = s$, and $s = 1/2$ is its smallest solution. Therefore, the probability of extinction is $1/2$.

(b) False. Take $p_0 = 0, p_1 = 1/2, p_2 = 1/2$, and $p_n = 0$, $n > 2$. Then the equation $P(s) = s$ reads $1/2s + 1/2s^2 = s$, and $s = 0$ is its smallest solution. Therefore, the probability of extinction is 0, but $P(1/2) = 1/4 + 1/8 > 1/4$.

(c) True. The condition $P'(1) < 1$, together with the convexity of the function P imply that the graph of P lies strictly above the 45-degree line for $s \in (0,1)$. Therefore, the only solution of the equation $P(s) = s$ is $s = 1$.

(d) False. Take the same example as in (a), so that $P'(1) = 4/3 > 1$. On the other hand, the extinction equation $P(s) = s$ reads $1/3 + 2/3s^2 = s$. Its smallest solution is $s = 1/2$.

(e) False.

Problem 7.3. Bacteria reproduce by cell division. In a unit of time, a bacterium will either die (with probability $\frac{1}{4}$), stay the same (with probability $\frac{1}{4}$), or split into 2 parts (with probability $\frac{1}{2}$). The population starts with 100 bacteria at time $n = 0$.

1. Write down the expression for the generating function of the distribution of the size of the population at time $n \in \mathbb{N}_0$. (*Note:* you can use n-fold composition of functions.)

2. Compute the extinction probability for the population.

3. Let m_n be the largest possible number of bacteria at time n. Find m_n and compute the probability that there are exactly m_n bacteria in the population at time n, $n \in \mathbb{N}_0$.

4. Given that there are 1000 bacteria in the population at time 50, what is the expected number of bacteria at time 51?

Solution:

1. The generating function for the offspring distribution is

$$P(s) = \tfrac{1}{4} + \tfrac{1}{4}s + \tfrac{1}{2}s^2.$$

We can think of the population Z_n at time n, as the sum of 100 independent populations, each one produced by one of 100 bacteria in the initial population. The generating function for the number of offspring of each bacterium is

$$P^{(n)}(s) = \underbrace{P(P(\dots P(s)\dots))}_{n \text{ Ps}}$$

Therefore, since sums of independent variables correspond to products in the world of generating functions, the generating function of Z_n is given by

$$P_{Z_n}(s) = \left[P^{(n)}(s)\right]^{100}.$$

2. What happens to the offspring of one of the 100 bacteria in the zero-th generation is independent of what happens to the offspring of the others. Therefore, whether the progeny of one of those bacteria goes extinct is independent of the extinction of the progeny of any other. Hence, the extinction probability of the entire population (which starts from 100) is the 100-th power of the extinction probability p_E of the population which starts from a single bacterium. To compute p_E, we write down the extinction equation

$$s = \tfrac{1}{4} + \tfrac{1}{4}s + \tfrac{1}{2}s^2.$$

Its solutions are $s_1 = 1$ and $s_2 = \frac{1}{2}$, so $p = s_2 = \frac{1}{2}$. Therefore, the extinction probability for the entire population is $(\frac{1}{2})^{100}$.

3. The maximum size of the population at time n will occur if each of the initial 100 bacteria splits into 2 each time, so $m_n = 100 \times 2^n$. For that to happen, there must be 100 splits to produce the first generation, 200 for the second, 400 for the third, ..., and $2^{n-1} \times 100$ for the n-th generation. Each one of those splits happens with probability $\frac{1}{2}$ and is independent of other splits in its generation, given the previous generations. It is clear that for $Z_n = m_n$, we must have $Z_{n-1} = m_{n-1}, \dots, Z_1 = m_1$, so

$$
\begin{aligned}
\mathbb{P}[Z_n = m_n] &= \mathbb{P}[Z_n = m_n, Z_{n-1} = m_{n-1}, \dots Z_1 = m_1, Z_0 = m_0] \\
&= \mathbb{P}[Z_n = m_n | Z_{n-1} = m_{n-1}, Z_{n-2} = m_{n-2}, \dots] \mathbb{P}[Z_{n-1} = m_{n-1}, Z_{n-2} = m_{n-2}, \dots] \\
&= \mathbb{P}[Z_n = m_n | Z_{n-1} = m_{n-1}] \times \\
&\qquad \times \mathbb{P}[Z_{n-1} = m_{n-1} | Z_{n-2} = m_{n-2}, Z_{n-3} = m_{n-3}, \dots] \mathbb{P}[Z_{n-2} = m_{n-2}, \dots] \\
&= \dots \\
&= \mathbb{P}[Z_n = m_n | Z_{n-1} = m_{n-1}] \mathbb{P}[Z_{n-1} = m_{n-1} | Z_{n-2} = m_{n-2}] \dots \mathbb{P}[Z_0 = m_0] \\
&= (\tfrac{1}{2})^{m_{n-1}} (\tfrac{1}{2})^{m_{n-2}} \dots (\tfrac{1}{2})^{m_0} \\
&= (\tfrac{1}{2})^{100(1+2+\dots+2^{n-1})} = (\tfrac{1}{2})^{100 \times (2^n - 1)}
\end{aligned}
$$

4. The expected number of offspring of each of the 1000 bacteria is given by $P'(1) = \frac{5}{4}$. Therefore, the total expected number of bacteria at time 51 is $1000 \times \frac{5}{4} = 1250$.

Problem 7.4. A branching process starts from 10 individuals, and each reproduces according to the probability distribution $(p_0, p_1, p_2, \dots)$, where $p_0 = 1/4$, $p_1 = 1/4$, $p_2 = 1/2$, $p_n = 0$, for $n > 2$. The extinction probability for the whole population is equal to

(a) 1

(b) $\frac{1}{2}$

(c) $\frac{1}{20}$

(d) $\frac{1}{200}$

(e) $\frac{1}{1024}$

Solution: The correct answer is (e). The extinction probability for the population starting from each of the 10 initial individuals is given by the smallest solution of $1/4 + 1/4s + 1/2s^2 = s$, which is $s = 1/2$. Therefore, the extinction probability for the whole population is $(1/2)^{10} = 1/1024$.

Problem 7.5. A (solitaire) game starts with 3 silver dollars in the pot. At each turn the number of silver dollars in the pot is counted (call it K) and the following procedure is repeated K times: a die is thrown, and according to the outcome the following four things can happen

- If the outcome is 1 or 2 the player takes 1 silver dollar from the pot.
- If the outcome is 3 nothing happens.

- If the outcome is 4 the player puts 1 extra silver dollar in the pot (you can assume that the player has an unlimited supply of silver dollars).
- If the outcome is 5 or 6, the player puts 2 extra silver dollars in the pot.

If there are no silver dollars on in the pot, the game stops.

1. Compute the expected number of silver dollars in the pot after turn $n \in \mathbb{N}$.
2. Compute the probability that the game will stop eventually.
3. Let m_n be the maximal possible number of silver dollars in the pot after the n-th turn? What is the probability that the actual number of silver dollars in the pot after n turns is equal to $m_n - 1$?

Solution: The number of silver dollars in the pot follows a branching process $\{Z_n\}_{n\in\mathbb{N}_0}$ with $Z_0 = 3$ and offspring distribution whose generating function $P(s)$ is given by

$$P(s) = \tfrac{1}{3} + \tfrac{1}{6}s + \tfrac{1}{6}s^2 + \tfrac{1}{3}s^3.$$

1. The generating function P_{Z_n} of Z_n is $(P(P(\dots P(s)\dots)))^3$, so
$$\mathbb{E}[Z_n] = P'_{Z_n}(1) = 3(P'(1))^n = 3(\tfrac{1}{6} + 2\tfrac{1}{6} + 3\tfrac{1}{3})^n = \frac{3^{n+1}}{2^n}$$
2. The probability p that the game will stop is the extinction probability of the process. We solve the extinction equation $P(s) = s$ to get the extinction probability corresponding to the case $Z_0 = 1$, where there is 1 silver dollar in the pot:
$$P(s) = s \;\Leftrightarrow\; 2 + s + s^2 + 2s^3 = 6s \;\Leftrightarrow\; (s-1)(s+2)(2s-1) = 0.$$
The smallest solution in $[0,1]$ of the equation above is $1/2$. To get the true (corresponding to $Z_0 = 3$) extinction probability we raise it to the third power to get $p = \frac{1}{8}$.
3. The maximal number m_n of silver dollars in the pot is achieved if we roll 5 or 6 each time. In that case, there will be 3 rolls in the first turn, $9 = 3^2$ in the second, $27 = 3^3$ in the third, etc. That means that after n turns, there will be at most $m_n = 3^{n+1}$ silver dollars in the pot. In order to have exactly $m_n - 1$ silver dollars in the pot after n turns, we must keep getting 5s and 6s throughout the first $n-2$ turns. The probability of that is
$$(\tfrac{1}{3})^{3^{n-1}+3^{n-2}+\dots+3} = (\tfrac{1}{3})^{\frac{1}{2}3^n - \frac{3}{2}}.$$

After that, we must get a 5 or a 6 in $3^n - 1$ throws and a 4 in a single throw during turn n. There are 3^n possible ways to choose the order of the throw which produces the single 4, so the later probability is

$$3^n (\tfrac{1}{3})^{3^n-1} (\tfrac{1}{6}) = \tfrac{1}{2} (\tfrac{1}{3})^{3^n-n}.$$

We multiply the two probabilities to get

$$\mathbb{P}[Z_n = m_n - 1] = \tfrac{1}{2} (\tfrac{1}{3})^{\frac{1}{2}3^{n+1}-n-\frac{3}{2}}.$$

Problem 7.6. It is a well-known fact(oid) that armadillos always have identical quadruplets (four offspring). Each of the 4 little armadillos has a $1/3$ chance of becoming a doctor, a lawyer or a scientist, independently of its 3 siblings. A doctor armadillo will reproduce further with probability $2/3$, a lawyer with probability $1/2$ and a scientist with probability $1/4$, again, independently of everything else. If it reproduces at all, an armadillo reproduces only once in its life, and then leaves the armadillo scene. (For the purposes of this problem assume that armadillos reproduce asexually.) Let us call the armadillos who have offspring *fertile*.

1. What is the distribution of the number of fertile offspring? Write down its generating function.

2. What is the generating function for the number of great-grandchildren an armadillo will have? What is its expectation? (*Note:* do not expand powers of sums)

3. Let the armadillo population be modeled by a branching process, and let's suppose that it starts from exactly one individual at time 0. Is it certain that the population will go extinct sooner or later?

Solution:

1. Each armadillo is fertile with probability p, where

$$\begin{aligned} p = \mathbb{P}[\,\textit{Fertile}\,] &= \mathbb{P}[\,\textit{Fertile} \mid \textit{Lawyer}\,]\,\mathbb{P}[\,\textit{Lawyer}\,] \\ &\quad + \mathbb{P}[\,\textit{Fertile} \mid \textit{Doctor}\,]\,\mathbb{P}[\,\textit{Doctor}\,] \\ &\quad + \mathbb{P}[\,\textit{Fertile} \mid \textit{Scientist}\,]\,\mathbb{P}[\,\textit{Scientist}\,] \\ &= \tfrac{1}{2} \times \tfrac{1}{3} + \tfrac{2}{3} \times \tfrac{1}{3} + \tfrac{1}{4} \times \tfrac{1}{3} = \tfrac{5}{12}. \end{aligned}$$

   Therefore, the number of fertile offspring is binomial with $n = 4$ and $p = \frac{5}{12}$. The generating function of this distribution is $P(s) = (\frac{7}{12} + \frac{5}{12}s)^4$.

2. To get the number of great-grandchildren, we first compute the generating function $Q(s)$ of the number of fertile grandchildren.

This is simply given by the composition of P with itself, i.e.,

$$P(P(s)) = (\tfrac{7}{12} + \tfrac{5}{12}(\tfrac{7}{12} + \tfrac{5}{12}s)^4)^4.$$

Finally, the number of great-grandchildren is the number of fertile grandchildren multiplied by 4. Therefore, its generating function is given by

$$Q(s) = P(P(s^4)) = (\tfrac{7}{12} + \tfrac{5}{12}(\tfrac{7}{12} + \tfrac{5}{12}s^4)^4)^4.$$

The compute the expectation, we need to evaluate $Q'(1)$:

$$Q'(s) = (P(P(s^4)))' = P'(P(s^4))P'(s^4)4s^3$$

and

$$P'(s) = 4\tfrac{5}{12}(\tfrac{7}{12} + \tfrac{5}{12}s)^3,$$

so that

$$Q'(1) = 4P'(1)P'(1) = \tfrac{100}{9}.$$

3. We need to consider the population of fertile armadillos. Its offspring distribution has generating function $P(s) = (\frac{7}{12} + \frac{5}{12}s)^4$, so the population will go extinct with certainty if and only if the extinction probability is 1, i.e., if $s = 1$ is the smallest solution of the extinction equation $s = P(s)$. We know, however, that $P'(1) = 5 \times \frac{5}{12} = \frac{10}{6} > 1$, so there exists a positive solution to $P(s) = s$ which is smaller than 1. Therefore, it is *not* certain that the population will become extinct sooner or later.

Problem 7.7. *Branching in alternating environments.* Suppose that a branching process $\{Z_n\}_{n\in\mathbb{N}_0}$ is constructed in the following way: it starts with one individual. The individuals in odd generations reproduce according to an offspring distribution with generating function $P_{odd}(s)$ and those in even generations according to an offspring distribution with generating function $P_{even}(s)$. All independence assumptions are the same as in the classical case.

1. Find an expression for the generating function P_{Z_n} of Z_n.
2. Derive the extinction equation.

Solution:

1. Since $n = 0$ is an "even" generation, the distribution of the number of offspring of the initial individual is given by $P_{even}(s)$. Each of the Z_1 individuals in the generation $n = 1$ reproduce according the generating function P_{odd}, so

$$P_{Z_2}(s) = P_{\sum_{k=1}^{Z_1} Z_{2,k}}(s) = P_{even}(P_{odd}(s)).$$

Similarly, $P_{Z_3}(s) = P_{even}(P_{odd}(P_{even}(s)))$, and, in general, for $n \in \mathbb{N}_0$,

$$P_{Z_{2n}}(s) = \underbrace{P_{even}(P_{odd}(P_{even}(\dots P_{odd}(s)\dots)),}_{2n \text{ Ps}} \text{ and}$$
$$P_{Z_{2n+1}}(s) = \underbrace{P_{even}(P_{odd}(P_{even}(\dots P_{even}(s)\dots)),}_{2n+1 \text{ Ps}}. \tag{7.4}$$

2. To compute the extinction probability, we must evaluate the limit

$$p_E = \mathbb{P}[E] = \lim_{n\to\infty} \mathbb{P}[Z_n = 0] = \lim_{n\to\infty} P_{Z_n}(0).$$

Since the limit exists (see the derivation in the classical case in the notes), we know that any subsequence also converges towards the same limit. In particular, $p_E = \lim_{n\to\infty} x_{2n}$, where

$$x_n = \mathbb{P}[Z_n = 0] = P_{Z_n}(0).$$

By the expression for P_{Z_n} above, we know that $x_{2n+2} = P_{even}(P_{odd}(x_{2n}))$, and so

$$p_E = \lim_{n\to\infty} x_{2n+2} = \lim_n P_{even}(P_{odd}(x_{2n})) = P_{even}(P_{odd}(\lim_n x_{2n}))$$
$$= P_{even}(P_{odd}(p_E)),$$

which identifies

$$p_E = P_{even}(P_{odd}(p_E)), \tag{7.5}$$

as the extinction equation. I will leave it up to you to figure out whether p_E is characterized as the smallest positive solution to (7.5).

Problem 7.8. In a branching process, the offspring distribution is given by its characteristic function

$$P(s) = as^2 + bs + c$$

where $a, b, c > 0$ and $a + b + c = 1$.

(i) Find the extiction probability for this branching process.

(ii) Give a necessary and sufficient condition for sure extinction.

Solution:

(i) By Proposition 7.5 in the lecture notes, the extinction probability is the smallest non-negative solution to

$$P(x) = x.$$

So, in this case (remembering that $b = 1 - a - c$):

$$ax^2 + bx + c = x \Rightarrow ax^2 - (a+c)x + c = 0,$$

i.e.,

$$(ax - c)(x - 1) = 0.$$

The solutions of this equation are 1 and c/a, and, so, the smallest solution in $[0, 1]$ is

- $p_E = 1$, if $c \geq a$, and
- $p_E = c/a$, if $c < a$.

(ii) The discussion above immediately yields the answer to question (ii): the necessary and sufficient condition for certain extinction is $c \geq a$.

Problem 7.9 (Optional)**.** The purpose of this problem is to describe a class of offspring distributions (pmfs) for which an expression for $P_{Z_n}(s)$ can be obtained.

An $\mathbb{N}_0$-valued distribution is said to be of *fractional-linear type* if its generating function P has the following form

$$P(s) = \frac{as + b}{1 - cs}, \tag{7.6}$$

for some constants $a, b, c \geq 0$. In order for P to be a generating function of a probability distribution we must have $P(1) = 1$, i.e. $a + b + c = 1$, which will be assumed throughout the problem.

1. What (familiar) distributions correspond to the following special cases (in each case identify the distribution and its parameters):
   (a) $c = 0$
   (b) $a = 0$
2. Let A be the following 2×2 matrix

$$A = \begin{bmatrix} a & b \\ -c & 1 \end{bmatrix} \text{ and let } A^n = \begin{bmatrix} a^{(n)} & b^{(n)} \\ c^{(n)} & d^{(n)} \end{bmatrix}$$

be its n^{th} power (using matrix multiplication, of course). Using mathematical induction prove that

$$\underbrace{P(P(\dots P(s)\dots)}_{n \text{ Ps}} = \frac{a^{(n)}s + b^{(n)}}{c^{(n)}s + d^{(n)}}. \tag{7.7}$$

3. Take $a = 0$ and $b = c = 1/2$. Show inductively that

$$A^n = \frac{1}{2^n} \begin{bmatrix} -(n-1) & n \\ -n & n+1 \end{bmatrix}. \tag{7.8}$$

Use that to write down the generating function of Z_n in the linear-fractional form (7.6).

4. Find the extinction probability as a function of a, b and c in the general case (don't forget to use the identity $a + b + c = 1$!).

Solution: *Note:* It seems that not everybody is familiar with the principle of mathematical induction. It is a logical device you can use if you have a conjecture and want to prove that it is true for all $n \in \mathbb{N}$. Your conjecture will be typically look like

For each $n \in \mathbb{N}$, the statement $P(n)$ holds.

where $P(n)$ is some assertion which depends on n. For example, $P(n)$ could be

$$\text{“ } 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \text{ ”} \tag{7.9}$$

The principle of mathematical induction says that you can prove that your statement is true for *all* $n \in \mathbb{N}$ by doing the following two things:

1. Prove the statement for $n = 1$, i.e., prove $P(1)$. (induction basis)
2. Prove that the implication $P(n) \Rightarrow P(n+1)$ always holds, i.e., prove the statement for $n+1$ if you are, additionally, allowed to use the statement for n as a hypothesis. (inductive step)

As an example, let us prove the statement (7.9) above. For $n = 1$, $P(1)$ reads "$1 = 1$", which is evidently true. Supposing that $P(n)$ holds, i.e., that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2},$$

we can add $n+1$ to both sides to get

$$\begin{aligned} 1 + 2 + \cdots + n + (n+1) &= \frac{n(n+1)}{2} + (n+1) = \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{n^2 + 3n + 2}{2} = \frac{(n+1)(n+2)}{2}, \end{aligned}$$

which is exactly $P(n+1)$. Therefore, we managed to prove $P(n+1)$ using $P(n)$ as a crutch. The principle of mathematical induction says that this is enough to be able to conclude that $P(n)$ holds for each n, i.e., that (7.9) is a ture statement for all $n \in \mathbb{N}$.

Back to the solution of the problem:

1. (a) When $c = 0$, $P(s) = as + b$ - Beronulli distribution with success probability $a = 1 - b$.
   (b) For $a = 0$, $P(s) = \frac{b}{1-cs}$ - Geometric distribution with success probability $b = 1 - c$.

2. For $n = 1$, $a^{(1)} = a$, $b^{(1)} = b$, $c^{(1)} = -c$ and $d^{(1)} = 1$, so clearly

$$P(s) = \frac{a^{(1)}s + b^{(1)}}{c^{(1)}s + d^{(1)}}.$$

Suppose that the equality (7.7) holds for some $n \in \mathbb{N}$. Then

$$\begin{aligned}\begin{bmatrix} a^{(n+1)} & b^{(n+1)} \\ c^{(n+1)} & d^{(n+1)} \end{bmatrix} &= A^{n+1} = A^n A = \begin{bmatrix} a^{(n)} & b^{(n)} \\ c^{(n)} & d^{(n)} \end{bmatrix} \begin{bmatrix} a & b \\ -c & 1 \end{bmatrix} \\ &= \begin{bmatrix} a\,a^{(n)} - c\,b^{(n)} & b\,a^{(n)} + b^{(n)} \\ a\,c^{(n)} - c\,d^{(n)} & b\,c^{(n)} + d^{(n)} \end{bmatrix}\end{aligned}$$

On the other hand, by the inductive assumption,

$$\begin{aligned}\underbrace{P(P(\dots P(s)\dots)}_{n+1 \text{ Ps}} &= \frac{a^{(n)}P(s) + b^{(n)}}{c^{(n)}P(s) + d^{(n)}} = \frac{a^{(n)}\frac{as+b}{1-cs} + b^{(n)}}{c^{(n)}\frac{as+b}{1-cs} + d^{(n)}} \\ &= \frac{(a\,a^{(n)} - c\,b^{(n)})s + b\,a^{(n)} + b^{(n)}}{(a\,c^{(n)} - c\,d^{(n)})s + b\,c^{(n)} + d^{(n)}}.\end{aligned}$$

Therefore, (7.7) also holds for $n + 1$. By induction, (7.7) holds for all $n \in \mathbb{N}$.

3. Here

$$A = \begin{bmatrix} 0 & 1/2 \\ -1/2 & 1 \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} = \tfrac{1}{2^1}\begin{bmatrix} (1-1) & 1 \\ -1 & 1+1 \end{bmatrix}$$

so the statement holds for $n = 1$ (induction basis). Suppose that (7.8) holds for some n. Then

$$\begin{aligned}A^{n+1} = A^n A &= \tfrac{1}{2^n}\begin{bmatrix} -(n-1) & n \\ -n & n+1 \end{bmatrix} \tfrac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \\ &= \frac{1}{2^{n+1}}\begin{bmatrix} -n & -(n-1)+2n \\ -n-1 & -n+2(n+1) \end{bmatrix} = \frac{1}{2^{n+1}}\begin{bmatrix} -n & n+1 \\ -(n+1) & n+2 \end{bmatrix}\end{aligned}$$

which is exactly (7.8) for $n + 1$ (inductive step). Thus, (7.8) holds for all $n \in \mathbb{N}$. By the previous part, the generating function of Z_n is given by

$$P_{Z_n}(s) = \frac{-\frac{1}{2}(n-1)s + \frac{1}{2}n}{-\frac{1}{2}ns + \frac{1}{2}(n+1)}.$$

We divide the numerator and the denominator by $\frac{1}{2}(n+1)$ to get the above expression into the form dictated by (7.6):

$$P_{Z_n}(s) = \frac{-\frac{n-1}{n+1}s + \frac{n}{n+1}}{1 - \frac{n}{n+1}s}.$$

Note that $a = -\frac{n-1}{n+1}$, $b = \frac{n}{n+1}$ and $c = \frac{n}{n+1}$ so that $a+b+c = \frac{-n+1+n+n}{n+1} = 1$, as required.

4. For the extinction probability we need to find the smallest solution of

$$s = \frac{as+b}{1-cs}$$

in $[0,1]$. The equation above transforms into a quadratic equation after we multiply both sides by $1-cs$

$$s - cs^2 = as + b, \text{ i.e., } cs^2 + (a-1)s + b = 0. \tag{7.10}$$

We know that $s=1$ is a solution and that $a-1 = -b-c$, so we can factor (7.10) as

$$cs^2 + (a-1)s + b = (s-1)(cs-b).$$

If $c=0$, the only solution is $s=1$. If $c \neq 0$, the solutions are 1 and $\frac{b}{c}$. Therefore, the extinction probability $\mathbb{P}[E]$ is given by

$$\mathbb{P}[E] = \begin{cases} 1, & c = 0, \\ \min(1, \frac{b}{c}), & \text{otherwise} . \end{cases}$$

Problem 7.10 (Optional)**.** For a branching process $\{Z_n\}_{n\in\mathbb{N}_0}$, denote by S the total number of individuals that ever lived, i.e., set

$$S = \sum_{n=0}^{\infty} Z_n = 1 + \sum_{n=1}^{\infty} Z_n.$$

(i) Assume that the offspring distribution has the generating function given by

$$P(s) = p + qs.$$

Find the generating function P_S in this case.

(ii) Assume that the offspring distribution has the generating function given by

$$P(s) = p/(1-qs).$$

Find P_S in this case.

(iii) Find the general expression for $\mathbb{E}[S]$ and calculate this expectation in the special cases (i) and (ii).

Solution: In general, the r.v. S satisfies the relationship

$$P_S(s) = \mathbb{E}[s^{1+\sum_{n=1}^{\infty} Z_n}] = \sum_{k=0}^{\infty} \mathbb{E}[s^{1+\sum_{n=1}^{\infty} Z_n}|Z_1 = k]\mathbb{P}[Z_1 = k]$$
$$= \sum_{k=0}^{\infty} s\mathbb{E}[s^{Z_1+\sum_{n=2}^{\infty} Z_n}|Z_1 = k]\mathbb{P}[Z_1 = k]$$

When $Z_1 = k$, the expression $Z_1 + \sum_{n=2}^{\infty} Z_n$ counts the total number of individuals in k separate and independent Branching processes - one for each of k members of the generation at time $k = 1$. Since this random variable is the sum of k independent random variables, each of which has the same distribution as S (why?), we have

$$\mathbb{E}[s^{Z_1+\sum_{n=2}^{\infty} Z_n}|Z_1 = k] = [P_S(s)]^k.$$

Consequently, P_S is a solution of the following equation

$$P_S(s) = s\sum_{k=0}^{\infty}[P_S(s)]^k\mathbb{P}[Z_1 = k] = sP(P_S(s)).$$

(i) In this case,

$$P_S(s) = s(p + qP_S(s)) \Rightarrow P_S(s) = \frac{sp}{1 - sq}.$$

(iii) Here, $P_S(s)$ must satisfy

$$P_S(s)(1 - qP_S(s)) - sp = 0,$$

i.e.,

$$qP_S(s)^2 - P_S(s) + sp = 0.$$

Solving the quadratic, we get as the only sensible solution (the one that has $P_S(0) = 0$):

$$P_S(s) = \frac{1 - \sqrt{1 - 4spq}}{2q}.$$

Note that $P_S(1) < 1$ if $p < q$. Does that surprise you? Should not: it is possible that $S = +\infty$.

(iii) Using our main "recursion" for $P_S(s)$, we get

$$P_S'(s) = (sP_{Z_1}(P_S(s)))' = P_{Z_1}(P_S(s)) + sP_{Z_1}'(P(s))P'(s).$$

So, for $s = 1$,

$$\mathbb{E}[S] = P_S'(1) = P_{Z_1}(P_S(1)) + P_{Z_1}'(P_S(1))P_s'(1) = 1 + \mu\mathbb{E}[S].$$

If $\mu \geq 1$, $\mathbb{E}[S] = +\infty$. If $\mu < 1$,

$$\mathbb{E}[S] = \frac{1}{1 - \mu}.$$