Course: Theory of Probability II
Term: Spring 2015
Instructor: Gordan Zitkovic

Lecture 16
Abstract Nonsense

Brownian motion is just an example (albeit a particularly important one) of a whole "zoo" of interesting and useful continuous-time stochastic processes. This chapter deals with the minimum amount of the general theory of stochastic processes[1] and related topics necessary for the rest of these notes. To ease the understanding, we keep track of the amount of structure a certain notion depends on; e.g., if it depends on the presence of a filtration or it can be defined without it.

[1] widely known as the "Théorie générale" in homage to the French school that developed it

Properties without filtrations or probability

We remind the reader that a **stochastic process** (in continuous time) is a collection $\{X_t\}_{t\in[0,\infty)}$ of random variables. Without any further assumptions, little can be said about the regularity properties of its paths $t \mapsto X_t(\omega)$. Indeed, not even Borel-measurability can be guaranteed; one only need to define $X_t(\omega) = f(t)$, where $f(t)$ is a non-measurable function. The notion of **(joint) measurability**, defined below, helps:

Definition 16.1 (Measurability)**.** A stochastic process $\{X_t\}_{t\in[0,\infty)}$ is said to be **(jointly) measurable** if the map $X : [0,\infty) \times \Omega \to \mathbb{R}$, given by, $X(t,\omega) = X_t(\omega)$ is (jointly) measurable from $\mathcal{B}([0,\infty)) \otimes \mathcal{F}$ to $\mathcal{B}(\mathbb{R})$.

Even though non-measurable processes clearly exist (and abound), they are never really necessary. For that reason, make the following standing assumption:

Assumption 16.2. All stochastic processes from now on are assumed to be (jointly) measurable.

This is not a significant assumption. All processes in the sequel will be constructed from the already existing ones in a measurable manner and the (joint) measurability will be easy to check. We will see quite soon that the (everywhere-continuous version of the) Brownian motion is always measurable.

An immediate consequence of this assumption is that the trajectories $t \mapsto X_t(\omega)$ are automatically Borel-measurable functions. This follows from the fact that sections of jointly-measurable functions are themselves measurable.

Remark 16.3. The measurability problem arises only in continuous time. In discrete time, every stochastic process $\{X_n\}_{n\in\mathbb{N}}$ is automatically jointly measurable. The reason is that the measurable structure on $\mathbb{N}$ is much simpler than that on $[0,\infty)$.

Definition 16.4 (Path-regularity classes)**.** A stochastic process $\{X_t\}_{t\in[0,\infty)}$ is said to be

1. **Continuous**, if all of its trajectories $t\mapsto X_t(\omega)$ are continuous functions on $[0,\infty)$,
2. **Right-continuous**, if all of its trajectories $t\mapsto X_t(\omega)$ are right-continuous functions, i.e., if $X_t(\omega)=\lim_{s\searrow t}X_s(\omega)$, for all $t\in[0,\infty)$.
3. **Left-continuous**, if all of its trajectories $t\mapsto X_t(\omega)$ are left-continuous functions, i.e., if $X_t(\omega)=\lim_{s\nearrow t}X_s(\omega)$, for all $t\in(0,\infty)$.
4. **RCLL** if all of its trajectories $t\mapsto X_t(\omega)$ have the following two properties
   (a) $X_t(\omega)=\lim_{s\searrow t}X_s(\omega)$, for all $t\in[0,\infty)$,
   (b) $\lim_{s\nearrow t}X_s(\omega)$, exists for all $t\in(0,\infty)$.
5. **LCRL** if all of its trajectories $t\mapsto X_t(\omega)$ have the following two properties
   (a) $X_t(\omega)=\lim_{s\nearrow t}X_s(\omega)$, for all $t\in(0,\infty)$, and
   (b) $\lim_{s\searrow t}X_t(\omega)$, exists for all $t\in[0,\infty)$.
6. **of finite variation** if almost all of its trajectories have finite variation on all segments $[0,t]$.
7. **bounded** if there exists $K\geq 0$ such that all of its trajectories are bounded on $[0,\infty)$ by K in absolute value.

Remark 16.5.

1. The acronym **RCLL** (right-continuous with left limits) is sometimes replaced by "càdlàg", which stands for the French phrase "**c**ontinue **à d**roite, limitée **à g**auche". Similarly, **LCRL** (left-continuous with right limits) is replaced by "càglàd", which stands for "**c**ontinue **à g**auche, limitée **à d**roite".
2. For a RCLL process $\{X_t\}_{t\in[0,\infty)}$, it is customary to denote the left limit $\lim_{s\nearrow t}X_s$ by X_{t-} (by convention $X_{0-}=0$). Similarly, $X_{t+}=\lim_{s\searrow t}X_s$, for a LCRL process. The random variable $\Delta X_t=X_t-X_{t-}$ (or $X_{t+}-X_t$ in the LCRL case) is called the **jump** of X at t.

Properties without probability

What really distinguishes the theory of stochastic processes from the theory of real-valued functions on product spaces, is the notion of adaptivity. The additional structural element comes in through the notion of a filtration and models the accretion of information as time progresses.

Filtrations and stopping times

Definition 16.6 (Filtrations and adaptedness)**.** A **filtration** is a family $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,\infty)}$ of sub-σ-algebras of $\mathcal{F}$ with the property that $\mathcal{F}_s \subseteq \mathcal{F}_t$, for $0 \leq s < t$. A stochastic process $X = \{X_t\}_{t\in[0,\infty)}$ is said to be **adapted** to $\mathbb{F}$ if $X_t \in \mathcal{F}_t$, for all t.

The **natural filtration** $\mathbb{F}^X$ of a stochastic process $\{X_t\}_{t\in[0,\infty)}$ is the smallest filtration with respect to which X is adapted, i.e.,

$$\mathcal{F}^X_t = \sigma(X_s; s \leq t),$$

and is related to the situation where the available information at time t is obtained by observing the values of the process (and nothing else) X up to time t.

Problem 16.1 (*)**.**

1. For a (possibly uncountable) family $(X_\alpha)_{\alpha\in A}$ of random variables, let $\mathcal{G}$ denote the σ-algebra generated by $(X_\alpha)_{\alpha\in A}$, i.e., $\mathcal{G} = \sigma(X_\alpha, \alpha \in A)$. Given a random variable $Y \in \mathcal{G}$, show that there exists a *countable* subset $C \subseteq A$, such that

$$Y \in \sigma(X_\alpha \,:\, \alpha \in C).$$

Hint: Use the Monotone-Class Theorem.

2. Let $\mathbb{F}^X$ be the natural filtration of the stochastic process $\{X_t\}_{t\in[0,T]}$. Show that for each $\mathcal{F}_1$-measurable random variable Y there exists a countable subset S of $[0,1]$ such that $Y(\omega) = Y(\omega')$, as soon as $X_t(\omega) = X_t(\omega')$, for all $t \in S$.

A random variable τ taking values in $[0,\infty]$ is called a **random time**. The additional element $+\infty$ is used as a placeholder for the case when τ "does not happen".

Definition 16.7 (Stopping and optional times)**.** Given a filtration $\mathbb{F}$, a random time τ is said to be

- an $\mathbb{F}$-**stopping time** if

$$\{\tau \leq t\} \in \mathcal{F}_t, \text{ for all } t \geq 0.$$

- an $\mathbb{F}$-**optional time** if

$$\{\tau < t\} \in \mathcal{F}_t, \text{ for all } t > 0.$$

Both stopping and optional times play the role of the "stopping time" in the continuous case. In discrete time, they collapse into the same concept, but there is a subtle difference in the continuous time. Each stopping time is an optional time, as is easily seen as follows: for $t > 0$, Suppose that $t_n \nearrow t$ with $t_n < t$, for all n. Then

$$\{\tau < t\} = \bigcup_{n \in \mathbb{N}} \{\tau \leq t_n\} \in \mathcal{F}_t,$$

because $\{\tau \leq t_n\} \in \mathcal{F}_{t_n} \subseteq \mathcal{F}_t$, for each $n \in \mathbb{N}$. The following example shows that the two concepts are not the same:

Example 16.8 (Optional but not stopping times)**.** Let $\{B_t\}_{t\in[0,\infty)}$ be the standard Brownian motion, and let $\mathbb{F} = \mathbb{F}_B$ be its natural filtration. Consider the random time τ, defined by

$$\tau(\omega) = \inf\{t \geq 0 \,:\, B_t(\omega) > 1\} \subseteq [0, \infty),$$

under the convention that $\inf \emptyset = +\infty$. We will see shortly (in Proposition 16.12), τ is an optional time. It is, however, not a stopping time. The rigorous proof will have to wait a bit, but here is a heuristic argument. Having only the information available at time 1, it is always possible to decide whether $\tau \leq 1$ or not. If $B_u > 1$ for some $u \leq 1$, then, clearly, $\tau \leq 1$. Similarly, if $B_u \geq 1$ for all $u < 1$ and $B_1 > 1$, we can also easily conclude that $\tau > 1$. The case when $B_u \leq 1$, for all $u < 1$ and $B_1 = 1$ is problematic. To decide whether $\tau = 1$ or not, we need to know about the behavior of B in some right neighborhood of 1. The process could enter the set $(1, \infty)$ right after 1, in which case $\tau = 1$. Alternatively, it could "bounce back" and not enter the set $(1, \infty)$ for a while longer. The problem is that the time τ is defined using the *infimum* of a set it does not have to be an element of.

It you need to see a more rigorous example, here is one. We work on the discrete measurable space $(\Omega, \mathcal{F})$, where $\Omega = \{-1, 1\}$ and $\mathcal{F} = 2^{\{-1,1\}}$. Let $X_t(\omega) = \omega t$, for $t \geq 0$ and let $\mathbb{F} = \mathbb{F}^X$ be the natural filtration of X. Clearly, $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_t = \mathcal{F}$, for $t > 0$. Consequently, the random time

$$\tau(\omega) = \inf\{t \geq 0 \,:\, X_t(\omega) > 0\} = \begin{cases} 0, & \omega = 1, \\ +\infty, & \omega = -1 \end{cases},$$

is not a stopping time, since $\{1\} = \{\tau \leq 0\} \notin \mathcal{F}_0$. On the other hand, it is an optional time. Indeed $\mathcal{F}_{t+} = \mathcal{F}$, for all $t \geq 0$.

The main difference between optional and stopping time - if Example 16.8 is to be taken as typical - is that no peaking is allowed for stopping times, while, for optional ones, a little bit of peaking is tolerated. What is exactly understood by "the future" is, in turn, dictated bt the filtration, and the difference between the "present" and the "immediate future" at time t is encoded in the difference between the σ-algebras: $\mathcal{F}_t$ and $\cap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon}$. If one is willing to add the extra information contained in the latter into one's information set, the difference between optional and stopping times would disappear. To simplify the discussion in the sequel, we introduce the following notation

$$\mathcal{F}_{t+} = \cap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon}, \text{ and } \mathbb{F}_+ = \{\mathcal{F}_{t+}\}_{t\in[0,\infty)},$$

and call the filtration $\mathbb{F}_+$ the **right-continuous augmentation of** $\mathbb{F}$. A filtration $\mathbb{F}$ with $\mathbb{F} = \mathbb{F}_+$ is said to be **right continuous**. The notion of right continuity only makes (nontrivial) sense in continuous time. In the discrete case, it formally implies that $\mathcal{F}_n = \mathcal{F}_0$, for all $n \in \mathbb{N}$.

Proposition 16.9 ($\mathbb{F}$-optional = $\mathbb{F}_+$-stopping)**.** *A random time τ is an $\mathbb{F}$-optional time if and only if is an $\mathbb{F}_+$-stopping time.*

Proof. Let τ be a $\mathbb{F}$-optional time. For $t \geq 0$, we pick a sequence $\{t_n\}_{n\in\mathbb{N}}$ with $t_n \searrow t$ and $t_n > t$. For $m \in \mathbb{N}$, we have

$$\{\tau \leq t\} = \cap_{n\geq m} A_n, \text{ where } A_n = \{\tau < t_n\} \in \mathcal{F}_{t_n} \subseteq \mathcal{F}_{t_m}.$$

Therefore, $\{\tau \leq t\} \in \cap_m \mathcal{F}_{t_m} = \mathcal{F}_{t+}$, and τ is an $\mathbb{F}_+$-stopping time.

Conversely, if τ is a $\mathbb{F}_+$ stopping time, then, for $t > 0$ and a sequence $t_n \nearrow t$, with $t_n < t$ we have

$$\{\tau < t\} = \cup_n B_n, \text{ where } B_n = \{\tau \leq t_n\} \in \mathcal{F}_{t_n+} \subseteq \mathcal{F}_t.$$

Consequently, $\{\tau < t\} \in \mathcal{F}_t$, and τ is an $\mathbb{F}$-optional time. $\square$

Corollary 16.10 (If $\mathbb{F} = \mathbb{F}_+$, then optional=stopping)**.** *If the filtration $\mathbb{F}$ is right continuous, the collections of optional and stopping times coincide.*

The above discussion is very useful when one tries to manufacture properties of optional times from the properties of stopping times. Here is an example:

Proposition 16.11 (Stability of optional and stopping times)**.** *If τ and σ are stopping (optional) times, then so are*

$$\sigma + \tau, \max(\sigma, \tau), \textit{ and } \min(\sigma, \tau).$$

Proof. By Proposition 16.9, it is enough to prove the statement in the case of stopping times. Indeed, to treat optional times, it will suffice to replace the filtration $\mathbb{F}$ with the right-continuous augmentation $\mathbb{F}_+$.

We focus on the case of the sum $\sigma + \tau$, and leave the other two to the reader. To show that $\sigma + \tau$ is a stopping time, we consider the event $\{\sigma + \tau > t\} = \{\sigma + \tau \leq t\}^c$ and note that

$$\{\sigma + \tau > t\} = \bigcup_{q \in \mathcal{Q}_+ \cap [0,t]} \Big(\{\sigma > q\} \cap \{\tau > t - q\} \Big).$$

It remains to observe that

$$\{\sigma > q\} \cap \{\tau > t - q\} \in \mathcal{F}_{\max(q,t-q)} \subseteq \mathcal{F}_t. \qquad \square$$

For a stochastic process $\{X_t\}_{t \in [0,\infty)}$ and a subset $A \subseteq \mathbb{R}$, we define the **hitting time** τ_A of A as

$$\tau_A = \inf\{t \geq 0 \,:\, X_t \in A\}.$$

Unlike in the discrete case, it is not immediately obvious that X is a stopping or an optional time. In fact, it is not even immediately clear that τ_A is even a random variable. Indeed, even if A is Borel-measurable, the set $\{\tau_A > t\}$ is a-priori defined as a combination of *uncountably* many restrictions. Under suitable regularity conditions, however, we do recover these intuitive properties (note, though, how the two cases below differ from each other):

Proposition 16.12 (Hitting times which are stopping times)**.** *If X is a* continuous *process, the map τ_A is*

1. *a stopping time when A is closed, and*
2. *an optional time when A is open or closed,*

Proof.

1. For $n \in \mathbb{N}$, define $A_n = \{x \in \mathbb{R} \,:\, d(x, A) < 1/n\}$. Since $d(\cdot, A)$ is a continuous function, A_n is an open set. The reader will easily check that

   $$\{\tau_A \leq t\} = \{X_t \in A\} \quad \bigcup \quad \left(\bigcap_{n \in \mathbb{N}} \bigcup_{q \in \mathcal{Q}_+ \cap [0,t)} \{X_q \in A_n\} \right),$$

   which implies directly that τ_A is a stopping time.

2. For $t > 0$, we clearly have

   $$\{\tau_A < t\} = \bigcup_{q \in \mathcal{Q} \cap [0,t)} \{X_q \in A\},$$

and so $\{\tau_A < t\} \in \mathcal{F}^X_t$, for all $t > 0$. It follows that $\{\tau_A \leq t\} = \cap_{n\geq m}\{\tau_A < t + 1/n\} \in \mathcal{F}^X_{t+1/m}$, for all $m \in \mathbb{N}$. Consequently, $\{\tau_A \leq t\} \in \mathcal{F}^{X+}_t$. □

Remark 16.13. The results in Proposition 16.12 are just the tip of the iceberg. First of all, the proof readily generalized to the case when the process X takes values in a d-dimensional Euclidean space. Also, one can show that the continuity assumption is not necessary. In fact, one needs minimal regularity on A (say, Borel), and minimal regularity on X (much less than, say, RCLL) to conclude that τ_A is an optional time.

Progressive measurability In addition to the notions of measurability mentioned above, one often uses a related notion which is better suited for the situation when a filtration is present and appears as a separate concept only in continuous time.

Definition 16.14 (Progressive measurability)**.** We say that the stochastic process $\{X_t\}_{t\in[0,\infty)}$ is **progressively measurable** (or, simply, **progressive**) if, when seen as a mapping on the product space $[0,\infty) \times \Omega$, it is measurable with respect to the σ-algebra Prog, where

$$\mathrm{Prog} = \Big\{A \in \mathcal{B}([0,\infty)) \times \mathcal{F} \,:\, A \cap ([0,T] \times \Omega) \in \mathcal{B}([0,T]) \times \mathcal{F}_T, \text{ for all } T \geq 0.\Big\}$$

Problem 16.2. Show that an adapted process $\{X_t\}_{t\in[0,\infty)}$ is progressively measurable if and only if, for each $T > 0$, the stopped process X^T is measurable when understood as a process on $(\Omega, \mathcal{F}_T)$.

Problem 16.3. Let $\{H_t\}_{t\in[0,\infty)}$ be a bounded progressively-measurable process. Show that the process

$$\{\textstyle\int_0^t X_u\, du\}_{t\in[0,\infty)}$$

is well-defined, continuous and progressively measurable. Argue that the assumption of uniform boundedness can be relaxed to that of integrability.

Clearly, a progressively measurable process is adapted and measurable, but the converse implication is not true.

Example 16.15 (*)(Measurable and adapted, but not progressively measurable)**.** Let Ω denote the set of all lower semicontinuous functions[2] $\omega : (0,\infty) \to \mathbb{R}$ such that

$$\int_0^T |\omega(t)|\, dt < \infty, \text{ for all } T > 0.$$

[2] A function $f : (0,\infty) \to \mathbb{R}$ is said to be **lower semicontinuous** if $\{f > c\} = f^{-1}((c,\infty))$ is open for each $c \in \mathbb{R}$.

For each pair $0 < a \leq b < \infty$, we define the map $I_{a,b} : \Omega \to \mathbb{R}$ by

$$I_{a,b}(\omega) = \inf_{a \leq t \leq b} \omega(t) \in (-\infty, \infty),$$

and let the σ-algebra $\mathcal{F}$ be generated by all $I_{a,b}$, $0 < a \leq b < \infty$. The filtration $\mathbb{F}$ is the natural filtration of the coordinate process $\{H_t\}_{t\in[0,T]}$, where $H_t(\omega) = \omega(t)$, i.e., $\mathcal{F}_t = \sigma(H_s, s \leq t)$. Hence, H is trivially adapted. To show that it is measurable it suffices to observe that for each $c \in \mathbb{R}$, we have

$$\{(t,\omega) \in (0,\infty) \times \Omega \,:\, H_t(\omega) > c\} = \bigcup [p,q] \times \{I_{p,q} > c\},$$

where the union is taken over all pairs $0 < p \leq q < \infty$ of rational numbers.

To show that H is not progressive, it will be enough (see Problem 16.3) to show that the process $\{X_t\}_{t\in[0,T]}$, defined by

$$X_t = \int_0^t H_u \, du, \; t \geq 0,$$

is *not adapted*. In fact, we have $X_t \not\in \mathcal{F}_t$, *for all* $t > 0$. Suppose, to the contrary, that for some $t > 0$, we have $X_t \in \mathcal{F}_t$. Then, by Problem 16.1, we could find a countable subset S of $[0,t]$ such that $X_t(\omega) = X_t(\omega')$ as soon as $\omega|_S = \omega'|_S$. That, however, leads to a contradiction. Indeed, we can always pick $\omega = 1$ and $\omega' = \mathbf{1}_O$, where O is an open subset of $(0,t]$ of Lebesgue measure strictly less than 1, which contains S.

Processes with mildly regular trajectories are progressive.

Proposition 16.16 (RCLL or LCRL + adapted → progressive)**.** *Suppose that the adapted stochastic process $\{X_t\}_{t\in[0,\infty)}$ has the property that all of its trajectories are right continuous or that all of its trajectories are left continuous. Then, X is progressively measurable.*

Proof. We assume that $\{X_t\}_{t\in[0,\infty)}$ is right continuous (the case of a left-continuous process is analogous). For $T \in (0,\infty)$ and $n \in \mathbb{N}$, $n \geq 1/T$, we define the process $\{X^n_t\}_{t\in[0,\infty)}$ by

$$X^n_t(\omega) = \begin{cases} X_T(\omega), & t \geq T, \\ \sum_{k=0}^{\lfloor nT \rfloor - 1} X_{\frac{k+1}{n}}(\omega) \mathbf{1}_{[\frac{k}{n}, \frac{k+1}{n})}(t), & t < T \end{cases}.$$

In words, we use the right-endpoint value of the process X throughout the interval $[\frac{k}{n}, \frac{k+1}{n})$. It is then easy to see that the restricted processes $\{X^n_t\}_{t\in[0,T]}$ are measurable with respect to $\mathcal{B}([0,T]) \times \mathcal{F}_T$, but note that it is not necessarily adapted.

It remains to show that X^n converges towards X. For $(t,\omega) \in (0,\infty) \times \Omega$ we have $X^n_t = X_{k_n(t)/n}$, where $k_n(t)$ is the smallest $k \in \mathbb{N}_0$ such that $k_n(t) \geq nt$. Since $k_n(t) - 1 < nt \leq k_n(t)$, we have $k_n(t)/n \searrow t$, as $n \to \infty$, and $X^n_t \to X_t$ by the RCLL property. □

Let $\{X_t\}_{t\in[0,\infty)}$ be a random process, and let τ be a random time. The **stopped process** $\{X^\tau_t\}_{t\in[0,\infty)}$ is defined by $X^\tau_t(\omega) = X_{t\wedge\tau(\omega)}(\omega)$, for $t \geq 0$.

Proposition 16.17 (A stopped progressive is still a progressive)**.** *Let $\{X_t\}_{t\in[0,\infty)}$ be a progressive process, and let τ be a stopping time. Then the stopped process X^τ is also progressive.*

Proof. We fix $T > 0$ and note that the map $(t,\omega) \to (\tau(\omega) \wedge t, \omega)$, from $([0,T] \times \Omega, \mathcal{B}([0,T]) \otimes \mathcal{F}_T)$ to itself is measurable. Then, we compose it with the map $(t,\omega) \to X_t(\omega)$, which, by the assumption, is jointly mesurable on $([0,T] \times \Omega, \mathcal{B}([0,T]) \otimes \mathcal{F}_T)$. The obtained mapping, namely the process $X^{T\wedge\tau} = (X^T)^\tau$, is thus measurable, and, since this holds for each $T > 0$, the process X^τ is progressive. □

In an analogy with the interpretation of $\mathcal{F}_t$ as the information available at time t, we define a σ-algebra which could be interpreted to contain all the information at a *stopping* time τ:

$$\mathcal{F}_\tau = \sigma\Big\{X^\tau_t \,:\, t \geq 0,\ X \text{ is a progressively-measurable process}\Big\}.$$

In words, $\mathcal{F}_\tau$ is generated by the values of all progressive processes stopped at τ.

Problem 16.4 (Properties of $\mathcal{F}_\tau$)**.** Let $\sigma, \tau, \{\tau_n\}_{n\in\mathbb{N}}$ be $\{\mathcal{F}_t\}_{t\in[0,\infty)}$-stopping times. Show that

1. $\mathcal{F}_\tau = \{A \in \bigvee_{t\geq 0} \mathcal{F}_t \,:\, A \cap \{\tau \leq t\} \in \mathcal{F}_t, \text{ for all } t \geq 0\}$, where $\bigvee_{t\geq 0} \mathcal{F}_t = \sigma(\cup_{t\geq 0} \mathcal{F}_t)$.
2. $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$, if $\sigma \leq \tau$,
3. $\mathcal{F}_\tau = \cap_{n\in\mathbb{N}} \mathcal{F}_{\tau_n}$ if $\tau_1 \geq \tau_2 \geq \dots \geq \tau$, $\tau = \lim_n \tau_n$, and the filtration $\{\mathcal{F}_t\}_{t\in[0,\infty)}$ is right-continuous.
4. If there exists a countable set $Q = \{q_k \,:\, k \in \mathbb{N}\} \subseteq [0,\infty)$ such that $\tau(\omega) \in Q$, for all $k \in \mathbb{N}$, then
$$\mathcal{F}_\tau = \{A \in \bigvee_{t\geq 0} \mathcal{F}_t \,:\, A \cap \{\tau = q_k\} \in \mathcal{F}_{q_k}, \text{ for all } k \in \mathbb{N}\}.$$
5. $X_\tau \mathbf{1}_{\{\tau<\infty\}}$ is a random variable in $\mathcal{F}_\tau$, whenever X is a progressively measurable process.

If one only wants X_τ to be a random variable, but not necessarily in $\mathcal{F}_\tau$, progressive measurability is too much to require:

Proposition 16.18 (Sampling of a measurable process at a stopping time)**.** *Let $\{X_t\}_{t\in[0,\infty)}$ be a (measurable) process and let τ be a $[0,\infty]$-valued random variable. Then $X_\tau \mathbf{1}_{\{\tau<\infty\}}$ is a random variable.*

Proof. Simply pick the trivial filtration $\mathcal{F}_t = \mathcal{F}$, $t \geq 0$. In this case the notions of measurability and progressive measurability coincide. $\square$

The progressive σ-algebra of Definition 16.14 has another nice property which will simplify our construction of the stochastic integral in the sequel.

Proposition 16.19 (Approximability of progressive processes)**.** *For a bounded progressive process X, there exists a sequence of continuous and adapted processes $\{X^{(n)}\}_{n\in\mathbb{N}}$ such that*

$$X_t^{(n)}(\omega) \to X_t(\omega), \lambda - a.e. \text{ in } t, \text{ for each } \omega \in \Omega.$$

Proof. We consider the process Y defined by $Y_t(\omega) = \int_0^t X_u(\omega)\, du$. By Problem 16.3, the process Y is continuous and progressively measurable, and, therefore, so are the processes $\{X^{(n)}\}_{n\in\mathbb{N}}$, defined by

$$X_t^{(n)} = n\Big(Y_t - Y_{t-1/n}\Big), \text{ for } t \geq 0,$$

where we use the convention that $Y_s = 0$, for $s \leq 0$. It remains to use the Lebesgue Differentiation Theorem for Lipschitz functions. $\square$

Continuous-time martingales

The definition of (super-, sub-) martingales in continuous time is the formally the same as the corresponding definition in discrete time:

Definition 16.20 (Continuous time (sub,super) martingales)**.** Given a filtration $\{\mathcal{F}_t\}_{t\in[0,\infty)}$, a stochastic process $\{X_t\}_{t\in[0,\infty)}$ is said to be an $\{\mathcal{F}_t\}_{t\in[0,\infty)}$-**supermartingale** if

1. $X_t \in \mathcal{F}_t$, for all $t \geq 0$
2. $X_t \in \mathcal{L}^1$, for all $t \in [0,\infty)$, and
3. $\mathbb{E}[X_t|\mathcal{F}_s] \leq X_s$, a.s., whenever $s \leq t$.

A process $\{X_t\}_{t\in[0,\infty)}$ is called a **submartingale** if $\{-X_t\}_{t\in[0,\infty)}$ is a supermartingale. A **martingale** is a process which is both a supermartingale and a submartingale at the same time, i.e., for which the equality holds in 3., above.

Problem 16.5 (Brownian motion is a martingale)**.** Show that the Brownian motion is a martingale with respect to its natural filtration $\mathbb{F}^B$, as well as with respect to its right-continuous augmentation $\mathbb{F}^B_+$.

Remark 16.21. An example of a continuous-time martingale that is not continuous is the **compensated Poisson process**, i.e., the process $N_t - \lambda t$, $t \geq 0$, where N_t is a Poisson process with parameter $\lambda > 0$ (see Problem 16.9). It is useful to keep $N_t - \lambda t$ in mind as a source of counterexamples.

The theory of continuous-time martingales (submartingales, supermartingales) largely parallels the discrete-time theory. There are two major tricks that help us transfer discrete-time results to the continuous time:

1. *Approximation*: a stopping time τ can be approximated from above by a sequence of stopping times $\{\tau_n\}_{n\in\mathbb{N}}$ each of which takes values in a countable set.
2. *Use of continuity properties of the paths*: typically, the right-continuity of the trajectories will imply that $X_{\tau_n} \to X_\tau$, and an apropriate limit theorem is then used.

Instead of giving detailed proofs of all the continuous-time counterparts of the optional sampling theorems, martingale-inequalities, etc., we only state most of our results. We do give a detailed proof of our first theorem, though, and, before we do that, here is some notation and a useful concept:

1. for $t \geq 0$, $\mathcal{S}_t$ denotes the set of all stopping times τ such that $\tau(\omega) \leq t$, for all $\omega \in \Omega$,
2. $\mathcal{S}_b = \cup_{t\geq 0}\mathcal{S}_t$ is the set of all bounded stopping times, and
3. $\mathcal{S}$ denotes the set of all stopping times.

Definition 16.22 (Classes (DL) and (D))**.**

1. A measurable process $\{X_t\}_{t\in[0,\infty)}$ is said to be **of class (DL)** if the family
$$\{X_\tau \,:\, \tau \in \mathcal{S}_t\} \text{ is uniformly integrable for all } t \geq 0.$$
2. A measurable process $\{X_t\}_{t\in[0,\infty)}$ is said to be **of class (D)** if the family
$$\{X_\tau \mathbf{1}_{\{\tau<\infty\}} \,:\, \tau \in \mathcal{S}\} \text{ is uniformly integrable.}$$

Proposition 16.23 (Bounded Optional Sampling). *Let $\{M_t\}_{t\in[0,\infty)}$ be a right-continuous martingale. Then, M is of class (DL) and*

$$\mathbb{E}[M_t|\mathcal{F}_\tau] = M_\tau \text{ and } \mathbb{E}[M_\tau] = \mathbb{E}[M_0], \text{ for all } \tau \in \mathcal{S}_t,\ t \geq 0.. \tag{16.1}$$

Proof. Let $\mathcal{S}_t^f$ be the set of all $\tau \in \mathcal{S}_t$ such that τ takes only finitely many values. We pick $\tau \in \mathcal{S}_t^f$, and take $0 \leq t_1 < t_2 < \dots < t_n \leq t$ to be the set of all the values it can take. Then for $A \in \mathcal{F}_\tau$, we have

$$\begin{aligned} \mathbb{E}[M_\tau \mathbf{1}_A] &= \sum_{k=1}^n \mathbb{E}[M_\tau \mathbf{1}_A \mathbf{1}_{\{\tau=t_k\}}] = \sum_{k=1}^n \mathbb{E}[M_{t_k} \mathbf{1}_A \mathbf{1}_{\{\tau=t_k\}}] \\ &= \sum_{k=1}^n \mathbb{E}[\mathbb{E}[M_t \mathbf{1}_A \mathbf{1}_{\{\tau=t_k\}}|\mathcal{F}_{t_k}]] = \mathbb{E}[M_t \mathbf{1}_A], \end{aligned} \tag{16.2}$$

so that $M_\tau = \mathbb{E}[M_t|\mathcal{F}_\tau]$. Therefore, the family $\{M_\tau : \tau \in \mathcal{S}_t^f\}$ is uniformly integrable, as each of its members can be represented as the conditional expectation of M_t with respect to some sub-σ-algebra of $\mathcal{F}_t$. Consider now a general stopping time $\tau \in \mathcal{S}_t$, and approximate it by the sequence $\{\tau_n\}_{n\in\mathbb{N}}$, given by

$$\tau_n = 2^{-n}\lceil 2^n \tau \rceil \wedge t,$$

so that each τ_n is in $\mathcal{S}_t$. Since $\tau_n \searrow \tau$, right continuity implies that $M_{\tau_n} \to M_\tau$ and, so, by uniform integrability of $\{M_{\tau_n}\}_{n\in\mathbb{N}}$, we conclude that M_τ is in the $\mathbb{L}^1$-closure of the uniformly integrable set $\{M_\tau : \tau \in \mathcal{S}_t^f\}$. In other words,

$$\{M_\tau : \tau \in \mathcal{S}_t\} \subseteq \overline{\{M_\tau : \tau \in \mathcal{S}_t^f\}}^{\mathbb{L}^1},$$

where $\overline{(\cdot)}^{\mathbb{L}^1}$ denotes the closure in $\mathbb{L}^1$. Using the fact that the $\mathbb{L}^1$-closure of a uniformly integrable set is uniformly integrable (why?), we conclude that $\{M_t\}_{t\in[0,\infty)}$ is of class (DL). To show (16.1), we note that uniform integrability (via the backward martingale convergence theorem - see Corollary 12.17) implies that

$$M_\tau = \lim_n M_{\tau_n} = \lim_n \mathbb{E}[M_t|\mathcal{F}_{\tau_n}],$$

where all the limits are in $\mathbb{L}^1$. Taking the conditional expectation with respect to $\mathcal{F}_\tau$ yields that

$$\begin{aligned} M_\tau = \mathbb{E}[M_\tau|\mathcal{F}_\tau] &= \mathbb{E}[\lim_n \mathbb{E}[M_t|\mathcal{F}_{\tau_n}]|\mathcal{F}_\tau] \\ &= \lim_n \mathbb{E}[\mathbb{E}[M_t|\mathcal{F}_{\tau_n}]|\mathcal{F}_\tau] = \mathbb{E}[M_t|\mathcal{F}_\tau], \end{aligned}$$

and, in particular, that $\mathbb{E}[M_\tau] = \mathbb{E}[M_t] = \mathbb{E}[M_0]$. $\square$

A partial converse and a useful martingality criterion is given in the following proposition:

Proposition 16.24 (A characterization of martingales). *Let $\{M_t\}_{t\in[0,\infty)}$ be an adapted and right-continuous process with the property that $M_\tau \in \mathbb{L}^1$ for all $\tau \in \mathcal{S}_b$. Then $\{M_t\}_{t\in[0,\infty)}$ is a martingale if and only if*

$$\mathbb{E}[M_\tau] = \mathbb{E}[M_0], \text{ for all } \tau \in \mathcal{S}_b. \tag{16.3}$$

Proof. Only sufficiency needs a proof. We consider stopping times of the form $\tau = s\mathbf{1}_{A^c} + t\mathbf{1}_A$, for $0 \le s \le t < \infty$ and $A \in \mathcal{F}_s$, as for such a τ, the condition (16.3) implies that

$$\mathbb{E}[M_0] = \mathbb{E}[M_\tau] = \mathbb{E}[M_s\mathbf{1}_{A^c}] + \mathbb{E}[M_t\mathbf{1}_A] = \mathbb{E}[M_s] + \mathbb{E}[(M_t - M_s)\mathbf{1}_A].$$

Since $\mathbb{E}[M_s] = \mathbb{E}[M_0]$, we have $\mathbb{E}[M_s\mathbf{1}_A] = \mathbb{E}[M_t\mathbf{1}_A]$, for all $A \in \mathcal{F}_s$, which, by definition, means that $M_s = \mathbb{E}[M_t|\mathcal{F}_s]$. □

Corollary 16.25 (Stopped martingales are martingales). *If $\{M_t\}_{t\in[0,\infty)}$ is a right-continuous martingale, then so is the stopped process $\{M^\tau_t\}_{t\in[0,\infty)}$, for each stopping time τ.*

Before we state a general optional sampling theorem, we give a criterion for uniform integrability for continuous-time martingales. The the proof is based on the same ideas as the corresponding discrete-time result (Proposition 12.12), so we omit it.

Theorem 16.26 (UI martingales). *Let $\{M_t\}_{t\in[0,\infty)}$ be a right-continuous martingale. The following are equivalent:*

1. *$\{M_t\}_{t\in[0,\infty)}$ is UI,*
2. *$\lim_{t\to\infty} M_t$ exists in $\mathbb{L}^1$, and*
3. *$\{M_t\}_{t\in[0,\infty)}$ has a last element, i.e., there exists a random variable M_∞ such that*

$$M_t = \mathbb{E}[M_\infty|\mathcal{F}_t], \text{ for all } t \in [0,\infty).$$

The version of the optional sampling theorem presented here

Theorem 16.27 (Optional Sampling and Convergence). *Let $\{X_t\}_{t\in[0,\infty)}$ be a right-continuous submartingale with a last element, i.e., such that there exists $X \in \mathbb{L}^1(\mathcal{F})$ with the property that*

$$X_t \le \mathbb{E}[X|\mathcal{F}_t], \text{ a.s}$$

Then,

1. *$X_\infty = \lim_{t\to\infty} X_t$ exists a.s., and $X_\infty \in \mathbb{L}^1$ is the a.s.-minimal last element for $\{X_t\}_{t\in[0,\infty)}$, and*

2. $X_\tau \leq \mathbb{E}[X_\infty|\mathcal{F}_\tau]$, *for all* $\tau \in \mathcal{S}$.

Corollary 16.28 (Optional sampling for nonnegative supermartingales)**.** *Let* $\{M_t\}_{t\in[0,\infty)}$ *be a nonnegative right-continuous supermartingale. Then* $\mathbb{E}[X_\tau|\mathcal{F}_\sigma] \leq X_\sigma$ *for all stopping times* τ, σ, *with* $\tau \geq \sigma$.

In complete analogy with discrete time (again), the (sub)martingale property stays stable under stopping:

Proposition 16.29 (Stability under stopping)**.** *Let* $\{M_t\}_{t\in[0,\infty)}$ *be a right-continuous martingale (submartingale), and let* τ *be a stopping time. Then the stopped process* $\{M_t^\tau\}_{t\in[0,\infty)}$ *is also a right-continuous martingale (submartingale).*

Most of the discrete-time results about martingales transfer to the continuous-time setting, provided that right-continuity of the paths is assumed. Such a condition is not as restrictive as it may seem at first glance, thanks to the following result. It is important to note, though, that it relies heavily on our standing assumption of right continuity for the filtration $\mathbb{F}$. It also needs a bit of completeness; more precisely, we assume that each probability-zero set in $\mathcal{F}$ belongs already to $\mathcal{F}_0$. These two assumption as so important that they even have a name:

Definition 16.30 (Usual conditions)**.** A filtration is said to satisfy the **usual conditions** if it is right continuous and complete in the sense that $A \in \mathcal{F}_0$, as soon as $A \in \mathcal{F}$ and $\mathbb{P}[A] = 0$.

Theorem 16.31 (Regularization of submartingales)**.** *Under usual conditions, a submartingale* $\{X_t\}_{t\in[0,\infty)}$ *has a RCLL modification if and only if the mapping* $t \mapsto \mathbb{E}[X_t]$ *is right continuous. In particular, each martingale has a RCLL modification.*

Various submartingale inequalities can also be extended to the case of continuous time. In order not to run into measurability problems, we associate the **maximal process**

$$X_t^* = \sup\{|X_q| \,:\, q = t \text{ or } q \text{ is a rational in } [0,t)\},$$

with a process X. Note that when X is RCLL or LCRL, $X_t = \sup_{s\in[0,t]} |X_s|$, a.s.

Theorem 16.32 (Doob's and Maximal Inequalities)**.** *Let* $\{X_t\}_{t\in[0,\infty)}$ *be a*

RCLL process which is either a martingale or a positive submartingale. Then,

$$\mathbb{P}[X^* \geq M] \leq \tfrac{1}{M^p} \sup_{t\geq 0} \mathbb{E}[|X_t|^p], \text{ for } M > 0 \text{ and } p \geq 1, \text{ and}$$
$$||X^*||_{\mathbb{L}^p} \leq \tfrac{p}{p-1} \sup_{t\geq 0} ||X_t||_{\mathbb{L}^p}, \text{ for } p > 1.$$

Proof. The main idea of the proof is to approximate $\sup_{t\geq 0} |X_t|$ by the random variables of the form $\sup_{t\in Q_n} |X_t|$, where $\{Q_n\}_{n\in\mathbb{N}}$ is an increasing sequence of finite sets whose union $Q = \cup_n Q_n$ is dense in $[0,\infty)$. By the right-continuity of the paths, $\sup_{t\geq 0} |X_t| = \sup_{t\in Q} |X_t|$, where Q is any countable dense set in $[0,\infty)$. To finish the proof, we can use the discrete-time inequalities in the pre-limit, and the monotone convergence theorem to pass to the limit. □

Additional Problems

Problem 16.6 (Predictable and optional processes)**.** A stochastic process $\{X_t\}_{t\in[0,\infty)}$ is said to be

- **optional**, if it is measurable with respect to the σ-algebra $\mathcal{O}$, where $\mathcal{O}$ is the smallest σ-algebra on $[0,\infty)\times\Omega$ with respect to which all RCLL and adapted adapted processes are measurable.
- **predictable**, if it is measurable with respect to the σ-algebra $\mathcal{P}$, where $\mathcal{P}$ is the smallest σ-algebra on $[0,\infty)\times\Omega$ with respect to which all LCRL adapted processes are measurable.

Show that

1. The predictable σ-algebra coincides with the σ-algebra generated by all *continuous* and piecewise linear adapted processes.
2. Show that $\mathcal{P} \subseteq \mathcal{O} \subseteq \text{Prog} \subseteq \mathcal{B}([0,\infty)) \otimes \mathcal{F}$.

Note: There are counterexamples which show that none of the implications above are equivalences. Some are very simple, and the others are quite involved.

Problem 16.7 (The total-variation process)**.** For each process $\{X_t\}_{t\in[0,\infty)}$ of finite variation, we define its **total-variation process** $\{|X|_t\}_{t\in[0,\infty)}$ as the process whose value at t is the total variation on $[0,t]$ of the path of $\{X_t\}_{t\in[0,\infty)}$.

If $\{X_t\}_{t\in[0,\infty)}$ is an RCLL adapted process of finite variation, show that $\{|X|_t\}_{t\in[0,\infty)}$ is

1. RCLL, adapted and of finite variation, and
2. continuous if $\{X_t\}_{t\in[0,\infty)}$ is continuous.

Problem 16.8 (The Poisson Point Process)**.** Let $(S,\mathcal{S},\mu)$ be a measurable space, i.e., S is a non-empty set, $\mathcal{S}$ is a σ-algebra, and μ is a positive measure on S. A mapping

$$\mathcal{N} : \Omega \times \mathcal{S} \to \mathbb{N} \cup \{\infty\},$$

where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, is called a **Poisson Point Process** (PPP) with **mean measure** μ if

- the mapping $\mathcal{N}(B)$ (more precisely, the mapping $\omega \mapsto \mathcal{N}(\omega, B)$) is $\mathcal{F}$-measurable (a random variable) for each $B \in \mathcal{S}$ and has the Poisson distribution[3] with parameter $\mu(B)$ (denoted by $P(\mu(B))$), whenever $\mu(B) < \infty$, and
- for each $\omega \in \Omega$, the mapping $B \mapsto N(\omega, B)$ is an $\mathbb{N} \cup \{\infty\}$-valued measure, and
- random variables $(\mathcal{N}(B_1), \mathcal{N}(B_2), \dots, \mathcal{N}(B_d))$ are independent when the sets $B_1, B_2, \dots, B_d$ are (pairwise) disjoint.

[3] A r.v. X is said to have the **Poisson** distribution with parameter $c > 0$, if $\mathbb{P}[X = n] = e^{-c} \frac{c^n}{n!}$, for $n \in \mathbb{N}_0$. When $c = 0$, $\mathbb{P}[X = 0] = 1$.

The purpose of this problem is to show that, under mild conditions on $(S, \mathcal{S}, \mu)$, a PPP with mean measure μ exists.

1. We assume first that $0 < \mu(S) < \infty$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space which supports a random variable N and an iid sequence $\{X_k\}_{k \in \mathbb{N}}$, independent of N and such that

   - $N \sim P(\mu(S))$, and
   - for each k, X_k takes values in S and $\mathbb{P}[X_k \in B] = \mu(B)/\mu(S)$, for all $B \in \mathcal{S}$ (technically, a measurable mapping from Ω into a measurable space is called a **random element**).

   Show that such a probability space exists.

2. For $B \in \mathcal{S}$, define

   $$\mathcal{N}(\omega, B) = \sum_{k=1}^{N(\omega)} \mathbf{1}_{\{X_k(\omega) \in B\}},$$

   i.e., $\mathcal{N}(\omega, B)$ is the number of terms in $X_1(\omega), \dots, X_{N(\omega)}(\omega)$ that fall into B. Show that $\mathcal{N}(B)$ is a random variable for each $B \in \mathcal{S}$.

3. Pick (pairwise) disjoint $B_1, \dots, B_d$ in $\mathcal{S}$ and compute

   $$\mathbb{P}[\mathcal{N}(B_1) = n_1, \dots, \mathcal{N}(B_d) = n_d | N = m], \text{ for } m, n_1, \dots, n_d \in \mathbb{N}_0.$$

4. Show that $\mathcal{N}$ is a PPP with mean measure μ.

5. Show that a PPP with mean measure μ exists when $(S, \mathcal{S}, \mu)$ is merely a **σ-finite** measure space, i.e., there exists a sequence $\{B_n\}_{n \in \mathbb{N}}$ in $\mathcal{S}$ such that $S = \cup_n B_n$ and $\mu(B_n) < \infty$, for all $n \in \mathbb{N}_0$.

Problem 16.9 (The Poisson Process)**.** A stochastic process $\{N_t\}_{t\in[0,\infty)}$ is called a **Poisson process with parameter** $c > 0$ if

- it has independent increments, i.e., $N_{t_1} - N_{s_1}$, $N_{t_2} - N_{s_2}$, $\dots$, $N_{t_d} - N_{s_d}$ are independent random variables when $s_1 \le t_1 \le s_2 \le t_2 \le \dots \le s_d \le t_d$,
- $N_t - N_s$ has the Poisson distribution with parameter $c(t-s)$, and
- all paths $t \mapsto N_t(\omega)$ are RCLL

Let $\mathcal{N} : \Omega \times \mathcal{B}([0,\infty)) \to \mathbb{N} \cup \{\infty\}$ be a PPP on $([0,\infty), \mathcal{B}([0,\infty)), c\lambda)$, where $c > 0$ is a constant and λ is the Lebesgue measure on $[0,\infty)$. Define the stochastic process $\{N_t\}_{t\in[0,\infty)}$ by $N_t = \mathcal{N}([0,t]), t \ge 0$.

1. Show that $\{N_t\}_{t\in[0,\infty)}$ is a Poisson process.
2. Show that processes $N_t - ct$ and $(N_t - ct)^2 - ct$ are RCLL $\mathcal{F}_t$-martingales, where $\mathcal{F}_t = \sigma(N_s, s \le t)$, $t \ge 0$.
3. Let $N^{(2)}$ be a PPP on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \lambda^2)$, where λ^2 stands for the 2-dimensional Lebesgue measure. Is the process $\{M_t\}_{t\in[0,\infty)}$ a Poisson process, where

$$M_t = N^{(2)}\left(\{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} \le t\}\right), t \ge 0\ ?$$

Problem 16.10 (Supermartingale Coalescence)**.** Let $\{X_t\}_{t\in[0,\infty)}$ be a nonnegative RCLL supermartingale, and let

$$\tau = \inf\{t \ge 0 : X_t = 0\}$$

be the first hitting time of level 0. Show that

$$X_t = 0 \text{ a.s., on } \{\tau \le t\}, \text{ for all } t \ge 0.$$

In words, once a nonnegative RCLL supermartingale hits 0, it stays there. Show, by means of an example, that the statement does not hold when X is a nonnegative RCLL *submartingale*.

Problem 16.11 (Hardy's inequality)**.** Using $([0,1], \mathcal{B}([0,1]), \lambda)$ ($\lambda =$ the Lebesgue measure) as the probability space, let $\mathcal{F}_t$ be the smallest sub sigma-field of $\mathcal{F}$ containing the Borel subsets of $[0,t]$ and all negligible sets of $[0,1]$, for each $t \in [0,1]$.

1. For $f \in L^1([0,1], \lambda)$, provide an explicit expression for the right-continuous version of the martingale

$$X_t = \mathbb{E}[f|\mathcal{F}_t]$$

2. Apply Doob's maximal inequality to the above martingale to conclude that if $g(t) = \frac{1}{1-t}\int_t^1 f(u)\,du$ then

$$||g||_{\mathbb{L}^p} \leq \tfrac{p}{p-1}||f||_{\mathbb{L}^p},$$

if $p > 1$. This inequality is know as **Hardy's inequality**.

3. Inspired by the construction above, give an example of a uniformly integrable martingale $\{X_t\}_{t\in[0,\infty)}$ for which $X^*_\infty \notin \mathbb{L}^1$.

Problem 16.12 (Polynomials that turn Brownian motion into a martingale)**.** For $c \in \mathbb{R}$ we define the function $F^c : [0,\infty) \times \mathbb{R} \to \mathbb{R}_+$ by

$$F^c(t,x) = \exp(cx - \tfrac{1}{2}c^2 t), \text{ for } (t,x) \in [0,\infty) \times \mathbb{R}.$$

1. For $n \in \mathbb{N}_0$, $(t,x) \in [0,\infty) \times \mathbb{R}$, define

$$F^{(n,c)}(t,x) = \tfrac{\partial^n}{(\partial c)^n} F^c(t,x),$$

where, by convention, $\frac{\partial^0}{(\partial c)^0} G = G$. Show that $P^{(n)}(t,x) = F^{(n,0)}(t,x)$ is a polynomial in t and x for each $n \in \mathbb{N}_0$, and write down expressions for $P^{(n)}(t,x)$, for $n = 0,1,2,3$.

2. Show that the process $\{Y^{(n)}_t\}_{t\in[0,\infty)}$, given by $Y^{(n)}_t = P^{(n)}(t,B_t)$, $t \in [0,\infty)$, is an $\{\mathcal{F}_t\}_{t\in[0,\infty)}$-martingale for each $n \in \mathbb{N}_0$.

Problem 16.13 (Hitting and exit from a strip for a Brownian motion)**.** Let $(B_t)_{0\leq t<\infty}$ be a standard Brownian motion. Set $\tau_x = \inf\{t \geq 0 : B_t = x\}$, for $x \in \mathbb{R}$, and compute

1. $\mathbb{P}[\tau_a < \tau_{-b}]$ and $\mathbb{E}[\tau_a \wedge \tau_{-b}]$ for $a,b > 0$.

2. $\mathbb{E}[e^{-\lambda\tau_x}]$, for $\lambda > 0$ and $x \in \mathbb{R}$.

Hint: Apply the optional sampling theorem to appopriately-chosen martingales.

Problem 16.14 (The maximum of a martingale that converges to 0)**.** Let $\{M_t\}_{t\in[0,\infty)}$ be a nonnegative continuous martingale with $M_0 = 1$ and $M_\infty = \lim_{t\to\infty} M_t = 0$, a.s. Find the distribution of the random variable $M^*_\infty = \sup_{t\geq 0} M_t$. *Hint:* For $x > 1$, set $\tau_x = \inf\{t \geq 0 : M^*_t \geq x\} = \inf\{t \geq 0 : M_t \geq x\}$, and note that $M^{\tau_x}_\infty \in \{0,x\}$, a.s.

Problem 16.15 (Convergence of paths of RCLL martingales)**.** Consider a sequence $\{M^n_t\}_{t\in[0,T]}$, $n \in \mathbb{N}$, of martingales with continuous paths (defined on the same filtered probability space). Suppose that $\{M^\infty_t\}_{t\in[0,T]}$ is an RCLL martingale such that $M^n_T \to M^\infty_T$ in $\mathcal{L}^1$, as $n \to \infty$. Show that M^∞ has continuous paths, a.s.