Course: Theory of Probability II
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Instructor: Gordan Zitkovic

Lecture 19
Semimartingales

Continuous local martingales

While tailor-made for the $\mathbb{L}^2$-theory of stochastic integration, martingales in $\mathcal{M}_0^{2,c}$ do not constitute a large enough class to be ultimately useful in stochastic analysis. It turns out that even the class of all martingales is too small. When we restrict ourselves to processes with continuous paths, a naturally stable family turns out to be the class of so-called local martingales.

Definition 19.1 (Continuous local martingales)**.** A continuous adapted stochastic process $\{M_t\}_{t\in[0,\infty)}$ is called a **continuous local martingale** if there exists a sequence $\{\tau_n\}_{n\in\mathbb{N}}$ of stopping times such that

1. $\tau_1 \leq \tau_2 \leq \dots$ and $\tau_n \to \infty$, a.s., and
2. $\{M_t^{\tau_n}\}_{t\in[0,\infty)}$ is a uniformly integrable martingale for each $n \in \mathbb{N}$.

In that case, the sequence $\{\tau_n\}_{n\in\mathbb{N}}$ is called the **localizing sequence** for (or is said to **reduce**) $\{M_t\}_{t\in[0,\infty)}$. The set of all continuous local martingales M with $M_0 = 0$ is denoted by $\mathcal{M}_0^{loc,c}$.

Remark 19.2.

1. There is a nice theory of local martingales which are not necessarily continuous (RCLL), but, in these notes, we will focus solely on the continuous case. In particular, a "martingale" or a "local martingale" will always be assumed to be continuous.
2. While we will only consider local martingales with $M_0 = 0$ in these notes, this is assumption is not standard, so we don't put it into the definition of a local martingale.
3. Quite often, instead for $\{M_t^{\tau_n}\}_{t\in[0,\infty)}$, the uniform integrability is required for $\{M_t^{\tau_n}\mathbf{1}_{\{\tau_n>0\}}\}_{t\in[0,\infty)}$, $n \in \mathbb{N}$. The difference is only significant when we allow $M_0 \neq 0$ and $\mathcal{F}_0$ contains non-trivial events,

and this modified definition leads to a very similar theory. It becomes important when one wants to study stochastic differential equations whose initial conditions are not integrable.

4. We do not have the right tools yet to (easily) construct a local martingale which is not a martingale, but many such examples exist.

Problem 19.1. Prove the following statements (remember, all processes with the word "martingale" in their name are assumed to be continuous),

1. Each martingale is a local martingale.
2. A local martingale is a martingale if and only if it is of class (DL).
3. A bounded local martingale is a martingale of class (D).
4. A local martingale bounded from below is a supermartingale.
5. For $M \in \mathcal{M}_0^{loc,c}$ and a stopping time τ, we have $M^\tau \in \mathcal{M}_0^{loc,c}$.
6. The set of all local martingales has the structure of a vector space.
   Note: Careful! The reducing sequence may differ from one local martingale to another.
7. For $M \in \mathcal{M}_0^{loc,c}$, the sequence
$$\tau_n = \inf\{t \geq 0 : |M_t| \geq n\}, \; n \in \mathbb{N},$$
   reduces $\{M_t\}_{t\in[0,\infty)}$. *Note:* This is not true if M is not continuous.

The following result illustrates one of the most important techniques in stochastic analysis - namely, *localization*.

Proposition 19.3 (Continuous local martingales of finite variation are constant)**.** *Let M be a continuous local martingale of finite variation. Then $M_t = 0$, for all $t \geq 0$, a.s.*

Proof. Suppose that M is a continuous local martingale of finite variation and let $\{\tau_n\}_{n\in\mathbb{N}}$ be its reducing sequence. We can choose $\{\tau_n\}_{n\in\mathbb{N}}$ so that the stopped processes M^{τ_n}, $n \in \mathbb{N}$, are bounded, and, therefore, in $\mathcal{M}_0^{2,c}$. By Theorem 18.10, for each $n \in \mathbb{N}$, we have $\langle M^{\tau_n}\rangle^{\Delta_k} \stackrel{\text{ucp}}{\to} \langle M^{\tau_k}\rangle = \langle M\rangle^{\tau_k}$, for any sequence of partitions Δ_k with $\Delta_n \to \mathrm{Id}$. On one hand,

$$\langle M^{\tau_n}\rangle_t^{\Delta_k} = \sum_{i=1}^{\infty}(M^{\tau_n}_{t^k_{i+1}\wedge t} - M^{\tau_n}_{t^k_i\wedge t})^2 \leq w^k_t \sum_{i=1}^{\infty} |M^{\tau_n}_{t^k_{i+1}\wedge t} - M^{\tau_n}_{t^k_i\wedge t}| \leq w^k_t \, |M|_t \, ,$$

where $|M|_t$ denotes the total variation of M on $[0,t]$, and

$$w_t^k = \sup_{i\in\mathbb{N}} \left| M_{t_{i+1}^k \wedge t} - M_{t_i^k \wedge t} \right|.$$

Since the paths of M are continuous (and, thus, uniformly continuous on compacts), we have $w_t^k \to 0$, a.s., as $k \to \infty$, for each $t \geq 0$. Therefore, $\langle M\rangle^{\tau_n} = 0$. This implies that $\mathbb{E}[(M_t^{\tau_n})^2] = 0$, for all $t \geq 0$, i.e., $M^{\tau_n} = 0$. Since $\tau_n \to \infty$, a.s., we have $M = 0$. □

The process of localization allows us to extend the notion of quadratic variation from square-integrable martingales, to local martingales. We need a result which loosely states that the ucp convergence and localization commute:

Problem 19.2. Let $\{X^n\}_{n\in\mathbb{N}}$ be a sequence of RCLL or LCRL processes, and let $\{\tau_k\}_{k\in\mathbb{N}}$ be a sequence of stopping times such that $\tau_k \to \infty$, a.s. If X is such that $(X^n)^{\tau_k} \overset{\text{ucp}}{\to} X^{\tau_k}$, for each k, then $X^n \overset{\text{ucp}}{\to} X$.

Theorem 19.4 (Quadratic variation of continuous local martingales)**.** *Let $\{M_t\}_{t\in[0,\infty)}$ be a continuous local martingale. Then there exists a unique process $\langle M\rangle \in \mathcal{A}_0^c$ such that $M^2 - \langle M\rangle$ is a local martingale. Moreover,*

$$\langle M\rangle^{\Delta_n} \overset{\text{ucp}}{\to} \langle M\rangle, \tag{19.1}$$

for any sequence $\{\Delta_n\}_{n\in\mathbb{N}}$ in $P_{[0,\infty)}$ with $\Delta_n \to \mathrm{Id}$.

Proof. The main idea of the proof is to use a localizing sequence for M to reduce the problem to the situation dealt with in Theorem 18.10 in the previous lecture. Problem 19.1, part 7. above implies that we can use the hitting-time sequence $\tau_k = \inf\{t \geq 0 : |M_t| \geq k\}$, $k \in \mathbb{N}$ to reduce M, i.e., that for each n, the process $M^{(k)}$, given by $M_t^{(k)} = M_t^{\tau_k}$ is a uniformly integrable martingale. Moreover, thanks to the choice of τ_k, each $M^{(k)}$ is bounded, and, therefore, in $\mathcal{M}_0^{2,c}$. For each $k \in \mathbb{N}$, Theorem 18.10 states that there exists $\langle M^{(k)}\rangle \in \mathcal{A}_0^c$ such that $(M^{(k)})^2 - \langle M^{(k)}\rangle$ is a UI martingale. The uniqueness of such a process implies that $\langle M^{(k)}\rangle_t = \langle M^{(k+1)}\rangle_t$ on $\{t \leq \tau_n\}$ (otherwise, we could stop the process $\langle M^{(k+1)}\rangle$ at τ_n and use it in lieu of $\langle M^{(k)}\rangle$). In words, $\langle M^{(k+1)}\rangle$ is an "extension" of $\langle M^{(k)}\rangle$. Therefore, by using the process $\langle M^{(k)}\rangle$ on $t \in [\tau_{k-1}, \tau_k)$ (where $\tau_0 = 0$), we can construct a continuous, adapted and non-decreasing process $\langle M\rangle$ with the property that $\langle M\rangle_t = \langle M^{(k)}\rangle_t$ for $\{t \leq \tau_k\}$ and all $k \in \mathbb{N}$. Such a process clearly has the property that $N_t = M_t^2 - \langle M\rangle_t$ is a martingale on $[0,\tau_k]$. Equivalently, the stopped process N^{τ_k} is a martingale on the whole $[0,\infty)$, which means that N_t is a local martingale. The uniqueness of $\langle M\rangle$ follows directly from Proposition 19.3.

We still need to establish 19.1. This follows directly from Problem 19.2 above and Theorem 18.10, which implies that $(\langle M\rangle^{\tau_k})^{\Delta_n} \overset{\text{ucp}}{\to} \langle M\rangle^{\tau_k}$ for each k, as $n \to \infty$. $\square$

Quadratic covariation of local martingales

The concept of the quadratic covariation between processes, already prescreened in Problem 18.1 for martingales in $\mathcal{M}_0^{2,c}$, rests on the following identity:

$$xy = \tfrac{1}{2}\big((x+y)^2 - x^2 - y^2\big),$$

is often refered to as the **polarization identity** and is used whenever one wants to produce a "bilinear" functional from a "quadratic" one. More precisely, we have the following definition:

Definition 19.5 (Quadratic covariation)**.** For $M, N \in \mathcal{M}_0^{loc,c}$, the finite-variation process $\{\langle M, N\rangle_t\}_{t\in[0,\infty)}$, given by

$$\langle M, N\rangle_t = \tfrac{1}{2}\Big(\langle M+N\rangle_t - \langle M\rangle_t - \langle N\rangle_t\Big),$$

is called the **quadratic covariation** (bracket) of M and N.

At this point, the reader should try his/her hand at the following problem:

Problem 19.3. Show that, for $M, N \in \mathcal{M}_0^{loc,c}$,

1. all conclusions of Problem 18.1, parts 1., 2., and 4. remain valid (now that M, N are assumed to be only local martingales).
2. $\langle M, N\rangle$ is the unique adapted and continuous process of finite variation such that $\langle M, N\rangle = 0$ amd $MN - \langle M, N\rangle$ is a local martingale.

We conclude the section on quadratic covariation with an imporant inequality (the proof is postponed for the Additional Problems section below). Since $\langle M, N\rangle$ is a continuous and adapted finite-variation process, its total-variation process $\{|\langle M, N\rangle|_t\}_{t\in[0,\infty)}$ is continuous, adapted and nondecreasing.

Theorem 19.6 (Kunita-Watanabe inequality)**.** *For $M, N \in \mathcal{M}_0^{loc,c}$ and any two measurable processes H and K, we have*

$$\int_0^\infty |H_t|\,|K_t|\; d\,|\langle M, N\rangle|_t \le \Big(\int_0^\infty H_t^2\, d\langle M\rangle_t\Big)^{1/2} \Big(\int_0^\infty K_t^2\, d\langle N\rangle_t\Big)^{1/2}, \text{ a.s.} \tag{19.2}$$

Moreover, for any $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\mathbb{E}\left[\int_0^\infty |H_t|\,|K_t|\; d\,|\langle M, N\rangle|_t\right] \leq$$
$$\leq \left\| \left(\int_0^\infty H_t^2 \, d\langle M\rangle_t \right)^{1/2} \right\|_{\mathbb{L}^p} \left\| \left(\int_0^\infty K_t^2 \, d\langle N\rangle_t \right)^{1/2} \right\|_{\mathbb{L}^q}. \quad (19.3)$$

Stochastic integration for local martingales

The restriction $H \in \mathbb{L}^2(M)$ on the integrand, and $M \in \mathcal{M}_0^{2,c}$ on the integrator in the definition of the stochastic integral $H \cdot M$ can be relaxed. For a continuous local martingale M, we define the class $L(M)$ which contains all predictable processes H with the property

$$\int_0^t H_u^2 \, d\langle M\rangle_u < \infty, \text{ for all } t \geq 0, \text{ a.s.}$$

In comparison with the space $\mathbb{L}^2(M)$, the $d\langle M\rangle$-integrals are on compact intervals, and the finite-expectation assumption is replaced by "finite-a.s." requirement.

We define the stochastic integral $H \cdot M$ for $H \in L(M)$ as follows: let $\{\tau_n\}_{n\in\mathbb{N}}$ be a sequence of stopping times defined by

$$\tau_n = \inf\{t \geq 0 \,:\, |M_t| \geq n\} \wedge \inf\{t \geq 0 \,:\, \int_0^t H_u^2 \, d\langle M\rangle_u \geq n\}.$$

This sequence clearly reduces[1] M. Moreover, we know that $M^{\tau_n} \in \mathcal{M}_0^{2,c}$ (because it is bounded) and $H\mathbf{1}_{[0,\tau_n]} \in \mathbb{L}^2(M^{\tau_n})$ for all $n \in \mathbb{N}$. Therefore, the integral $H\mathbf{1}_{[0,\tau_n]} \cdot M^{\tau_n}$ is well-defined in the $\mathbb{L}^2$-sense for each n, and the sequence has the property that

$$H\mathbf{1}_{[0,\tau_n]} \cdot M^{\tau_n} \text{ and } H\mathbf{1}_{[0,\tau_{n+1}]} \cdot M^{\tau_{n+1}} \text{ coincide on } [0, \tau_n].$$

[1] By reducing each process separately, and then taking the minimum of all stopping times over a (finite) family of processes, we can always find a single sequence that reduces each process in the family. We will continue doing this in the sequel, with minimal explanation.

As in the proof of Theorem 19.4 we can patch the "stopped integrals" together to obtain a stochastic process which we denote by $H \cdot M$. The process $H \cdot M$ is clearly continuous and adapted, and we have

$$(H \cdot M)^{\tau_n} = H\mathbf{1}_{[0,\tau_n]} \cdot M^{\tau_n} \in \mathcal{M}_0^{2,c}.$$

Hence, the sequence $\{\tau_n\}_{n\in\mathbb{N}}$ can be used as a reducing sequence for $H \cdot M$, yielding that $H \cdot M \in \mathcal{M}_0^{loc,c}$. Summarizing all of the above, we have the following result (the reader will easily supply the proofs of all the remaining statements):

Theorem 19.7. *For $M \in \mathcal{M}_0^{loc,c}$ and $H \in L(M)$, there exists a stochastic process $H \cdot M$ in $\mathcal{M}_0^{loc,c}$ with the following properties*

1. *$H \cdot M$ conincides with the $\mathbb{L}^2$-stochastic integral, for $M \in \mathcal{M}_0^{2,c}$ and $H \in \mathbb{L}^2(M)$.*

2. *For each stopping time τ, we have $(H \cdot M)^\tau = H\mathbf{1}_{[0,\tau]} \cdot M^\tau$.*

The two properties above are enough to characterize the $\mathcal{M}_0^{loc,c}$-integral, and to transfer many properties from the $\mathcal{M}_0^{2,c}$ to the $\mathcal{M}_0^{loc,c}$ setting (the reader will be asked to do this, in a more general framework in Problem ? below).

Here is another characterization of the $\mathcal{M}_0^{loc,c}$-integral, this time in terms of the quadratic-covariation processes. It also tells us how to compute quadratic (co)variations of stochastic integrals:

Proposition 19.8. *For $M, L \in \mathcal{M}_0^{loc,c}$ and $H \in L(M)$, we have*

$$L = H \cdot M \text{ iff } \langle L, N\rangle = \int_0^\cdot H_u \, d\langle M, N\rangle, \text{ for all } N \in \mathcal{M}_0^{loc,c}. \tag{19.4}$$

Proof. First, we note that, since $\int_0^t H_u^2 \langle M\rangle_u < \infty$, a.s., we can use the Kunita-Watanabe inequality (19.2) (with $K = \mathbf{1}_{[0,t]}$) to conclude that

$$\int_0^t |H_u| \, d\, |\langle M, N\rangle|_u < \infty, \text{ a.s.,}$$

and that the integral $\int_0^\cdot H_u \, d\langle M, N\rangle_u$ is well-defined.

Assuming that $M, N \in \mathcal{M}_0^{2,c}$, $H \in \mathcal{H}^\infty_{simp}$, the elementary manipulations show that

$$\mathbb{E}[(H \cdot M)_\infty N_\infty] = \mathbb{E}[\int_0^\infty H_u \, d\langle M, N\rangle_u]. \tag{19.5}$$

The "in-expectation" form (19.3) of the Kunita-Watanabe inequality with $p = q = \frac{1}{2}$, implies immediately that both sides of (19.5) are linear continuous functions of $H \in \mathbb{L}^2(M)$. Since they coincide on a dense subset $\mathcal{H}^\infty_{simp}$ of $\mathbb{L}^2$, they must conincide everywhere on $\mathbb{L}^2(M)$. Finally, for $M, N \in \mathcal{M}_0^{loc,c}$, we first localize into $\mathcal{M}_0^{2,c}$, and then apply the dominated-convergence theorem (with its use justified by the Kunita-Watanabe inequality) to "remove the localization" and conclude that the left-hand side of (19.4) implies the right-hand side.

For the other implication, we assume that $L \in \mathcal{M}_0^{loc,c}$ satisfies the equality on the right-hand side of (19.4). Let τ_n be a localizing sequence for L; without loss of generality, we can also suppose that, stopped at each τ_n, the processes L, M and $\int_0^\cdot H_u^2 \, d\langle M\rangle_u$ are bounded. For $N \in \mathcal{M}_0^{2,c}$, the process $L^{\tau_n} N - \langle L^{\tau_n}, N\rangle$ is a UI martingale, so

$$\mathbb{E}[L^{\tau_n}_\infty N_\infty] = \mathbb{E}[\int_0^\infty H_t \mathbf{1}_{[0,\tau_n]}(t) \, d\langle M^{\tau_n}, N\rangle_t],$$

for all $N \in \mathcal{M}_0^{2,c}$. On the other hand, we also have

$$\mathbb{E}[(H \cdot M)^{\tau_n} N_\infty] = \mathbb{E}[\int_0^\infty H_t \mathbf{1}_{[0,\tau_n]}(t) \, d\langle M^{\tau_n}, N\rangle_t],$$

for all n, so

$$\mathbb{E}[L_\infty^{\tau_n} N_\infty] = \mathbb{E}[(H \cdot M)^{\tau_n} N_\infty],$$

for all $N \in \mathcal{M}_0^{2,c}$. In particular, the choice $N = L^{\tau_n} - (H \cdot M)^{\tau_n}$ implies that $||L^{\tau_n} - (H \cdot M)^{\tau_n}||_{\mathcal{M}_0^{2,c}} = 0$, i.e., that $L^{\tau_n} = (H \cdot M)^{\tau_n}$. Letting $n \to \infty$ gives the left-hand side of (19.4). □

The semimartingale integral

At this point we know how to use 1) processes of finite variation, and 2) local martingales as integrators. One can show (it is beyond the scope of these notes, though) that linear combinations of those are, in a sense, all processes that can be used as integrators. For this reason, we give them a name, but before we do, we introduce one more piece of notation: $\mathcal{V}_0^c$ denotes the family of all continuous and adapted processes with paths of finite variation which vanish at $t = 0$.

Definition 19.9. A stochastic process X is called a **continuous semimartingale** if there exist processes $A \in \mathcal{V}_0^c$ and $M \in \mathcal{M}_0^{loc,c}$ such that

$$X_t = X_0 + M_t + A_t, \text{ for all } t \geq 0, \text{ a.s.}$$

Using the fact that there are no non-trivial continuous local martingales of finite variation one can show that that for a continuous semimartingale the decomposition $X = X_0 + M + A$ into the initial value, a continuous local martingale and a continuous adapted process of finite variation is unique. This decomposition is called the **semimartingale decomposition** of X.

Problem 19.4. Show that, for a continuous semimartingale X with the decomposition $X = X_0 + M + A$, we have

$$\langle X \rangle^{\Delta_n} \stackrel{\text{ucp}}{\to} \langle M \rangle,$$

for any sequence $\{\Delta_n\}_{n \in \mathbb{N}} \in P_{[0,\infty)}$, with $\Delta_n \to \mathrm{Id}$.

Problem 19.4 above makes it natural to define the quadratic-variation process $\langle X \rangle$ of the semimartingale $X = X_0 + M + A$ by

$$\langle X \rangle_t = X_0^2 + \langle M \rangle_t.$$

For a continuous semimartingale X with the semimartingale decomposition $X = X_0 + A + M$, let $L(X)$ denote the set of all predictable processes with the property that

$$\int_0^t |H_u| \, d\,|A|_u + \int_0^t H_u^2 \, d\langle M \rangle_u < \infty \text{ for all } t \geq 0, \text{ a.s.}$$

For $H \in L(X)$ we can define both the Lebesgue-Stieltjes integral $\int_0^t H_u \, dA_u$ (which we also denoted by $H \cdot A$) and the stochastic integral $H \cdot M$; thus, we define the stochastic integral $H \cdot X$ of H with respect to X by

$$(H \cdot X)_t = (H \cdot A)_t + (H \cdot M)_t, \text{ for all } t \geq 0.$$

It is immediately clear that $H \cdot A$ is an adapted process of finite variation and that $H \cdot M$ is a local martingale, so that $H \cdot X$ is a continuous semimartingale and $H \cdot X = H \cdot A + H \cdot M$ is its semimartingale decomposition. As the reader can check, the stochastic integral for semimartingales has the following properties:

Problem 19.5. Let X be a continuous semimartingale with the semimartingale decomposition $X = A + M$. Then

1. Both maps $H \mapsto (H \cdot X)$ and $X \mapsto (H \cdot X)$ are linear on their natural domains.
2. For $H \in L(X)$ and $K \in L(H \cdot X)$, we have $KH \in L(X)$ and
$$KH \cdot X = K \cdot (H \cdot X).$$
3. For $H \in L(X)$ and a stopping time T, we have $H^T \in L(X)$ and
$$(H \cdot X)^T = (H\mathbf{1}_{[0,T]} \cdot X) = H \cdot X^T.$$
4. For $H \in \mathcal{H}_{simp}$ with representation $H_t = \sum_{n=0}^{\infty} K_n \mathbf{1}_{(\tau_n, t_{n+1}]}(t)$, we have $H \in L(X)$ and
$$(H \cdot X)_t = \sum_{n=0}^{\infty} K_n (X_{t \wedge t_{n+1}} - X_{t \wedge \tau_n}).$$

We conclude the lecture on the semimartingale integration with a a very useful version of the dominated convergence theorem for stochastic integration. A stochastic process H is said to be **locally bounded** if it can be reduced to a (uniformly) bounded process, i.e., if there exists a nondecreasing sequence $\{\tau_n\}_{n \in \mathbb{N}}$ of stopping times with $\tau_n \to \infty$, a.s., such that H^{τ_n} is a (uniformly) bounded process.

Problem 19.6. Let X be a continuous semimartingale. Show that

1. Show that each adapted LCRL process is locally bounded, and that
2. each predictable and locally bounded process H is in $L(X)$.
3. Construct an example of an adapted RCLL process which is not locally bounded.

Proposition 19.10 (A Stochastic Dominated Convergence Theorem). *Let $X = M + A$ be a continuous semimartingale, and let $\{H^n\}_{n\in\mathbb{N}}$ be a sequence of progressively-measurable processes with the property that there exists a càglàd process H such that $|(H^n)_t| \leq H_t$ for all $t \geq 0$, a.s., for all $n \in \mathbb{N}$. Then $H^n \in L(X)$ for all $n \in \mathbb{N}$ and*

$$\lim_n H^n_t \to 0 \text{ for all } t \geq 0, \text{ a.s., implies } H^n \cdot X \overset{\text{ucp}}{\to} 0, \text{ as } n \to \infty.$$

Proof. Combine Problems 19.2 and 19.9. □

Additional Problems

Problem 19.7 (Square integrability and convergence via quadratic variation). For $M \in \mathcal{M}_0^{loc,c}$, the following statements hold:

1. M is in $\mathcal{M}_0^{2,c}$ iff $\mathbb{E}[\langle M\rangle_\infty] < \infty$, where $\langle M\rangle_\infty = \lim_{t\to\infty}\langle M\rangle_t$. *Hint:* Use Fatou's lemma to show that $\mathbb{E}[M_t^2] \leq \mathbb{E}[\langle M\rangle_\infty]$, for each t.

2. The limit $\lim_{t\to\infty} M_t$ exists a.s. on $\{\langle M\rangle_\infty < \infty\}$. *Hint:* Define $\tau_n = \inf\{t \geq 0 : \langle M\rangle_t \geq n\}$ and observe that M^{τ_n} converges a.s. Think about the behavior of the sequence $\{\tau_n\}_{n\in\mathbb{N}}$ on $\{\langle M\rangle_\infty < \infty\}$.

Problem 19.8 (The Kunita-Watanabe inequality).

1. Let $F, G : [0,\infty) \to [0,\infty)$ be two non-decreasing continuous functions with $F(0) = G(0) = 0$, and let $L : [0,\infty) \to \mathbb{R}$ be a continuous function of finite variation with the property that $L(0) = 0$ and

$$|L(t) - L(s)| \leq \sqrt{F(t) - F(s)}\sqrt{G(t) - G(s)},$$

for all $0 \leq s \leq t < \infty$. Show that, for any two measurable functions $h, k : [0,\infty) \to \mathbb{R}$, we have

$$\int_0^\infty |h(t)|\,|k(t)|\; d\,|L|\,(t) \leq$$
$$\leq \left(\int_0^\infty h^2(t)\, dF(t)\right)^{1/2} \left(\int_0^\infty k^2(t)\, dG(t)\right)^{1/2},$$

where $|L| : [0,\infty) \to [0,\infty)$ denotes the total-variation of L. *Hint:* Approximate.

2. Prove that, for all $M, N \in \mathcal{M}_0^{loc,c}$, we have $|\langle M,N\rangle_t - \langle M,N\rangle_s| \leq \sqrt{\langle M\rangle_t - \langle M\rangle_s}\sqrt{\langle N\rangle_t - \langle N\rangle_s}$, for all $s \leq t$, a.s., and use it to establish the Kunita-Watanabe inequality. *Hint:* $\langle M + rN\rangle_t \geq \langle M + rN\rangle_s$, a.s., for all rational r.

Problem 19.9 (Continuity of stochastic integration).

1. Let $\{M^n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}_0^{loc,c}$ with the property that

$$\langle M^n\rangle_\infty \to 0 \text{ in probability.}$$

Show that $(M^n)^*_\infty \to 0$ in probability, where $(M^n)^*_\infty = \sup_{t\geq 0} |M^n_t|$.
Hint: Define $\tau^\varepsilon_n = \inf\{t \geq 0 : \langle M^n_t\rangle \geq \varepsilon\}$ and use the fact that $(M^n)^{\tau^\varepsilon_n}$ is in $\mathcal{M}_0^{2,c}$, with the norm bounded by $\sqrt{\varepsilon}$. Use Doob's inequality.

2. Let X be a continuous semimartingale with the semimartingale decomposition $X = X_0 + A + M$. For $H \in L(X)$, we define the $[0,\infty]$-valued random variable

$$[H]_{L(X)} = \int_0^\infty |H_u| \, d\,|A|_u + \int_0^\infty (H_u)^2 \, d\langle M\rangle_u.$$

Show the following continuity property of stochastic integration: let $\{H^n\}_{n\in\mathbb{N}}$ be a sequence in $L(X)$ such that $[H^n]_{L(X)} \xrightarrow{\mathbb{P}} 0$. Then $(H^n \cdot X)^*_\infty \xrightarrow{\mathbb{P}} 0$.