L-spaces and left-orderability

Cameron McA. Gordon

(joint with Steve Boyer and Liam Watson)

Geometry and Topology Down Under

Melbourne, July 2011

Left Orderability

A group $G \neq 1$ is *left-orderable* (LO) if $\exists$ strict total order $<$ on G such that $g < h \Rightarrow fg < fh \; \forall \, f \in G$

- $\mathbb{R}$ is LO
- G LO, $1 \neq H < G \Rightarrow H$ LO
- $G \ni g \; (\neq 1)$ finite order $\Rightarrow G$ not LO
- G locally indicable $\Rightarrow G$ LO
- G, H LO $\Rightarrow G * H$ LO (Vinogradov, 1949)
- G (countable) LO $\Leftrightarrow \exists$ embedding $G \subset \mathrm{Homeo}_+(\mathbb{R})$
- braid group B_n is LO (Dehornoy, 1994)

- M compact, orientable, prime 3-manifold (poss. with boundary)

  Then $\pi_1(M)$ is LO $\Leftrightarrow$ $\pi_1(M)$ has an LO quotient

  (Boyer-Rolfsen-Wiest, 2005)

Hence $\beta_1(M) > 0 \Rightarrow \pi_1(M)$ LO

So interesting case is when

M is a $\mathbb{Q}$-homology 3-sphere (QHS)

Suppose M has a co-orientable taut foliation $\mathcal{F}$

$\pi_1(M)$ acts on leaf space $\mathcal{L}$ of universal covering of M

If $\mathcal{L} \cong \mathbb{R}$ ($\mathcal{F}$ is $\mathbb{R}$-covered) then we get non-trivial homomorphism

$\pi_1(M) \to \mathrm{Homeo}_+(\mathbb{R}) \quad \therefore \quad \pi_1(M)$ is LO

Theorem (BRW, 2005)

M a Seifert fibered QHS. Then $\pi_1(M)$ is LO $\Leftrightarrow$ M has base orbifold $S^2(a_1, \ldots, a_n)$ and admits a horizontal foliation.

Theorem (Calegari-Dunfield, 2003)

M a prime, atoroidal QHS with a co-orientable taut foliation, $\widetilde{M}$ the universal abelian cover of M. Then $\pi_1(\widetilde{M})$ is LO.

Thurston's universal circle construction gives

$$\rho : \pi_1(M) \subset \mathrm{Homeo}_+(S^1)$$

Central extension

$$1 \to \mathbb{Z} \to \widetilde{\mathrm{Homeo}}_+(S^1) \to \mathrm{Homeo}_+(S^1) \to 1$$

Restriction of ρ to $\pi_1(\widetilde{M})$ lifts to $\widetilde{\mathrm{Homeo}}_+(S^1) \subset \mathrm{Homeo}_+(\mathbb{R})$

Heegaard Floer Homology (Ozsváth-Szabó)

M a QHS

$\widehat{HF}(M)$: finite dimensional $\mathbb{Z}_2$-vector space

$$\dim \widehat{HF}(M) \geq |H_1(M)|$$

M is an L-space if equality holds

E.g. lens spaces are L-spaces

Is there a “topological” characterization of L-spaces?

Conjecture

M a prime QHS. Then

$$M \text{ is an } L\text{-space} \Leftrightarrow \pi_1(M) \text{ is not LO}$$

E.g.

$$\pi_1(M) \text{ finite } \begin{array}{l} \nearrow \; M \text{ is an } L\text{-space} \\ \searrow \; \pi_1(M) \text{ not LO} \end{array}$$

Theorem (OS, 2004)

If M is an L-space then M does not admit a co-orientable taut foliation.

So Conjecture $\Rightarrow$: if M has a co-orientable taut foliation then $\pi_1(M)$ is LO (virtually true by Calegari-Dunfield)

M ZHS graph manifold admits a taut foliation, horizontal in each Seifert piece. Hence M not an L-space, $\pi_1(M)$ LO (Boileau-Boyer, 2011)

(A) Seifert manifolds

Theorem

The Conjecture is true if M is Seifert fibered.

Base orbifold is either

$S^2(a_1, \ldots, a_n)$:

M an L-space $\Leftrightarrow$ M does not admit a horizontal foliation

(Lisca-Stipsicz, 2007)

$\Leftrightarrow$ $\pi_1(M)$ not LO (BRW, 2005)

(also observed by Peters)

$P^2(a_1, \ldots, a_n)$: $\pi_1(M)$ not LO (BRW, 2005)

Show M is an L-space by inductive surgery argument using:

N compact, orientable 3-manifold, ∂N a torus; $\alpha, \beta \subset \partial N$, $\alpha \cdot \beta = 1$,

such that

$$|H_1\big(N(\alpha+\beta)\big) = |H_1\big(N(\alpha)\big)| + |H_1\big(N(\beta)\big)|$$

Then $$N(\alpha), N(\beta) \ L\text{-spaces} \Rightarrow N(\alpha+\beta) \ L\text{-space} \qquad (*)$$

(OS, 2005)

(uses $\widehat{HF}$ surgery exact sequence of a triad)

(B) Sol manifolds

$N =$ twisted I-bundle/Klein bottle

N has two Seifert structures:

base Möbius band; fiber φ_0

base $D^2(2,2)$; fiber φ_1

$\varphi_0 \cdot \varphi_1 = 1$ on ∂N

$f : \partial N \to \partial N$ homeomorphism, $M = N \cup_f N$

Assume M a QHS $(f(\varphi_0) \neq \pm\varphi_0)$

M Seifert $\Leftrightarrow f(\varphi_i) = \pm\varphi_j$ (some $i, j \in \{0, 1\}$)

Otherwise, M is a Sol manifold

$\pi_1(M)$ is not LO (BRW, 2005)

Theorem

M is an L-space

$$f_* = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad (c \neq 0) \text{ with respect to basis } \varphi_0, \varphi_1$$

(1) True if $f_* = \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix}$

$f(\varphi_1) = \varphi_0$, so M Seifert

(2) True if $f_* = \begin{bmatrix} a & b \\ 1 & d \end{bmatrix} = \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix} (t_0)_*^d$

where $t_0 : \partial N \to \partial N$ is Dehn twist along φ_0

Write $W(f) = N \cup_f N$

Bordered $\widehat{HF}$ calculation shows $\widehat{HF}(W(f)) \cong \widehat{HF}(W(f \circ t_0))$

So reduced to case (1)

(3) In general, induct on $|c|$: do surgery on suitable simple closed curves $\subset \partial N$ and use $(*)$

(C) Dehn surgery

Theorem (OS, 2005)

K a hyperbolic alternating knot. Then $K(r)$ is not an L-space $\forall\ r \in Q$

Theorem (Roberts, 1995)

K an alternating knot.

(1) *If K is not special alternating then $K(r)$ has a taut foliation $\forall\ r \in Q$.*

(2) *If K is special alternating then $K(r)$ has a taut foliation either $\forall\ r > 0$ or $\forall\ r < 0$.*

$K(1/q)$ is a ZHS $\quad\therefore\quad$ foliation is co-orientable

$K(1/q)$ atoroidal $\quad\therefore\quad \pi_1(K(1/q)) \subset \mathrm{Homeo}_+(S^1)$

$H^2(\pi_1(K(1/q))) = 0$; so lifts to $\pi_1(K(1/q)) \subset \mathrm{Homeo}_+(\mathbb{R})$

$\therefore\quad \pi_1(K(1/q))$ is LO $\quad$ ($\forall\, q \neq 0$ in (1), $\forall\, q > 0$ or $\forall\, q < 0$ in (2))

Theorem

Let K be the figure eight knot. Then $\pi_1(K(r))$ is LO for $-4 < r < 4$.

Uses representations $\quad \rho : \pi_1(S^3 \setminus K) \to PSL_2(\mathbb{R})$

(Also true for $r = \pm 4$ $\quad$ (Clay-Lidman-Watson, 2011))

(D) 2-fold branched covers

L a link in S^3

$\Sigma(L)$ =2-fold branched cover of L

Theorem (OS, 2005)

If L is a non-split alternating link then $\Sigma(L)$ is an L-space.

(uses (*) ;

L L_0 L_∞

$\Longrightarrow$ $\Sigma(L),\ \Sigma(L_0),\ \Sigma(L_\infty)$ a surgery triad

with $\det L = \det L_0 + \det L_\infty$)

Theorem

If L is a non-split alternating link then $\pi_1(\Sigma(L))$ is not LO.

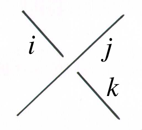
(Also proofs by Greene, Ito)

L a link in S^3, D a diagram of L

Define group $\pi(D)$:

generators $a_1, \ldots, a_n \longleftrightarrow$ arcs of D

relations $\longleftrightarrow$ crossings of D



$$a_j^{-1} a_i a_j^{-1} a_k$$

Theorem (Wada, 1992)

$$\pi(D) \cong \pi_1(\Sigma(L)) * \mathbb{Z}$$

$\therefore \quad \pi(D)$ LO $\iff \pi_1(\Sigma(L))$ LO (if $L \neq$ unknot)

Suppose $\pi(D)$ LO

$$a_j^{-1}a_ia_j^{-1}a_k = 1 \quad \Longleftrightarrow \quad a_j^{-1}a_i = a_k^{-1}a_j$$

$$a_i < a_j \quad \Longleftrightarrow \quad a_j^{-1}a_i < 1$$

$\therefore$ at each crossing either

$$a_i < a_j < a_k$$

$$\text{or} \quad a_i > a_j > a_k$$

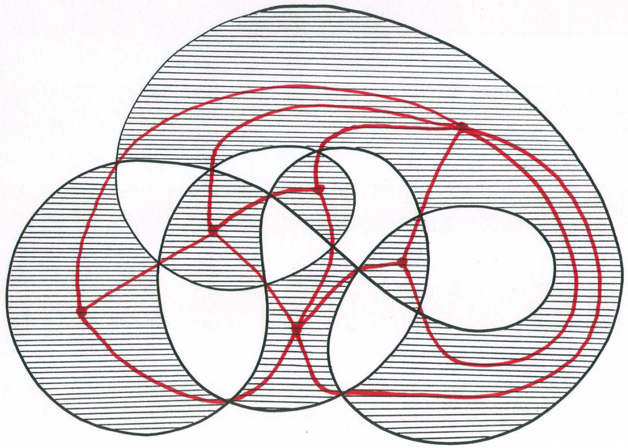
$$\text{or} \quad a_i = a_j = a_k$$

Shade complementary regions of D alternately Black/White

Define graph $\Gamma(D) \subset S^2$:

$$\text{vertices} \quad \longleftrightarrow \quad B\text{-regions}$$

$$\text{edges} \quad \longleftrightarrow \quad \text{crossings}$$



Assume D connected, alternating

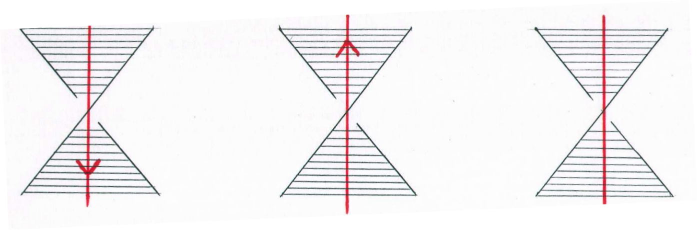
We want to show $\pi_1(\Sigma(L))$ not LO

True if $L =$ unknot; so assume $L \neq$ unknot

Then $\pi_1(\Sigma(L))$ LO $\Leftrightarrow$ $\pi(D)$ LO

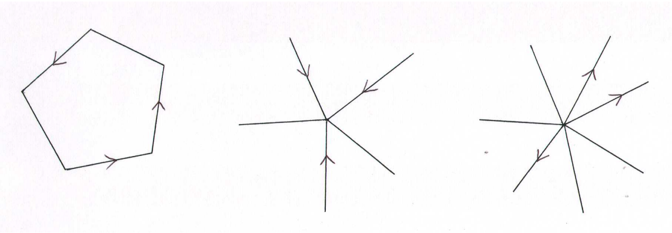
So assume $\pi(D)$ LO

Orient edges of $\Gamma(D)$



$a_i < a_j < a_k$ $a_i > a_j > a_k$ $a_i = a_j = a_k$

Γ a connected, **semi-oriented** graph $\subset S^2$



cycle **sink** **source**

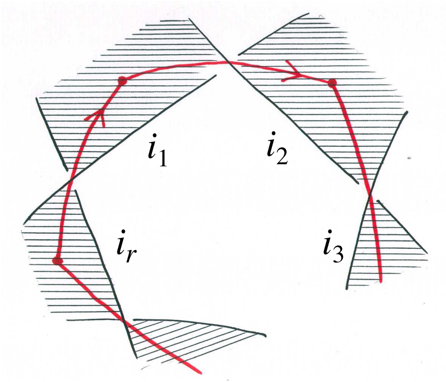
where, in each case, there is at least one oriented edge

Lemma

Let $\Gamma \subset S^2$ *be a connected semi-oriented graph with at least one oriented edge. Then* Γ *has a sink, source or cycle.*

Let $\quad \Gamma = \Gamma(D)$

cycle:

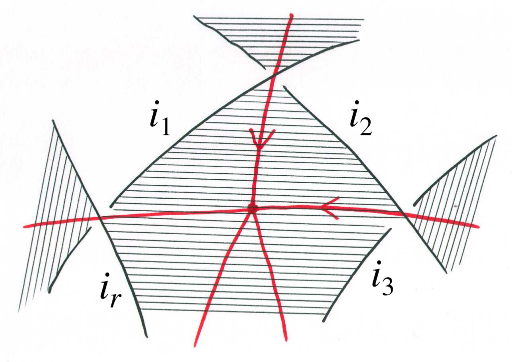


$$a_{i_1} \leq a_{i_2} \leq \cdots \leq a_{i_r} \leq a_{i_1}$$

$$\therefore \quad a_1 = a_2 = \cdots = a_r$$

a contradiction, since at least one oriented edge

sink:



$$a_{i_1} \leq a_{i_2} \leq \cdots \leq a_{i_r} \leq a_{i_1}$$

$$\therefore \quad a_1 = a_2 = \cdots = a_r$$

a contradiction, since at least one oriented edge

Similarly for a source

$$a_{i_1} \geq a_{i_2} \geq \cdots \geq a_{i_r} \geq a_{i_1} \ , \quad \text{contradiction}$$

$\therefore$ by Lemma, all edges of $\Gamma(D)$ are unoriented

$\therefore$ (since D connected) $a_1 = a_2 = \cdots = a_n$

$$\therefore \quad \pi(D) \cong \mathbb{Z}$$

$$\therefore \quad \pi_1(\Sigma(K)) = 1$$

$$\therefore \quad L = \text{ unknot, } \quad \text{contradiction}$$

L **quasi-alternating** $\implies$ $\Sigma(L)$ an L-space

Question

Does L quasi-alternating $\implies$ $\pi_1(\Sigma(L))$ not LO?

HAPPY BIRTHDAY, HYAM !