KNOTS

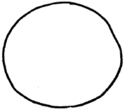
Cameron McA. Gordon

UT Math Club

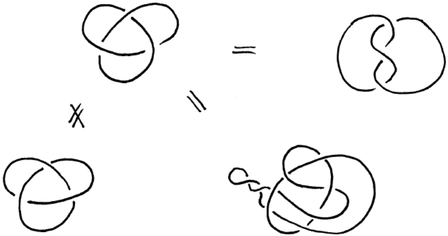
October 29, 2019

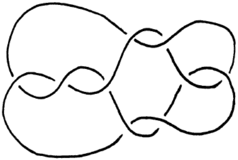
A knot is a closed loop in space.

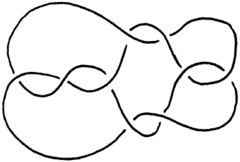
A knot is a closed loop in space.



unknot

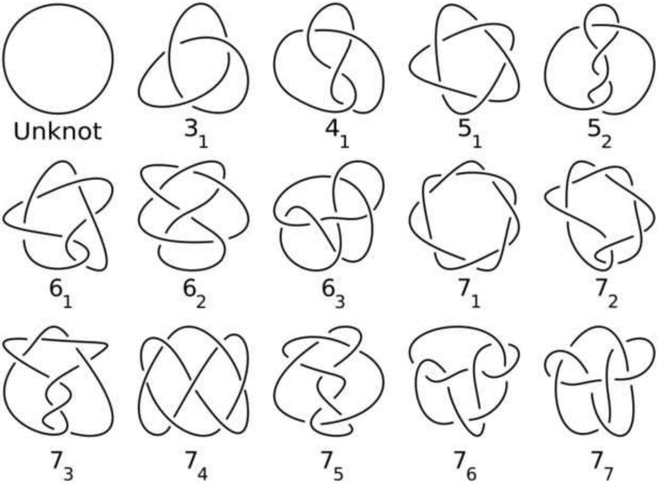


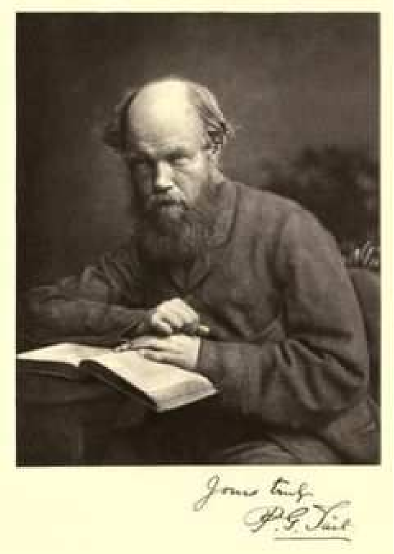




$c(K)$ = crossing number of K = minimum number of crossings in any diagram of K.

$c(K)$ = crossing number of K = minimum number of crossings in any diagram of K.





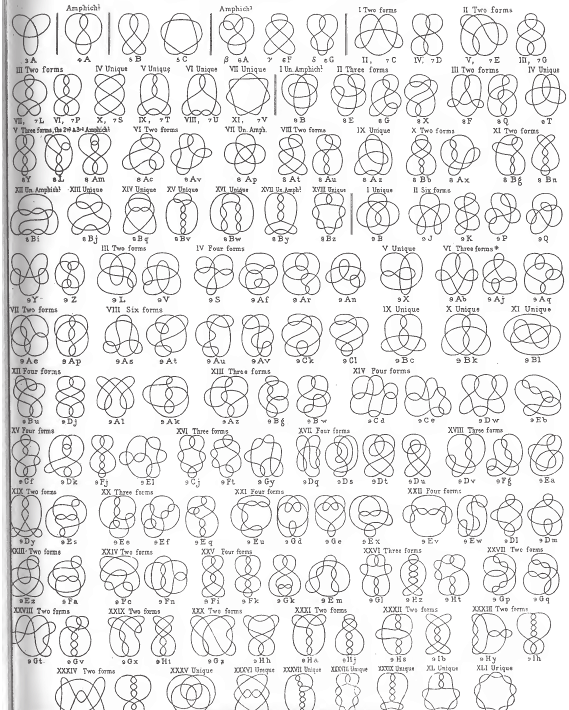
Peter Guthrie Tait (1831-1901)

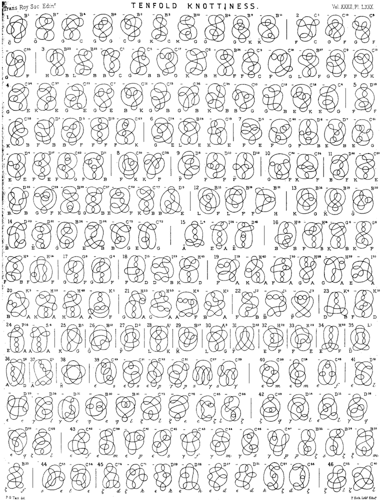
Vortex Atoms
(Lord Kelvin, 1867)

$c(K)$ = order of knottiness of K

THE FIRST SEVEN ORDERS OF KNOTTINESS.

Plate VI.




Trans Roy Soc Edin^r
TENFOLD KNOTTINESS.
Vol. XXXII, Pl. LXXX.
P G Tait del.
F Huth Lith^r Edin^r


Trans. Roy. Soc. Edin^r
TENFOLD KNOTTINESS.
Vol. XXXII, Pl. LXXXI.
P G Tait del.
F Huth Lith^r Edin^r

$c(K)$	# of knots
3	1
4	1
5	2
6	3
7	7
8	21
9	49
10	165

$c(K)$	# of knots
11	552
12	2, 176
13	9, 988
14	46, 972
15	253, 293
16	1, 388, 705
17	8, 053, 378

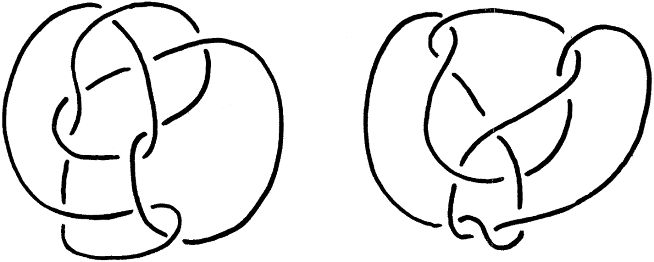
$c(K)$	# of knots
3	1
4	1
5	2
6	3
7	7
8	21
9	49
10	165

$c(K)$	# of knots
11	552
12	2, 176
13	9, 988
14	46, 972
15	253, 293
16	1, 388, 705
17	8, 053, 378

9, 755, 313 prime knots with $c(K) \leq 17$.

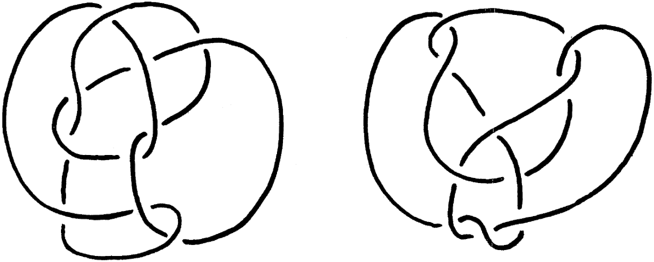
How can you prove that two knots are different?

How can you prove that two knots are different?



The Perko Pair

How can you prove that two knots are different?



The Perko Pair

How can you determine whether a given knot is the unknot or not?

There are many knot invariants that help with these questions.

There are many knot invariants that help with these questions.

Example. Alexander polynomial (1928)

There are many knot invariants that help with these questions.

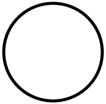
Example. Alexander polynomial (1928)

Can associate to K a polynomial $\Delta(t) = \Delta_K(t)$.

There are many knot invariants that help with these questions.

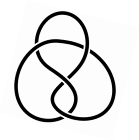
Example. Alexander polynomial (1928)

Can associate to K a polynomial $\Delta(t) = \Delta_K(t)$.



$\Delta(t) = 1$

$\Delta(t) = 1 - t + t^2$



$\Delta(t) = 1 - 3t + t^2$

James Waddell Alexander
(1888-1971)

If $\Delta_K(t) \neq \Delta_{K'}(t)$ then $K \neq K'$.

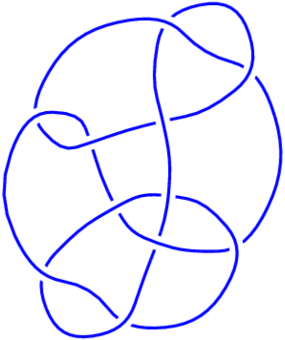
If $\Delta_K(t) \neq \Delta_{K'}(t)$ then $K \neq K'$.

But if $=$, can't conclude anything.

If $\Delta_K(t) \neq \Delta_{K'}(t)$ then $K \neq K'$.

But if $\quad = \quad$, can't conclude anything.

E.g.



$\Delta(t) = 1$
But $K \neq$ unknot

Is there an algorithm (systematic procedure, computer program, Turing machine, . . .) to decide whether or not any given knot is the unknot?

Is there an algorithm (systematic procedure, computer program, Turing machine, . . .) to decide whether or not any given knot is the unknot?

Fundamental dichotomy in mathematics:

solvable/decidable unsolvable/undecidable

Is there an algorithm (systematic procedure, computer program, Turing machine, . . .) to decide whether or not any given knot is the unknot?

Fundamental dichotomy in mathematics:

solvable/decidable unsolvable/undecidable

Example 1

Given words W_1, W_2 in A, B, can you get from W_1 to W_2 using the substitution rule $AB = BA$?

Is there an algorithm (systematic procedure, computer program, Turing machine, . . .) to decide whether or not any given knot is the unknot?

Fundamental dichotomy in mathematics:

solvable/decidable   unsolvable/undecidable

Example 1

Given words W_1, W_2 in A, B, can you get from W_1 to W_2 using the substitution rule $AB = BA$?

E.g. $ABAAB \longleftrightarrow AABAB \longleftrightarrow AABBA$, etc.

Is there an algorithm (systematic procedure, computer program, Turing machine, . . .) to decide whether or not any given knot is the unknot?

Fundamental dichotomy in mathematics:

solvable/decidable unsolvable/undecidable

Example 1

Given words W_1, W_2 in A, B, can you get from W_1 to W_2 using the substitution rule $AB = BA$?

E.g. $ABAAB \longleftrightarrow AABAB \longleftrightarrow AABBA$, etc.

There is an algorithm to decide:

Is there an algorithm (systematic procedure, computer program, Turing machine, ...) to decide whether or not any given knot is the unknot?

Fundamental dichotomy in mathematics:

solvable/decidable unsolvable/undecidable

Example 1

Given words W_1, W_2 in A, B, can you get from W_1 to W_2 using the substitution rule $AB = BA$?

E.g. $ABAAB \longleftrightarrow AABAB \longleftrightarrow AABBA$, etc.

There is an algorithm to decide:

Yes iff W_1 and W_2 each have the same number of A's and same number of B's.

Example 2

Same, but with words in A, B, C, D, E, and substitution rules

$$AB = BA, \quad AD = DA, \quad CB = BC, \quad CD = DC$$

$$DAE = ED, \quad BCE = EB, \quad DBAD = EDBD$$

Example 2

Same, but with words in A, B, C, D, E, and substitution rules

$$AB = BA, \quad AD = DA, \quad CB = BC, \quad CD = DC$$

$$DAE = ED, \quad BCE = EB, \quad DBAD = EDBD$$

E.g $\qquad DABCDADE \longleftrightarrow EDBDCED$

Example 2

Same, but with words in A, B, C, D, E, and substitution rules

$$AB = BA, \quad AD = DA, \quad CB = BC, \quad CD = DC$$

$$DAE = ED, \quad BCE = EB, \quad DBAD = EDBD$$

E.g

$$DABCDADE \longleftrightarrow EDBDCED$$

$$(DABCDADE \longleftrightarrow DBACDADE \longleftrightarrow DBADCADE$$

$$EDBDCADE \longleftrightarrow EDBDCDAE \longleftrightarrow EDBDCED)$$

Example 2

Same, but with words in A, B, C, D, E, and substitution rules

$$AB = BA, \quad AD = DA, \quad CB = BC, \quad CD = DC$$

$$DAE = ED, \quad BCE = EB, \quad DBAD = EDBD$$

E.g

$$DABCDADE \longleftrightarrow EDBDCED$$

$$(DABCDADE \longleftrightarrow DBACDADE \longleftrightarrow DBADCADE$$

$$EDBDCADE \longleftrightarrow EDBDCDAE \longleftrightarrow EDBDCED)$$

$$DABABCD \not\longleftrightarrow ABCDED$$

Example 2

Same, but with words in A, B, C, D, E, and substitution rules

$$AB = BA, \quad AD = DA, \quad CB = BC, \quad CD = DC$$

$$DAE = ED, \quad BCE = EB, \quad DBAD = EDBD$$

E.g

$$DABCDADE \longleftrightarrow EDBDCED$$

$$(DABCDADE \longleftrightarrow DBACDADE \longleftrightarrow DBADCADE$$

$$EDBDCADE \longleftrightarrow EDBDCDAE \longleftrightarrow EDBDCED)$$

$$DABABCD \not\longleftrightarrow ABCDED$$

There is no algorithm to decide.

"... A similar problem which might well be unsolvable is the one concerning knots ..."

(Turing, 1954)

Alan Turing (1912-1954)

Wolfgang Haken (1928-)

There is an algorithm to decide whether or not a given knot is the unknot.

(Haken, 1957)

Used 3-dimensional topology.

There are now other proofs.

Wolfgang Haken (1928-)

There is an algorithm to decide whether or not a given knot is the unknot.

(Haken, 1957)

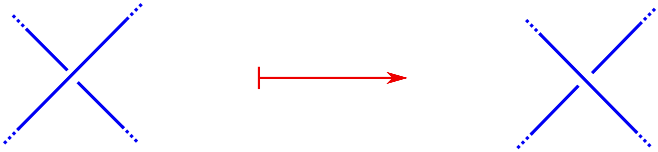
Used 3-dimensional topology.

There are now other proofs.

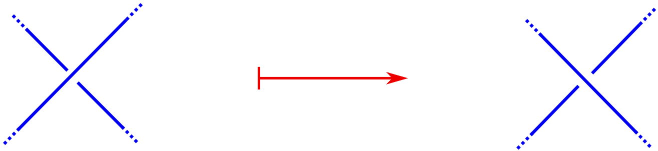
There is an algorithm to decide whether or not two given knots are the same.

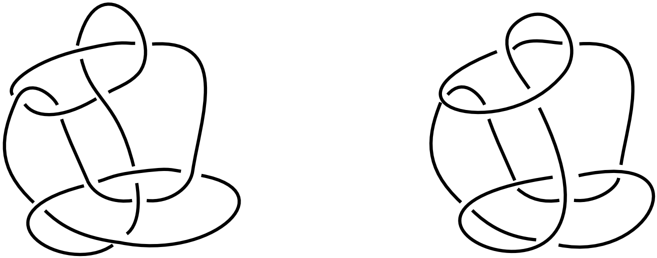
Every knot can be unknotted if it is allowed to pass through itself

Every knot can be unknotted if it is allowed to pass through itself



Every knot can be unknotted if it is allowed to pass through itself





The unknotting number of K, $u(K)$, is the minimum number of such pass moves needed to unknot K.

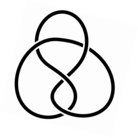
The unknotting number of K, $u(K)$, is the minimum number of such pass moves needed to unknot K.

"In what follows the term Beknottedness will be used to signify the peculiar property in which knots, even when of the same order of knottiness, may thus differ: and we may define it, at least provisionally, as the smallest number of changes of sign which will render all the crossings in a given scheme nugatory. The question is, as we shall soon see, a delicate and difficult one."
(Tait, 1877)

$u(K) = 0$ iff K is the unknot

$u(K) = 0$ iff K is the unknot

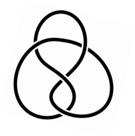
$u(K) = 1$



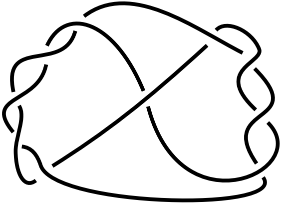
$u(K) = 1$

$u(K) = 0$ iff K is the unknot

$u(K) = 1$



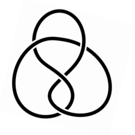
$u(K) = 1$



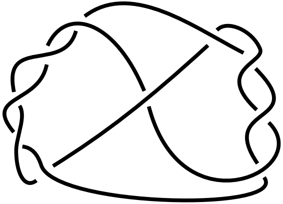
$u(K) = 2$

$u(K) = 0$ iff K is the unknot

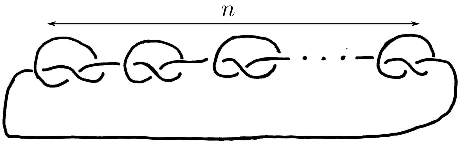
$u(K) = 1$



$u(K) = 1$



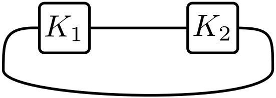
$u(K) = 2$



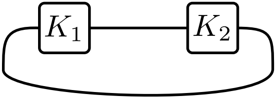
$u(K) = n$

Sum of knots:

$$K_1 + K_2 =$$



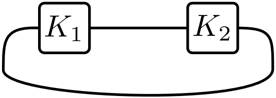
Sum of knots:

$$K_1 + K_2 =$$ 

Is $u(K_1 + K_2) = u(K_1) + u(K_2)$?

(Certainly $\leq$)

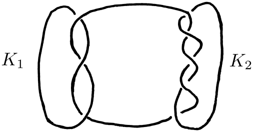
Sum of knots:

$K_1 + K_2 =$ 

Is $u(K_1 + K_2) = u(K_1) + u(K_2)$?

(Certainly $\leq$)

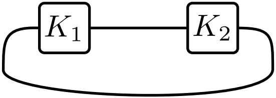
Example:



$K = K_1 + K_2$

Sum of knots:

$$K_1 + K_2 =$$

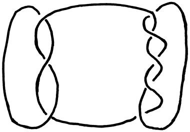


Is $u(K_1 + K_2) = u(K_1) + u(K_2)$?

(Certainly $\leq$)

Example:

K_1



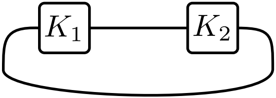
K_2

$K = K_1 + K_2$

$$u(K_1) = 1, u(K_2) = 2, \text{ so } u(K) \leq 3$$

Sum of knots:

$$K_1 + K_2 =$$

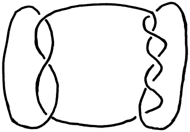


Is $u(K_1 + K_2) = u(K_1) + u(K_2)$?

(Certainly $\leq$)

Example:

K_1



K_2

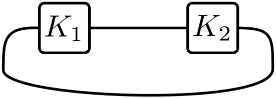
$K = K_1 + K_2$

$$u(K_1) = 1, u(K_2) = 2, \text{ so } u(K) \leq 3$$

Also, $\quad u(K) > 1$ (hard!)

Sum of knots:

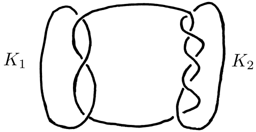
$$K_1 + K_2 =$$



Is $u(K_1 + K_2) = u(K_1) + u(K_2)$?

(Certainly $\leq$)

Example:



$K = K_1 + K_2$

$$u(K_1) = 1, u(K_2) = 2, \text{ so } u(K) \leq 3$$

Also, $u(K) > 1$ (hard!)

Is $u(K) = 2$ or 3?

Is there an algorithm to compute $u(K)$?

Is there an algorithm to compute $u(K)$?

Is there an algorithm to decide whether or not $u(K) = 1$?

Is there an algorithm to compute $u(K)$?

Is there an algorithm to decide whether or not $u(K) = 1$?

(Haken's theorem says there is an algorithm to decide whether or not $u(K) = 0$!)

Is there an algorithm to compute $u(K)$?

Is there an algorithm to decide whether or not $u(K) = 1$?

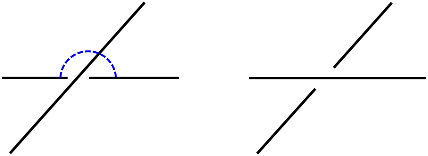
(Haken's theorem says there is an algorithm to decide whether or not $u(K) = 0$!)

Is $c(K_1 + K_2) = c(K_1) + c(K_2)$?

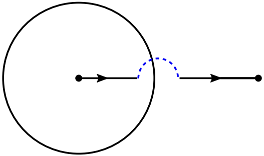
(Certainly $\leq$)

Felix Klein (1849-1925)

Every knot can be unknotted in 4 dimensions.



Analog in 3 dimensions.



So $u(K) =$ minimum # of “jumps” into the 4th dimension needed to unknot K.

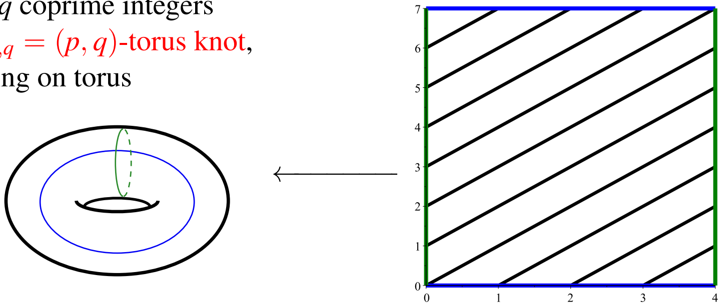
So $u(K) =$ minimum # of “jumps” into the 4th dimension needed to unknot K.

Consequently, 4-dimensional topological methods give information about $u(K)$.

So $u(K)$ = minimum # of "jumps" into the 4th dimension needed to unknot K.

Consequently, 4-dimensional topological methods give information about $u(K)$.

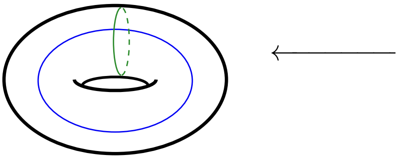
p, q coprime integers
$T_{p,q} = (p, q)$-torus knot,
lying on torus

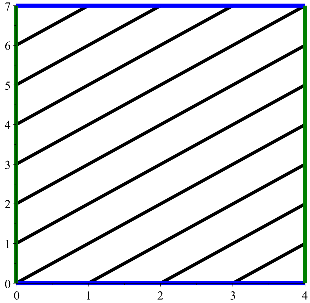


So $u(K)$ = minimum # of "jumps" into the 4th dimension needed to unknot K.

Consequently, 4-dimensional topological methods give information about $u(K)$.

p, q coprime integers
$T_{p,q} = (p, q)$-torus knot,
lying on torus

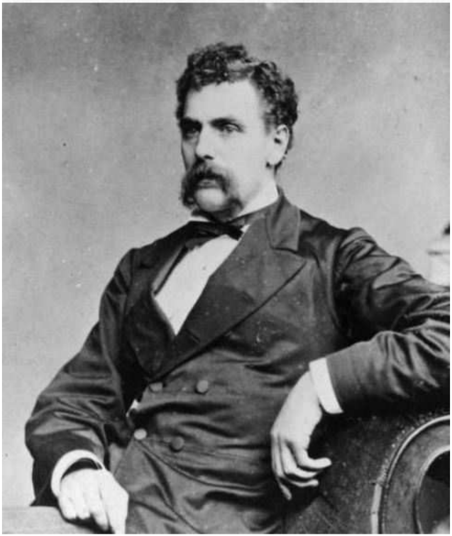




$$u(T_{p,q}) = \frac{(p-1)(q-1)}{2}$$ (Milnor Conjecture; proved in 1993)

My soul's an amphicheiral knot
Upon a liquid vortex wrought
By Intellect in the Unseen residing,
While thou dost like a convict sit
With marlinspike untwisting it
Only to find my knottiness abiding,
Since all the tools for my untying
In four-dimensioned space are lying,
Where playful fancy intersperces,
Whole avenues of universes;
Where Klein and Clifford fill the void
With one unbounded, finite homaloid,
Whereby the Infinite is hopelessly
destroyed.

James Clerk Maxwell
(1831-1879)



Henry Slade (1835-1905)

Arthur Conan Doyle
1918.

TRANSCENDENTAL PHYSICS.

An Account of Experimental Investigations.

From the Scientific Treatises

Shows once for all that Slade was a true medium.

OF

JOHANN CARL FRIEDRICH ZÖLLNER,

Professor of Physical Astronomy at the University of Leipsic;
Member of the Royal Saxon Society of Sciences;
Foreign Member of the Royal Astronomical Society of London;
of the Imperial Academy of Natural Philosophers at Moscow;
Honorary Member of the Physical Association at Frankfort-on-the-Main;
of the "Scientific Society of Psychological Studies," at Paris;
and of the "British National Association of Spiritualists," at London.

Translated from the German, with a Preface and Appendices, by

CHARLES CARLETON MASSEY,

OF LINCOLN'S INN, BARRISTER-AT-LAW.

SECOND EDITION.

LONDON:
W. H. HARRISON, 33 MUSEUM STREET, W.C.
1882.

Johann Zöllner (1834-1882)

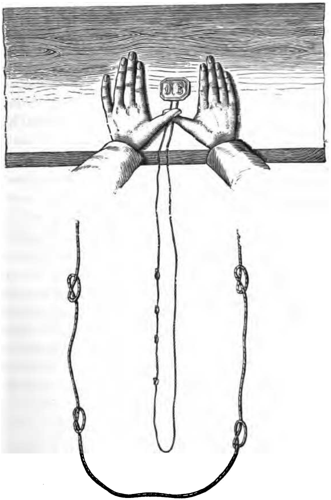


PLATE I.

"If a single cord has its ends tied together and sealed, an intelligent being, having the power voluntarily to produce on this cord four-dimensional bendings and movements, must be able, *without* loosening the seal, to tie one or more knots in this endless cord."

(Zöllner, 1879)

G Grey

THE

UNSEEN UNIVERSE

OR

PHYSICAL SPECULATIONS

ON A

FUTURE STATE

BY

B. STEWART, AND P. G. TAIT.

——the things which are seen are temporal, but the things which are not seen are eternal.

FIFTH EDITION.
(*Revised and Enlarged.*)

London
MACMILLAN AND CO.
1876.

“There must be some very simple method of determining the amount of beknottedness for any given knot; but I have not hit upon it.”

(Tait, 1877)