L-spaces and left-orderability

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(joint with Steve Boyer and Liam Watson)

Alexandroff Readings

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Left Orderability

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- G, H LO $\iff G * H$ LO (Vinogradov, 1949)
- G (countable) LO $\iff \exists$ embedding $G \subset \mathrm{Homeo}_+(\mathbb{R})$

Theorem

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Proof.

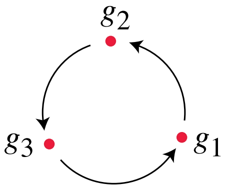
Let $x_1, x_2, \ldots$ be a dense subset of $\mathbb{R}$

$$g, h \in \mathrm{Homeo}_+(\mathbb{R})\ , \qquad g \neq h$$

Let $n =$ least index such that $g(x_n) \neq h(x_n)$.

Define $g < h \quad \iff \quad g(x_n) < h(x_n)$ □

G is left circularly orderable (LCO) if $\exists$ strict circular order on G, $T \subset G^3$, such that $(g_1, g_2, g_3) \in T \Rightarrow (fg_1, fg_2, fg_3) \in T \; \forall f \in G$



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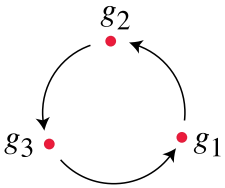

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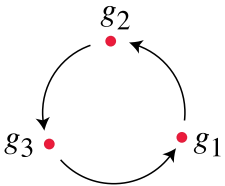

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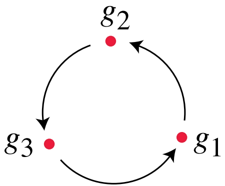

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- G LO $\Rightarrow$ G LCO

Theorem (Boyer-Rolfsen-Wiest, 2005)

M a compact, orientable, prime 3-manifold (poss. with boundary). Then $\pi_1(M)$ is LO $\Leftrightarrow$ $\pi_1(M)$ has an LO quotient.

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So interesting case is when

$$H_*(M;\mathbb{Q}) \cong H_*(S^3;\mathbb{Q})$$

M is a $\mathbb{Q}$-homology 3-sphere ($\mathbb{Q}$HS)

Foliations

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$\exists$ arithmetic characterization of those $S^2(a_1, \ldots, a_n)$'s that admit $\left\{ \begin{array}{c} \text{horizontal foliations} \\ \equiv \text{ CTF's} \end{array} \right\}$ (Jankins-Neumann, 1985, Naimi, 1994).

Theorem (Calegari-Dunfield, 2003)

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Restriction of ρ to $\pi_1(\widetilde{M})$ lifts to $\widetilde{\mathrm{Homeo}_+}(S^1) \subset \mathrm{Homeo}_+(\mathbb{R})$

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Conjecture

M a prime $\mathbb{Q}$HS. Then

$$M \text{ is an } L\text{-space} \iff \pi_1(M) \text{ is not LO}$$

E.g.

$$\begin{tikzcd}[column sep=small] & \pi_1(M) \text{ finite} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\ \pi_1(M) \text{ not LO} & & M \text{ an } L\text{-space} \end{tikzcd}$$

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Remark

By BRW and Agol (2012): M prime, $\pi_1(M)$ infinite $\Longrightarrow$ $\pi_1(M)$ virtually bi-orderable.

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Example (Calegari)

M Seifert fibered $\mathbb{Q}$HS $\quad S^2(p_1/q_1,\, p_2/q_2,\, p_3/q_3), \quad \sum 1/q_i < 1$

$\exists$ central extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(M) \longrightarrow \Delta \longrightarrow 1$$

$$\Delta = (q_1, q_2, q_3)\text{-triangle group}$$

$$\Delta \subset \mathrm{Isom}_+(\mathbb{H}^2) \cong PSL_2(\mathbb{R}) \subset \mathrm{Homeo}_+(S^1)$$

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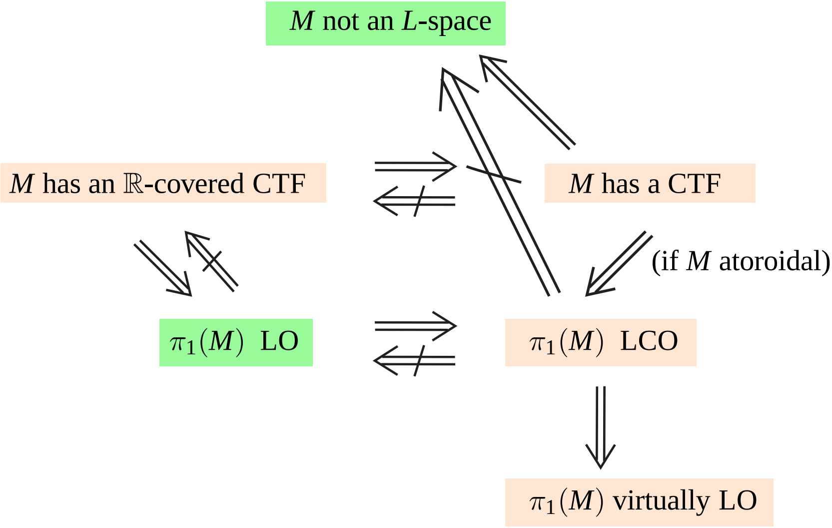
But $\pi_1(M)$ LO $\iff$ M admits a horizontal foliation, and this doesn't always hold.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(A) Seifert manifolds

Theorem

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The Conjecture is true if M is Seifert fibered.

Base orbifold is either

$S^2(a_1, \ldots, a_n)$:

M admits a horizontal foliation

(Lisca-Stipsicz, 2007) $\Longleftrightarrow$ $\Longleftrightarrow$ (BRW, 2005)

M not an L-space $\qquad\qquad$ $\pi_1(M)$ LO

$P^2(a_1, \ldots, a_n)$: $\pi_1(M)$ not LO (BRW, 2005)

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Show M an L-space by induction on n; using (OS, 2005):

X compact, orientable 3-manifold, ∂X a torus;

$\alpha, \beta \subset \partial X$, $\alpha \cdot \beta = 1$, and

$$|H_1\big(X(\alpha + \beta)\big)| = |H_1\big(X(\alpha)\big)| + |H_1\big(X(\beta)\big)|$$

Then $X(\alpha), X(\beta)$ L-spaces $\Rightarrow X(\alpha + \beta)$ L-space $(*)$

(uses $\widehat{HF}$ surgery exact sequence of a triad)

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So Conjecture true for $\mathbb{Z}$HS graph manifolds

(C) Sol manifolds

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N = twisted I-bundle/Klein bottle

N has two Seifert structures:

base Möbius band; fiber φ_0

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$\pi_1(M(f))$ not LO (BRW, 2005)

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where $t_0 : \partial N \longrightarrow \partial N$ is Dehn twist along φ_0

Bordered $\widehat{HF}$ calculation shows $\widehat{HF}(M(f)) \cong \widehat{HF}(M(f \circ t_0))$

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Hence: Conjecture is true for all non-hyperbolic geometric manifolds.

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Theorem (Roberts, 1995)

K an alternating knot. Then $K(r)$ has a CTF $\forall r \in \mathbb{Q}$ if K is not special, and either $\forall r > 0$ or $\forall r < 0$ if K is special.

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So Conjecture $\implies$ $\pi_1(K(r))$ LO

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K an alternating knot. Then $K(r)$ has a CTF $\forall r \in \mathbb{Q}$ if K is not special, and either $\forall r > 0$ or $\forall r < 0$ if K is special.

Corollary

The Conjecture is true for $K(1/q)$, K alternating, either $\forall q \in \mathbb{Q}$, or $\forall q > 0$ or $\forall q < 0$.

$$K(1/q) \text{ atoroidal} \qquad \therefore \quad \pi_1(K(1/q)) \subset \mathrm{Homeo}_+(S^1)$$

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Also true for $r \in \mathbb{Z}$: $K(r)$ has an $\mathbb{R}$-covered CTF (Fenley)

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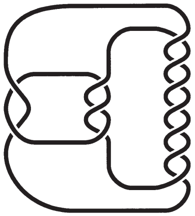
$K(17)$ has finite π_1
$K(18)$, $K(19)$ lens spaces



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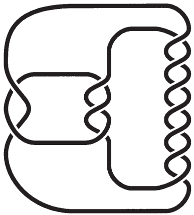

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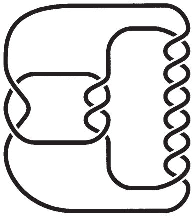

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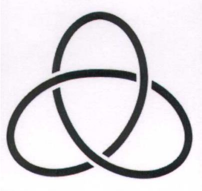

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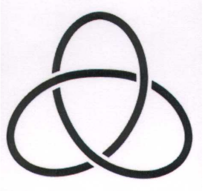

Question

For an arbitrary L-space knot K, what happens for $r \in [1, 2g(K) - 1)$?

(E) 2-fold branched covers

L a link in S^3

$\Sigma(L)$ =2-fold branched cover of L

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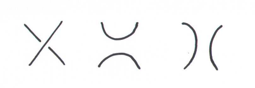

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$\Longrightarrow \Sigma(L),\ \Sigma(L_0),\ \Sigma(L_\infty)$ a surgery triad

with $\det L = \det L_0 + \det L_\infty$)

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(Also proofs by Greene, Ito)

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Define group $\pi(D)$:

$$\text{generators } a_1, \ldots, a_n \longleftrightarrow \text{ arcs of } D$$

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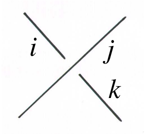

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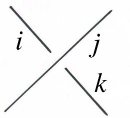

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Theorem (Wada, 1992)

$$\pi(D) \cong \pi_1(\Sigma(L)) * \mathbb{Z}$$

Suppose $\pi(D)$ LO

$$\begin{aligned} a_j^{-1}a_ia_j^{-1}a_k = 1 \quad &\Longleftrightarrow \quad a_j^{-1}a_i = a_k^{-1}a_j \\ a_i < a_j \quad &\Longleftrightarrow \quad a_j^{-1}a_i < 1 \end{aligned}$$

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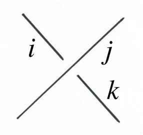

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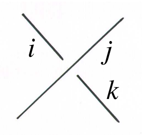

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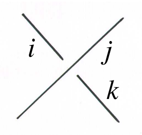

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True if $L =$ unknot; so assume $L \neq$ unknot

Then $\pi_1(\Sigma(L))$ LO $\iff$ $\pi(D) \cong \pi_1(\Sigma(L)) * \mathbb{Z}$ LO

So assume $\pi(D)$ LO.   Let $a = \min\{a_1, a_2, \ldots, a_n\}$.

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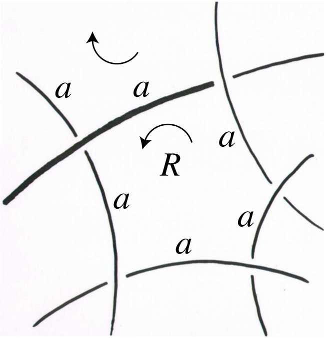
If an arc of D adjacent to a region R has value a then all arcs adjacent to R have value a.



$\therefore \quad a_1 = a_2 = \cdots = a_n$

$\therefore \quad \pi(D) \cong \mathbb{Z}$

$\therefore \quad \pi_1(\Sigma(L)) = 1$

$\therefore \quad L =$ unknot; contradiction

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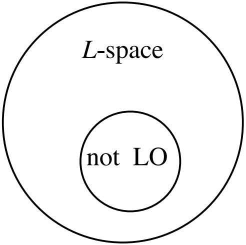

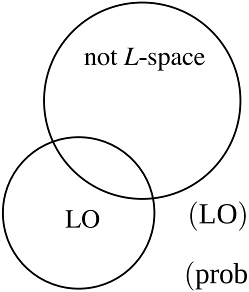

$$(\text{LO}) \cap (L\text{-space}) = \emptyset$$

$$(\text{probability} \sim 10^{-29}!)$$

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Conjecture $\implies$ Q's 1, 2 and 3 have answer "yes"

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If M is a $\mathbb{Q}$HS with $\pi_1(M)$ *LO* does M admit a CTF? (Maybe “no”?)

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Question 6

M a hyperbolic $\mathbb{Z}$HS. Is $\pi_1(M)$ LO?