Seifert Fibered Dehn Filling

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December 2012

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Example. $M = S^3 - \overset{\circ}{N}$ (figure eight knot)

$M(1/0) = S^3$

$M(0) = T^2$-bundle over S^1

$M(\pm 1), M(\pm 2), M(\pm 3)$ small Seifert; orbifold $S^2(a, b, c)$, $a, b, c > 1$

$M(\pm 4)$ toroidal

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For all M, $\#$ exceptional slopes for $M \leq 10$, and α, β exceptional $\implies \Delta(\alpha, \beta) \leq 8$.

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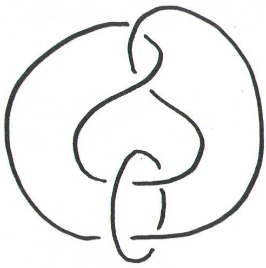
Least well understood: small Seifert

Wh = Whitehead link exterior

$Wh(1) = M_1 =$ figure eight exterior

$Wh(-5) = M_2 =$ figure eight sister

$Wh(2) = M_3$, $Wh(-5/2) = M_4$



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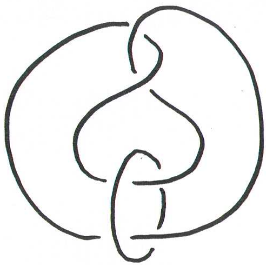

M_1, M_2, M_3, M_4 have pairs of toroidal fillings $M_i(\alpha_i)$, $M_i(\beta_i)$, $1 \leq i \leq 4$, with $\Delta(\alpha_i, \beta_i) = 8, 8, 7, 6$, respectively (Hodgson-Weeks)

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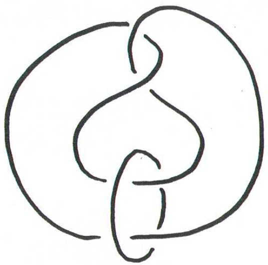

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Conjecture 1

M_1, M_2, M_3, M_4 are the only hyperbolic 3-manifolds with exceptional slopes α, β where $\Delta(\alpha, \beta) \geq 6$.

(1) Known to be true unless $M(\alpha)$ is small Seifert

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Remains to do: $M(\alpha)$ small Seifert and $M(\beta)$

- reducible (almost done, Boyer-Culler-Shalen-Zhang, 2008)
- toroidal
- Seifert

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Consider case $M(\alpha)$ small Seifert, $M(\beta)$ toroidal

Conjecture 1 becomes

Conjecture 2

M hyperbolic with slopes α, β *such that* $M(\alpha)$ *is small Seifert,* $M(\beta)$ *is toroidal, and* $\Delta(\alpha, \beta) \geq 6$. *Then M is the figure eight knot exterior.*

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$T \subset M(\beta)$ incompressible torus with $m = |T \cap \partial M|$ minimal $(m \geq 1)$

$T \cap M = F =$ essential m-punctured torus $\subset M$ with ∂-slope β

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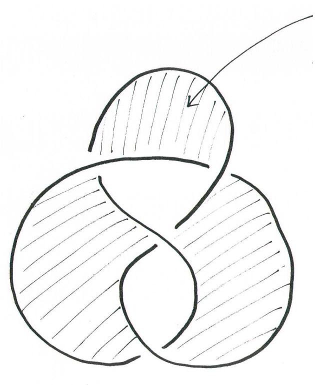
Theorem (Boyer-G-Zhang)

Conjecture 2 is true unless $m \geq 3$ and M is an F-bundle or F-semi-bundle.



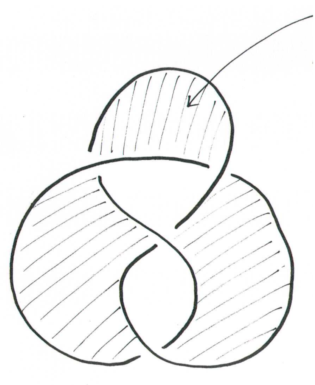
once-punctured Klein bottle

$\subset M_1$, ∂-slope 4

$\longmapsto$ Klein bottle $B \subset M_1(4)$

torus $T = \partial N(B) \subset M_1(4)$

$m = 2$



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$$M_1(4) = N(B) \cup_T D^2(2,3)$$

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$m = 1$ or 2: gives a lot of topological information

e.g. if T is separating (so $m = 2$), get

$$M(\beta) = D^2(p_1, q_1) \cup_T D^2(p_2, q_2)$$

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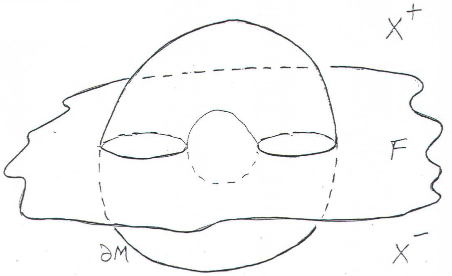

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If $\quad S^+$ incompressible in X^+

then $\quad S^+$ incompressible in M

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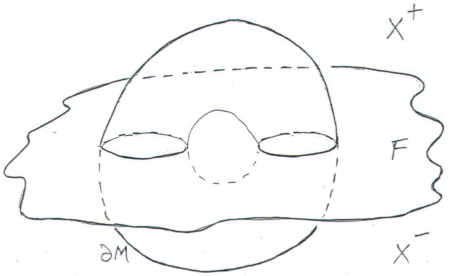

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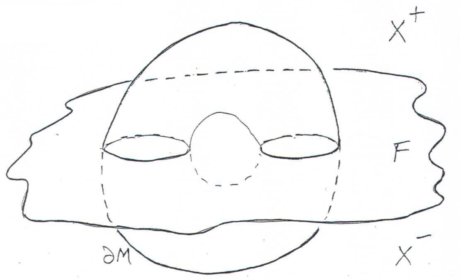

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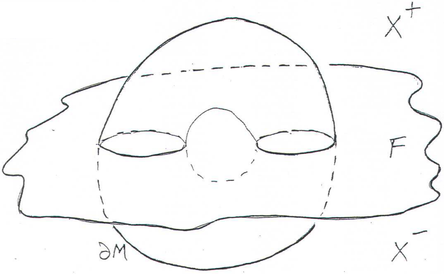

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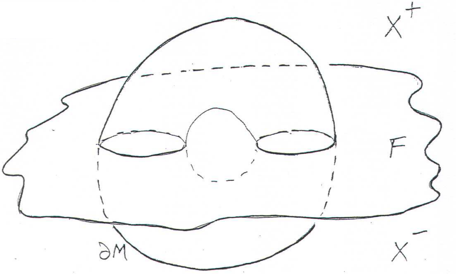

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$\therefore\ X^+$ is a genus 2 handlebody

Similarly X^- is a genus 2 handlebody

$\therefore\ \exists$ involution $\tau : M \to M$, such that $\tau \mid \partial M$ is the elliptic involution

τ extends to

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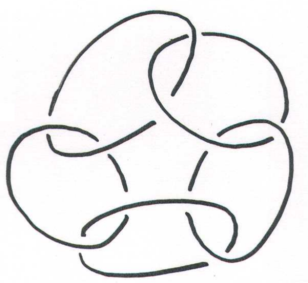

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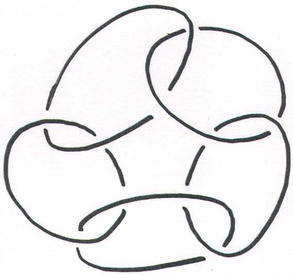

Roukema (2011) $\Longrightarrow$

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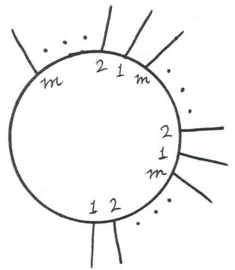
(3) γ a component of $f^{-1}(F) \Longrightarrow f \mid \; : (\gamma, \partial\gamma) \to (F, \partial F)$ essential

Get graph $\Gamma \subset T^2$

vertices of $\Gamma \longleftrightarrow$ components of $f^{-1}(V_\alpha)$

edges of $\Gamma \longleftrightarrow$ arc components of $f^{-1}(F)$



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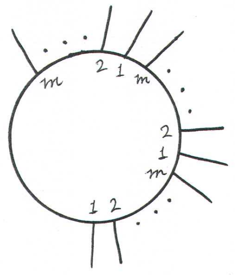

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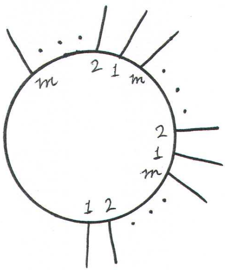
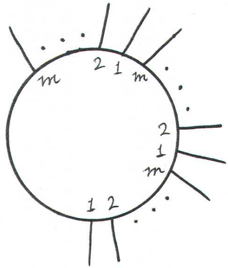


$|\partial F| = m$; so $|\partial F \cap$ meridian of $V_\alpha| = m\Delta(\alpha, \beta)$

$\therefore$ each vertex of Γ has valency $m\Delta$



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Number components of ∂F $1, 2, \ldots, m$ in order around ∂M

Label endpoints of edges of Γ = points of $\partial F \cap f(T^2)$ with corresponding component of ∂F

Assume F separates M : $\quad M = X^+ \cup_F X^-$

f(faces of Γ) lie alternately in $X^\pm$

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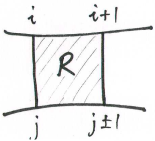
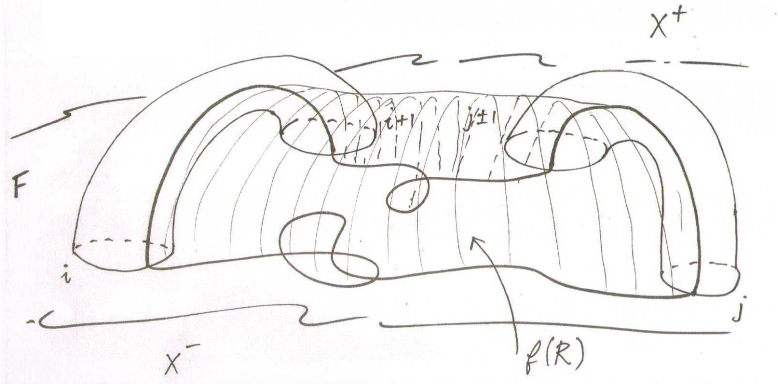


R bigon face of Γ



$f \mid R$ gives essential homotopy

$$H : (\bigcirc\!\!-\!\!\bigcirc) \times (I, \partial I) \to (X^{\varepsilon}, F) \qquad (\varepsilon = \pm)$$

$$H_0, H_1 \ \text{ not homotopic into } \ \partial F$$

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Jaco-Shalen-Johannson: $\exists$ characteristic I-bundle

$(\Sigma^{\varepsilon}, \Phi^{\varepsilon}) \subset (X^{\varepsilon}, F)$ such that

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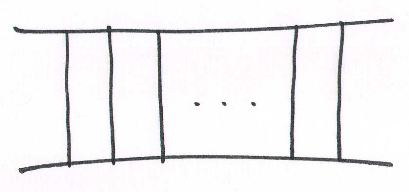
Jaco-Shalen-Johannson: $\exists$ characteristic I-bundle $(\Sigma^\varepsilon, \Phi^\varepsilon) \subset (X^\varepsilon, F)$ such that

(1) $(\Sigma^\varepsilon, \Phi^\varepsilon)$ is an $(I, \partial I)$-bundle

(2) any essential homotopy H as above is homotopic into $(\Sigma^\varepsilon, \Phi^\varepsilon)$

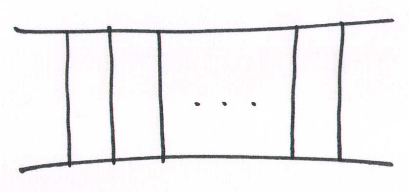
(3) $(\Sigma^\varepsilon, \Phi^\varepsilon)$ is minimal w.r.t. (2).



$n + 1$ parallel edges in Γ

gives essential homotopy of length n

$$H : (\bigcirc\!\!-\!\!\bigcirc) \times \Big(I, \{i/n : 0 \le i \le n\}\Big) \to (M, F)$$



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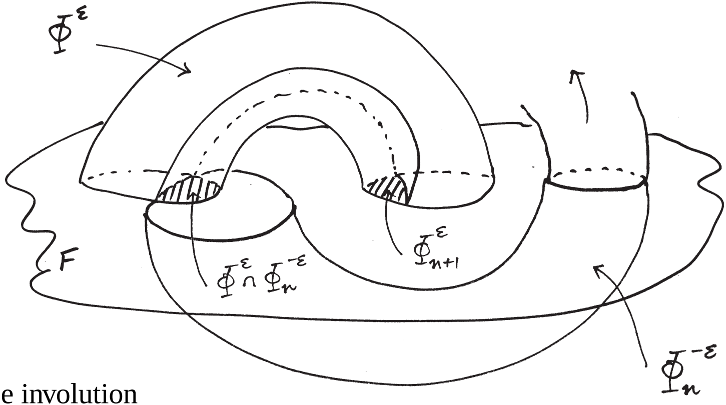
Get surfaces in F $\Phi^\varepsilon = \Phi_1^\varepsilon \supset \Phi_2^\varepsilon \supset \Phi_3^\varepsilon \supset \cdots$

minimal w.r.t. property

$$\left.\begin{array}{c} H \text{ essential homotopy of} \\ \text{length } n \text{ starting in } X^\varepsilon \end{array}\right\} \Longrightarrow H_0 \simeq \text{ into } \Phi_n^\varepsilon$$



Free involution

$\tau_\varepsilon : \Phi^\varepsilon \to \Phi^\varepsilon$

$$\Phi^\varepsilon_{n+1} = \tau_\varepsilon(\Phi^\varepsilon \cap \Phi^{-\varepsilon}_n)$$

Proposition (BCSZ)

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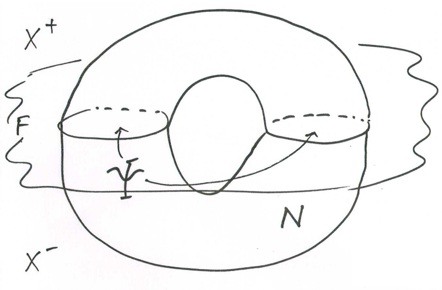
Then get $N \subset M$, $\partial N = \coprod$ tori, $N \cap F = \Psi$,



$(N \cap X^{\varepsilon}, \Psi)$ an

I-bundle, $\varepsilon = \pm$

M hyperbolic

$\Longrightarrow N = M$

$\therefore\ (X^{\varepsilon}, F)$ I-bundle, $\varepsilon = \pm$

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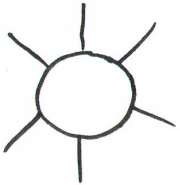
$\overline{\Gamma}$ reduced graph : amalgamate parallel edges



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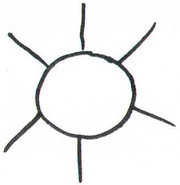

$\therefore\ \Delta \geq 6 \Longrightarrow \exists$ family of $\geq m$ parallel edges in Γ

$M(\beta) = \widehat{X}^{+} \cup_T \widehat{X}^{-}$; analysis of Φ_k^{ε} for small k gives information about $\widehat{X}^{\pm}$; e.g. consider

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Faces of Γ give (singular) disks in $X^\pm \longmapsto$ topological information about $\widehat{X}^\pm$

$\cdots\cdot \rightsquigarrow$ eventually get contradiction to $\Delta \geq 6$ if $m \geq 3$.

To complete proof of Conjecture 1, need to show:

if $M(\alpha)$ is small Seifert then

(1) if M is a bundle or semi-bundle and $M(\beta)$ is reducible or toroidal then $\Delta(\alpha, \beta) \leq 5$;

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(2) if $M(\beta)$ is Seifert then

either $\Delta(\alpha, \beta) \leq 5$

or M is the figure eight exterior