L-spaces and left-orderability

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(joint with Steve Boyer and Liam Watson)

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- G, H LO $\iff G * H$ LO (Vinogradov, 1949)
- G (countable) LO $\iff \exists$ embedding $G \subset \mathrm{Homeo}_+(\mathbb{R})$

Theorem (Boyer-Rolfsen-Wiest, 2005)

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So interesting case is when

$$H_*(M;\mathbb{Q}) \cong H_*(S^3;\mathbb{Q})$$

M is a $\mathbb{Q}$-homology 3-sphere ($\mathbb{Q}$HS)

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Restriction of ρ to $\pi_1(\widetilde{M})$ lifts to $\widetilde{\text{Homeo}_+}(S^1) \subset \text{Homeo}_+(\mathbb{R})$

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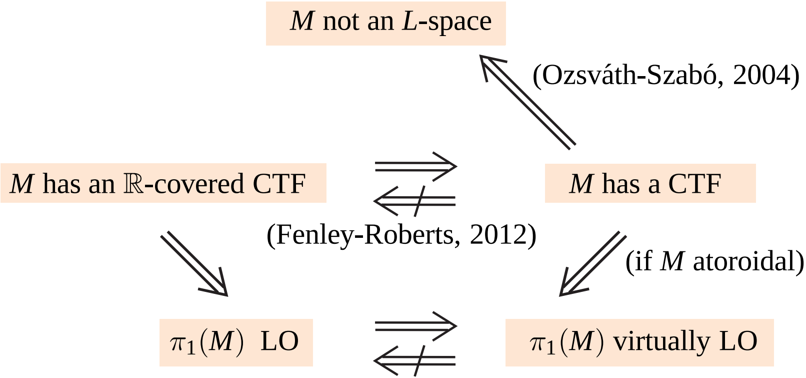
Is there a "non-Heegaard Floer" characterization of L-spaces?

Conjecture

M a prime $\mathbb{Q}$HS. Then

$$M \text{ is an } L\text{-space} \iff \pi_1(M) \text{ is not LO}$$


M not an L-space
(Ozsváth-Szabó, 2004)
M has an $\mathbb{R}$-covered CTF
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(Fenley-Roberts, 2012)
(if M atoroidal)
$\pi_1(M)$ LO
$\pi_1(M)$ virtually LO

(A) Seifert manifolds

Theorem

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M does not admit a horizontal foliation

(Lisca-Stipsicz, 2007) $\Updownarrow$ $\Updownarrow$ (BRW, 2005)

M an L-space $\pi_1(M)$ not LO

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Show M an L-space by induction on n; using (OS, 2005):

X compact, orientable 3-manifold, ∂X a torus;

$\alpha, \beta \subset \partial X$, $\alpha \cdot \beta = 1$, and

$$|H_1(X(\alpha + \beta))| = |H_1(X(\alpha))| + |H_1(X(\beta))|$$

Then $X(\alpha), X(\beta)$ L-spaces $\Rightarrow X(\alpha + \beta)$ L-space $(*)$

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So Conjecture true for $\mathbb{Z}$HS graph manifolds

(C) Sol manifolds

N = twisted I-bundle/Klein bottle

N has two Seifert structures:

base Möbius band; fiber φ_0

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$f : \partial N \to \partial N$ homeomorphism, $M(f) = N \cup_f N$

$$f_* = \begin{bmatrix} a & b \\ c & d \end{bmatrix} ; \quad M(f) \ \mathbb{Q}\mathrm{HS} \Longrightarrow c \neq 0$$

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$\pi_1(M(f))$ not LO (BRW, 2005)

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where $t_0 : \partial N \longrightarrow \partial N$ is Dehn twist along φ_0

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Hence: Conjecture is true for all non-hyperbolic geometric manifolds.

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Corollary

The Conjecture is true for $K(1/q)$, K alternating, either $\forall q \in \mathbb{Q}$, or $\forall q > 0$ or $\forall q < 0$.

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$\exists$ some results in this direction (Clay-Teragaito; Clay-Watson)

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(Also proofs by Greene, Ito)

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Conjecture $\implies$ Q's 1, 2 and 3 have answer "yes"

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