UNSOLVABLE PROBLEMS ABOUT HIGHER-DIMENSIONAL KNOT GROUPS

joint with

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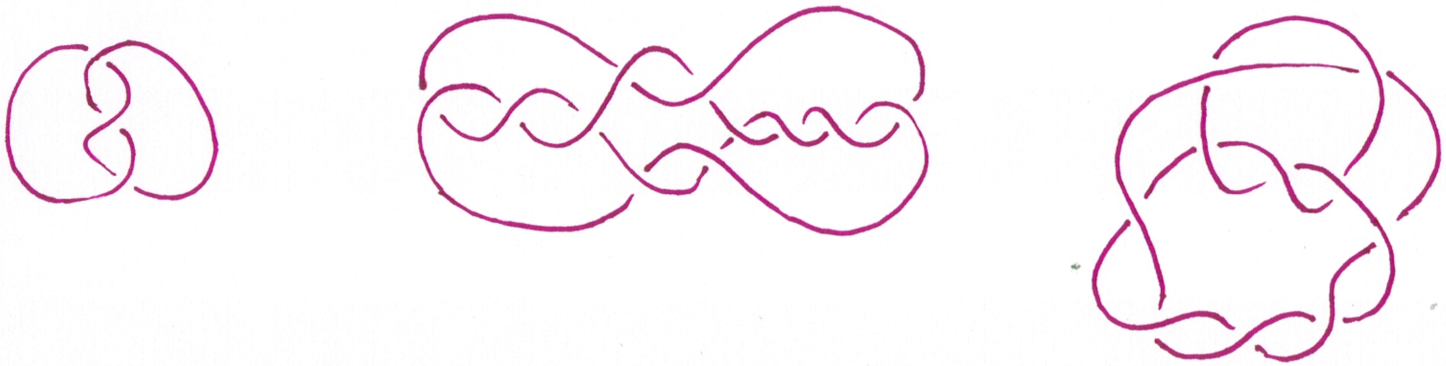
UQAM, Dec. 14, 2011

n-knot : $K \subset S^{n+2}$, $K \cong S^n$ (PL, locally flat)

e.g. $n=1$:



$\underline{K_1 \sim K_2} \iff \underline{(S^{n+2}, K_1) \overset{PL}{\cong} (S^{n+2}, K_2)}$

trivial knot (unknot) : standard inclusion

$S^n \subset S^{n+2}$

$G(K) = \pi_1(S^{n+2} - K)$, the group of K

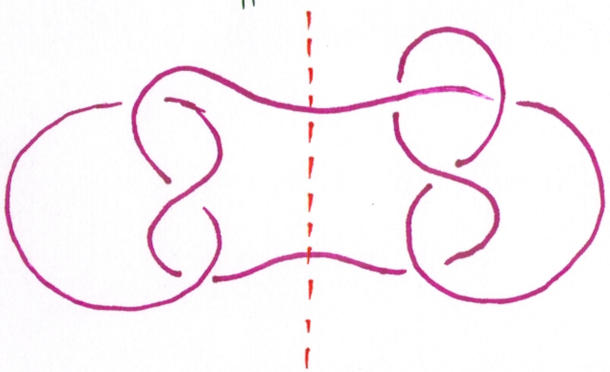
$$K_1 \sim K_2 \Rightarrow G(K_1) \cong G(K_2)$$

$$G(\text{unknot}) \cong \mathbb{Z}$$

$n=1$: $\quad G(K) \cong \mathbb{Z} \Rightarrow K = \text{unknot}$

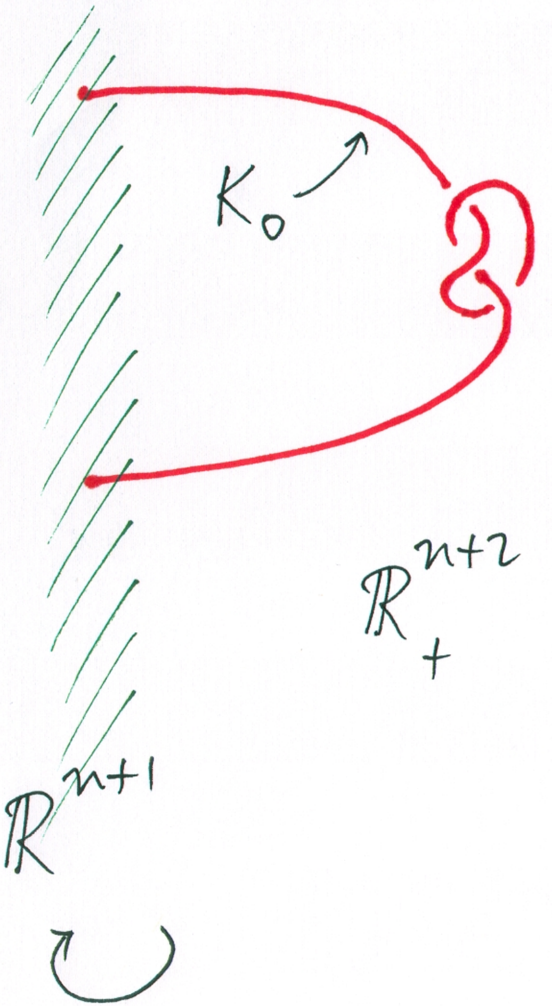
K_1 prime, $G(K_1) \cong G(K_2) \Rightarrow K_1 \sim K_2$

$\#$



Spinning (Artin, 1925)

n-knot $K \subset S^{n+2} \mapsto K_0 \subset \mathbb{R}^{n+2}_+$, $K_0 \cong D^n$, $\partial K_0 \subset \mathbb{R}^{n+1}$

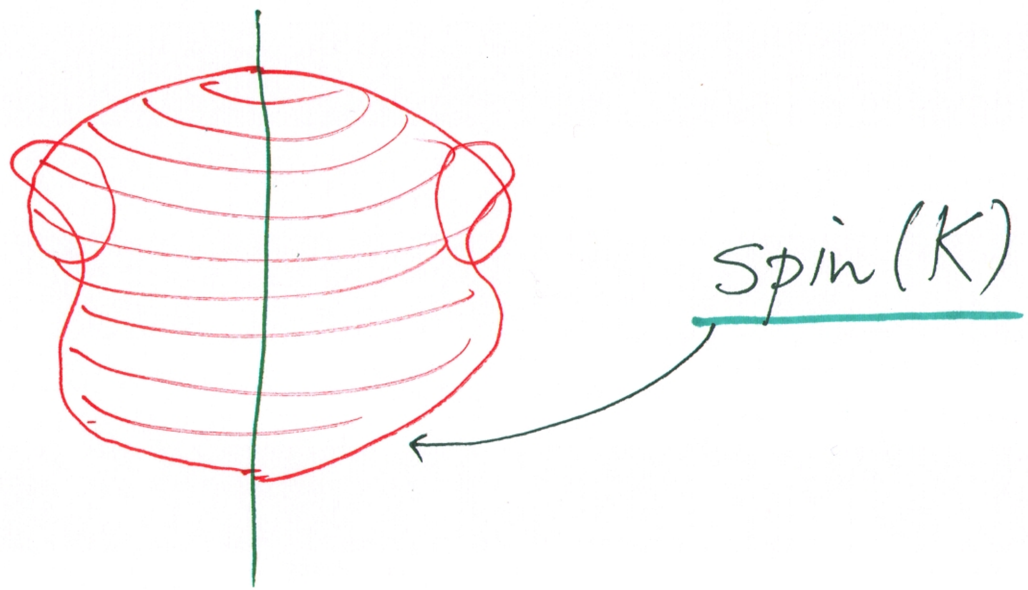


Rotate $\mathbb{R}^{n+2}_+$ about $\mathbb{R}^{n+1}$;

sweeps out $\mathbb{R}^{n+3}$

K_0 sweeps out an $(n+1)$-sphere;

the spin of $K \subset \mathbb{R}^{n+3}$ $(\subset S^{n+3})$



$\underline{G(\mathrm{spin}\, K)} \cong \underline{G(K)}$

So $\exists$ non-trivial n-knots $\forall n \geq 1$.

$\underline{n=2}$: $G(K) \cong \mathbb{Z} \Rightarrow K$ TOPologically $\sim$ unknot

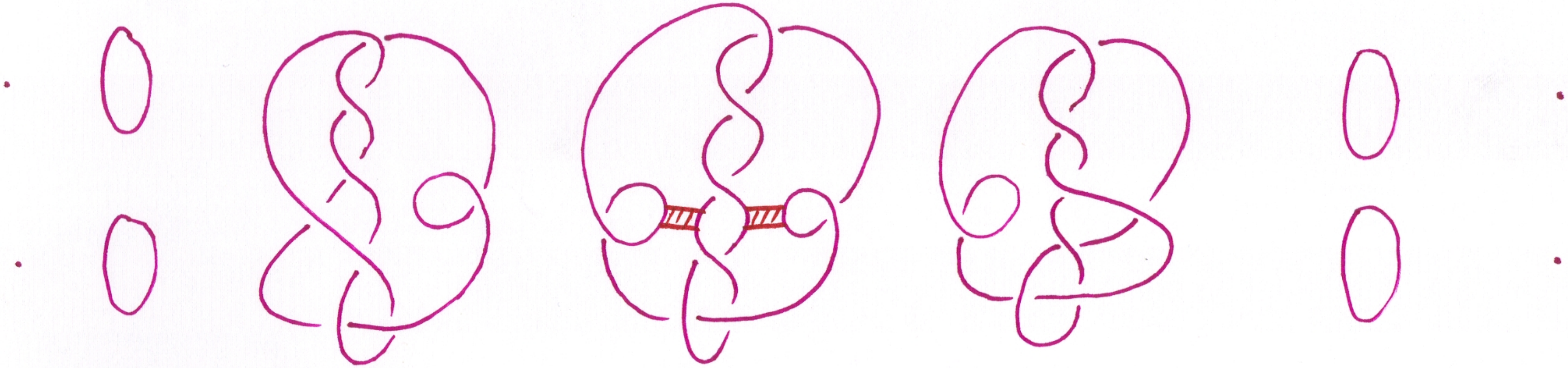
(Freedman, 1982)

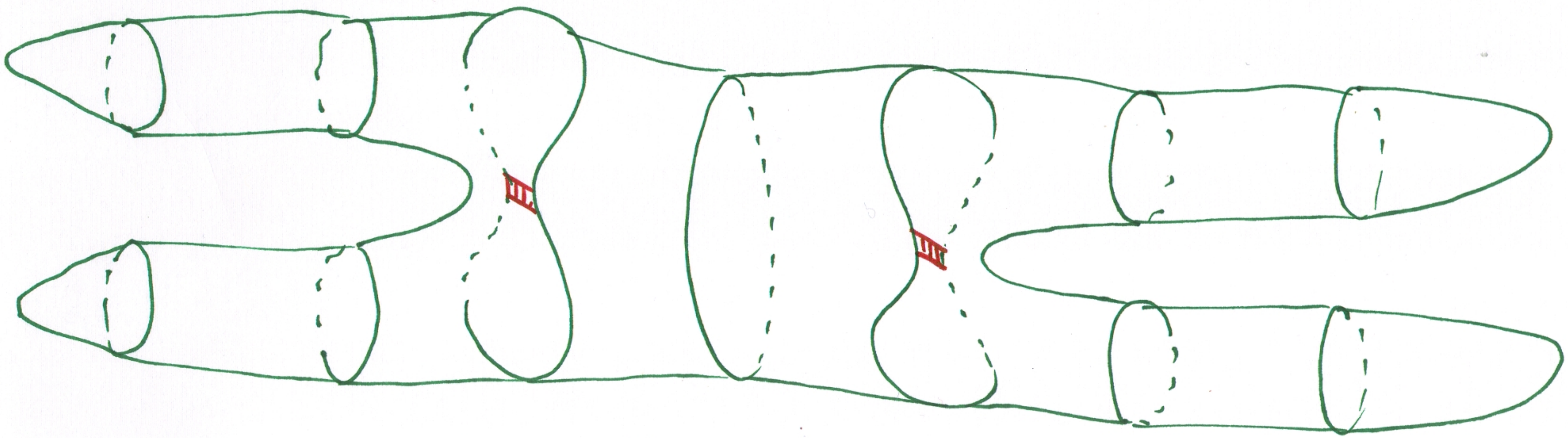
$\underline{\text{Q}}$. $\underline{\text{Is } K \text{ PL} \sim \text{unknot ?}}$

$\exists$ (TOP) prime 2-knots K_1, K_2 with $G(K_1) \cong G(K_2)$,

$K_1 \not\sim K_2$

$\underline{n \geq 3}$: $\exists$ non-trivial n-knots with $G(K) \cong \mathbb{Z}$





2-knot $K \subset S^4$ (FOX, 1962)

$$G(K) = \langle a, t : t^{-1}at = a^{-1},\ a^3 = 1 \rangle$$

Let $\mathcal{K}_n = \{n\text{-knot groups}\}$

$\mathcal{K}_0 = \{\mathbb{Z}\}$; spinning shows $\mathcal{K}_n \subset \mathcal{K}_{n+1}$ $\forall n \geq 0$.

⌈ G group ; Y complex with $\pi_1(Y) \cong G$, $\pi_n(Y) = 0$, $n \neq 1$.

Then $H_k(G) = H_k(Y)$ ⌋

$K \subset S^{n+2}$ an n-knot

$X = S^{n+2} - K$

$G = G(K) = \pi_1(X)$

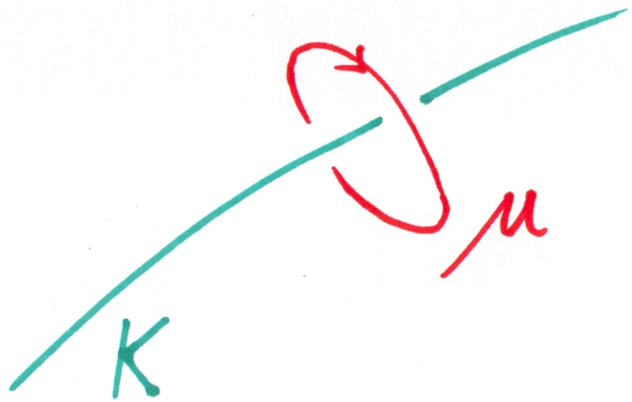
$X \simeq$ finite complex $\Rightarrow$ $\underline{G \in \mathcal{G}}$ $=$ {$\underline{\text{finitely presented groups}}$}

$$H_*(X) \cong H_*(S^1)$$

$$H_1(X) \cong \mathbb{Z} \Rightarrow \underline{H_1(G)}\ (= G/[G,G])\ \underline{\cong \mathbb{Z}}$$

$$H_2(X) = 0 \Rightarrow \underline{H_2(G) = 0}$$

van Kampen $\Rightarrow$



$$\underline{G/\langle\langle \mu \rangle\rangle} = \pi_1(S^{n+2}) = \underline{1}.$$

(G has $\underline{\text{weight } 1}$)

μ = meridian of K

Thm (Kervaire, 1965) For $n \geq 3$, $G \in \mathcal{K}_n$ iff

(1) $G \in \mathcal{G}$; (2) $H_1(G) \cong \mathbb{Z}$; (3) $H_2(G) = 0$;

(4) G has weight 1.

So $\{\mathbb{Z}\} = \mathcal{K}_0 \subset \mathcal{K}_1 \subset \mathcal{K}_2 \subset \mathcal{K}_3 = \mathcal{K}_n$, $n \geq 3$

$\mathcal{K}_0 \neq \mathcal{K}_1$:  $G = \langle x, y : x^2 = y^3 \rangle \not\cong \mathbb{Z}$

$\mathcal{K}_1 \neq \mathcal{K}_2$: $\langle t, a : t^{-1}at = a^2, a^3 = 1 \rangle \in \mathcal{K}_2 - \mathcal{K}_1$

$\mathcal{K}_2 \neq \mathcal{K}_3$: $\langle t, a : t^{-1}at = a^2, a^5 = 1 \rangle \in \mathcal{K}_3 - \mathcal{K}_2$

Q. Is there a "nice" characterization of $\mathcal{K}_2$?

$\mathcal{S}_n = \{ \pi_1(S^{n+2} - N) :$ N conn., closed, orientable n-mfld $\}$

$\mathcal{M}_n = \{ \pi_1(M^{n+2} - N) :$ M closed, 1-conn. $(n+2)$-mfld;

N.... $\}$

$\mathcal{S}_0 = \mathcal{M}_0 = \mathcal{K}_0$; $\mathcal{S}_1 = \mathcal{M}_1 = \mathcal{K}_1$

$\mathcal{S}_2 = \mathcal{S}_n , \; n \geq 2 \quad (= \mathcal{S})$

$\mathcal{M}_2 = \mathcal{M}_n , \; n \geq 2 \quad (= \mathcal{M})$

So $\mathcal{S} = \{ \pi_1(S^4 - N) :$ N conn. closed, orientable surface $\}$

Thm (Simon, 1980). $G \in \mathcal{S}$ iff (1) $G \in \mathcal{G}$; (2) $H_1(G) \cong \mathbb{Z}$; (3) $\exists\, t \in G$ such that $\langle\langle t \rangle\rangle = G$ & $t \wedge C_t = H_2(G)$.

(C_t = centralizer of t; $\wedge$ = Pontryagin product)

$G \in \mathcal{M}$ iff (1) $G \in \mathcal{G}$, & (2) G has weight 1.

$$\{\mathbb{Z}\} = \mathcal{K}_0 \subset \mathcal{K}_1 \subset \mathcal{K}_2 \subset \mathcal{K}_3 \subset \mathcal{S} \subset \mathcal{M} \subset \mathcal{G}$$

All inclusions proper: $A_5 \times \mathbb{Z} \in \mathcal{S} - \mathcal{K}_3$;

$$\langle x, y, s : x^2 = y^3 = (xy)^7,\ s^{-1} x s = x^2 \rangle \in \mathcal{M} - \mathcal{S}$$

$$\mathbb{Z} \times \mathbb{Z} \in \mathcal{G} - \mathcal{M}$$

Let $\mathcal{B} \subset \mathcal{A} \subset \mathcal{G}$, where

$\{$finite presentations of groups $\in \mathcal{A}\}$ is recursively enumerable.

The recognition problem $\mathrm{Rec}(\mathcal{A}, \mathcal{B})$ is solvable if there is an algorithm to decide, given a finite presentation of a group $A \in \mathcal{A}$, whether or not $A \in \mathcal{B}$.

Thm (Adjan, Rabin, 1958)

$\mathrm{Rec}(\mathcal{G}, \{1\})$ is unsolvable.

i.e. there is no algorithm to decide whether or not the group defined by a given finite presentation is trivial.

Based on

Thm (Novikov, Boone, 1955–56). **$\exists\, G \in \mathcal{G}$ with unsolvable word problem.**

i.e. there is no algorithm to decide, given a word w in a set of generators of G, whether or not $[w] = 1 \in G$.

Pyotr Sergeyevich Novikov
1901–1975

William Werner Boone
1920–1983

It has recently been announced from Russia that the 'word problem in groups' is not solvable. This is a decision problem not unlike the 'word problem in semi-groups', but very much more important, having applications in topology: attempts were being made to solve this decision problem before any such problems had been proved unsolvable. ...Another problem which mathematicians are very anxious to settle is known as 'the decision problem of the equivalence of manifolds'.... It is probably unsolvable, but has never been proved to be so. *A similar decision problem which might well be unsolvable is the one concerning knots which has already been mentioned.* (Turing, 1954)

Thm (Haken, 1962) $\exists$ algorithm to decide whether or not a given 1-knot $K \subset S^3$ is trivial.

Since K trivial $\Leftrightarrow G(K) \cong \mathbb{Z}$:

Cor. $Rec(\mathcal{K}_1, \mathcal{K}_0)$ is solvable.

— // —

$$\mathbb{G} = \{\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{S}, \mathcal{M}, \mathcal{G}\}$$

If $\mathcal{A} \in \mathbb{G}$ then $\{$finite presentations of groups $\in \mathcal{A}\}$ is recursively enumerable.

Thm (Gonzalez-Acuña - G. - Simon). Let $\mathcal{A}, \mathcal{B} \in \mathcal{G}$, $\mathcal{B} \not\subset \mathcal{A}$, $\mathcal{A} \supset \mathcal{K}_3$. Then $Rec(\mathcal{A}, \mathcal{B})$ is unsolvable.

Q. Does this also hold for $\mathcal{A} = \mathcal{K}_2$?

Most cases proved by using unsolvability of $Rec(\mathcal{G}, \{1\})$.

e.g.

Prop^n. $\exists$ effective procedure which takes a finite presentation of a group G and produces a finite presentation of a group H such that

(1) $H \in \mathcal{K}_3$; (2) $G = 1 \Rightarrow H \cong \mathbb{Z}$; (3) $G \neq 1 \Rightarrow H \notin \mathcal{K}_2$.

Cor. If $\mathcal{K}_0 \subset \mathcal{B} \subset \mathcal{K}_2$ then $Rec(\mathcal{K}_3, \mathcal{B})$ is unsolvable.

(a sol'n. to $Rec(\mathcal{K}_3, \mathcal{B})$ wd. $\mapsto$ sol'n. to $Rec(\mathcal{G}, \{1\})$.)

Addendum. In Prop^n. G embeds in H.

Cor. $\exists\, H \in \mathcal{K}_3$ with unsolvable word problem.

Cor. $\exists\, H \in \mathcal{K}_3$ which contains a copy of every $G \in \mathcal{G}$.

Q. Does every $G \in \mathcal{G}$ embed in some $H \in \mathcal{K}_2$?

n-knot triviality problem

$\{n\text{-knots}\}$ is recursively enumerable. So can ask:

"given an n-knot, is it trivial?"

Solvable if $n=1$.

Thm (G-A-G-S.) If $\exists\ G \in \mathcal{K}_n$ with unsolvable word problem then the n-knot triviality problem is unsolvable.

Cor (Nabutovsky-Weinberger, 1996) If $n \geq 3$ the n-knot triviality problem is unsolvable.

Q. Is there a $G \in \mathcal{K}_2$ with unsolvable word problem?

Sketch proof. (S^{n+2}, K) n-knot, $n \geqslant 2$, with

$G = G(K)$ having unsolvable word problem.

Surgery on K: $M^{n+2} = (S^{n+2} - N(K)) \cup (D^{n+1} \times S^1)$

$$N(K) \cong S^n \times D^2$$

$C = * \times S^1 \subset D^{n+1} \times S^1 \subset M$

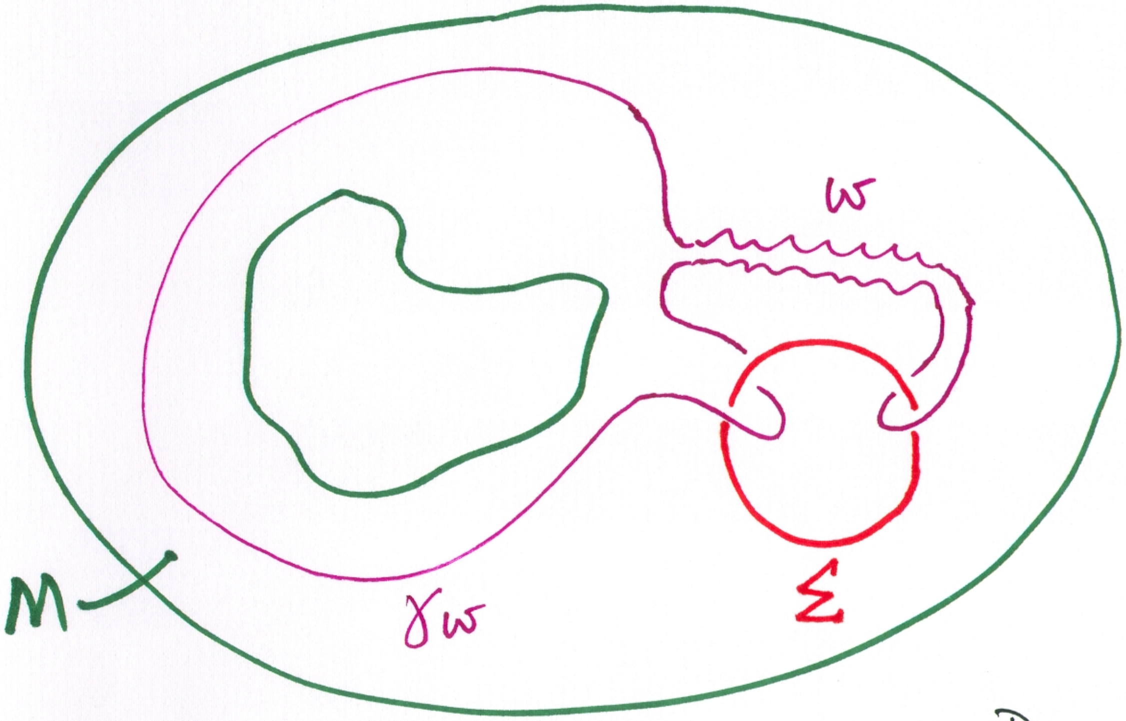
$\pi_1(M) \cong G$; $[C] = \mu =$ meridian of K

Let Σ_1 be a trivial n-sphere $\subset M$

$\pi_1(M - \Sigma_1) \cong G * \mathbb{Z}(\sigma)$, $\sigma =$ meridian of Σ_1.

w word in the generators of G , $[w] \in [G,G]$.

γ_w loop in $M-\Sigma$ such that $[\gamma_w] = \sigma^{-1}w^{-1}\sigma w \mu$
$\in \pi_1(M-\Sigma)$



$[\gamma_w] = \mu \in \pi_1(M)$

$\therefore$ γ_w isotopic to C in M

Do surgery on γ_w :

$M \mapsto S^{n+2}$

$\Sigma \mapsto$ n-sphere $\Sigma_w \subset S^{n+2}$

$\underline{[w] = 1 \in G}$: $\quad [\gamma_w] = \mu \in \pi_1(M - \Sigma_1)$

$\therefore \gamma_w$ isotopic to C in $M - \Sigma_1$

$\therefore \underline{(S^{n+2}, \Sigma_w) \text{ trivial}}$

$\underline{[w] \neq 1 \in G}$: $\quad \pi_1(S^{n+2} - \Sigma_w) = \langle G, \sigma : \sigma^{-1} w \sigma = w\mu \rangle$

$=$ HNN extension of G ; $\quad \therefore \not\cong \mathbb{Z}$

$\therefore \underline{(S^{n+2}, \Sigma_w) \text{ non-trivial}}$.

$\therefore$ a solution to the n-knot triviality problem would give a solution to the word problem for G. $\#$

simple connectivity of 3-manifolds problem:

"given a connected, closed 3-mfld. M, is $\pi_1(M)=1$?" (*)

Let $\mathcal{M}_3 = \{ \pi_1(M) : M \text{ closed 3-mfld.} \}$

{ finite presentations of groups in $\mathcal{M}_3$ } is rec. enumerable.

Can show

$$\text{if } G \in \mathcal{M}_3 \text{ then } G=1 \Leftrightarrow H_1(G)=H_3(G)=0$$

(*) is solvable: (1) $\pi_1(M)=1 \Leftrightarrow M \cong S^3$ (Perelman, 2003)

(2) $\exists$ algorithm to decide whether or not a given 3-mfld. M is $\cong S^3$ (Rubinstein-Thompson, 1994)

$G \in \mathcal{G} \Rightarrow H_1(G), H_2(G)$ finitely generated abelian groups

($H_k(G)$, $k \geq 3$, not nec. fin. gen. (Stallings, 1963)).

Clearly $\exists$ algorithm to compute $H_1(G)$ for $G \in \mathcal{G}$.

Thm (G, 1980) There is no algorithm to decide, given $G \in \mathcal{G}$, whether or not $H_2(G) = 0$.

More generally:

Thm (G-A-G-S). Let $I \subset \mathbb{N}$, $I \neq \emptyset, \{1\}$. Then "given $G \in \mathcal{G}$, is $H_k(G) = 0$ $\forall k \in I$" is unsolvable.