ARTIN GROUPS, 3-MANIFOLDS AND COHERENCE

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Dedicated to Fico on the occasion of his 60th birthday.

1. **Introduction.**

By a *labeled graph* we shall mean a finite (non-empty) graph Γ, without loops or multiple edges, each of whose edges is labeled by an integer greater than or equal to 2. Let the vertices of Γ be $s_1, s_2, \ldots, s_n$, and let the label on an edge with endpoints s_i and s_j be $m_{ij} \geq 2$. Define $\langle ab \rangle^m$ to be the word $abab\ldots$ of length m. Then the *Artin group* $A\Gamma$ associated with the labeled graph Γ is the group with generators $s_1, s_2, \ldots, s_n$, and relations $\langle s_i s_j \rangle^{m_{ij}} = \langle s_j s_i \rangle^{m_{ij}}$, one for each edge of Γ. In particular, if $m_{ij} = 2$ then the generators s_i and s_j commute. Note also that if Γ is the disjoint union of graphs Γ_1 and Γ_2 then $A\Gamma \cong A\Gamma_1 * A\Gamma_2$.

A *3-manifold group* is a group that is isomorphic to $\pi_1(M)$ for some (connected) 3-manifold M. Note that we do not assume that M is orientable, or compact, or without boundary. Taking a connected sum shows that if G_1 and G_2 are 3-manifold groups then so is $G_1 * G_2$. If Γ is a tree, then $A\Gamma$ is the fundamental group of the complement of a link L in S^3, where L is a connected sum of $(2, m)$ torus links; see [Bru], [HM]. Thus $A\Gamma$ is a 3-manifold group. If Γ is a triangle with each edge labeled 2, then $A\Gamma \cong \mathbb{Z}^3 \cong \pi_1(T^3)$ is also a 3-manifold group. In this note we confirm the suspicion of Hermiller and Meier [HM, p.143] that these are the only connected graphs whose Artin groups are 3-manifold groups.

Theorem 1.1. *For an Artin group* $A\Gamma$ *the following are equivalent.*

(1) $A\Gamma$ *is a 3-manifold group.*
(2) $A\Gamma$ *is virtually a 3-manifold group.*
(3) *Each component of* Γ *is either a tree, or a triangle with each edge labeled 2.*

The equivalence of (1) and (3) was proved by Droms [D] in the case of *all right* Artin groups, or *graph groups*, that is, when all labels are 2, and by Hermiller and Meier [HM] in the case when all labels are even.

In Section 3 we make some additional remarks about coherence. In particular we show that $\mathrm{Aut}(F_2)$ and the braid group B_4 are incoherent, although neither has a subgroup of the form $F_2 \times F_2$. The latter fact for B_4 ws originally proved by Akimenkov [A], using different methods.

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2. **Artin groups and 3-manifolds.**

Recall that a group is *coherent* if every finitely generated subgroup is finitely presented.

The following is proved in [HM, Proposition 5.7(ii)].

Lemma 2.1 (Hermiller and Meier)**.** *Let Γ be a cycle of length at least 4. Then $A\Gamma$ is incoherent.*

If Γ is a labeled graph, we shall say that Γ is of *infinite* or *finite type* according as the Coxeter group corresponding to the Artin group $A\Gamma$ is infinite or finite. We will use (p, q, r) to denote a triangle with edge labels p, q and r. The triangles of finite type are then $(2, 2, m)$, $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$.

If Γ is a triangle, the simplicial complex K_0 defined in [CD2] is either a triangle or a 2-simplex, according as Γ is of infinite or finite type. The Main Conjecture of [CD1] and [CD2] therefore holds for $A\Gamma$, by [CD1]. The following lemma is then a consequence of [CD2, Corollary 1.4.2 and Corollary 2.2.5].

Lemma 2.2 (Charney and Davis)**.** *Let Γ be a triangle.*

(i) *If Γ is of infinite type then $A\Gamma$ has geometric dimension 2 and $\chi(A\Gamma) = 1$.*
(ii) *If Γ is of finite type then $A\Gamma$ has geometric dimension 3 and $\chi(A\Gamma) = 0$.*

For the three triangles $(2, 3, 6)$, $(2, 4, 4)$ and $(3, 3, 3)$ of Euclidean type, (i) also follows from the descriptions of $A\Gamma$ given in [Sq1].

Lemma 2.3. *Let Γ be a triangle of infinite type. Then $A\Gamma$ is incoherent.*

Proof. Let $\varphi : A\Gamma \to \mathbb{Z}$ be the epimorphism that maps each generator s_i to 1. By [Me, Proposition 5.1 and Corollary 5.3] $K = \ker\varphi$ is finitely generated. Now $A\Gamma$ has geometric dimension 2 (Lemma 2.2), and hence K has geometric dimension ≤ 2. Suppose K were finitely presented. Then K would be of type FP [Bro, p.199], and so $\chi(K)$ would be defined. We would then have $\chi(A\Gamma) = \chi(K)\chi(\mathbb{Z}) = 0$ [Bro, p.250], [St2], (compare [G]), contradicting Lemma 2.2. $\square$

In [W], Wall asked whether a group of the form $F *_C F'$, where F and F' are free and C has finite index in F and F', is coherent. This was answered negatively by Gersten [G], who showed that the double of a free group of rank ≥ 2 along a subgroup of finite index ≥ 3 is always incoherent. We remark that Lemma 2.3 also provides examples, which are not doubles, since Squier has shown [Sq1] that $A(2, 3, 6)$ and $A(3, 3, 3)$ can each be expressed as a free product with amalgamation $F *_C F'$, where $\operatorname{rank} F = 4$, $\operatorname{rank} F' = 3$, and C has index 2 in F and index 3 in F'.

Lemma 2.4. *$A(2, 3, 3)$ and $A(2, 3, 4)$ are incoherent.*

Proof. Since $A(2, 3, 4)$ embeds in $A(2, 3, 3)$ (as a subgroup of finite index) [La], it is enough to show that $A(2, 3, 4)$ is incoherent. One way to do this is to use the fact that $A(3, 3, 3)$ embeds in $A(2,3,4)$ [KP], together with Lemma 2.3. Another argument is that the commutator subgroup A' of $A(2,3,4)$ is finitely generated but, since $H_2(A') \cong \mathbb{Z}^\infty$, not finitely presented [Sq2]. $\square$

Note that $A(2,2,m) \cong A(m) \times \mathbb{Z}$, where $A(m)$ is the Artin group of a single edge with label m. Since $A(m)$ is a 3-manifold group, it is coherent by [Sc] (it is also easy to show this directly), and hence $A(2,2,m)$ is coherent.

Lemma 2.5. *$A(2,2,m)$, $m > 2$, and $A(2,3,5)$ are not virtually 3-manifold groups.*

Proof. Let G be a finitely generated group, with an epimorphism $\varphi : G \to \mathbb{Z}$ such that $\ker\varphi$ is finitely generated, and let H be a subgroup of G of finite index. Then φ induces an epimorphism $\psi : H \to \mathbb{Z}$, where $\ker\psi$ has finite index in $\ker\varphi$. Now suppose that H is a 3-manifold group. Since H is finitely generated, it is the fundamental group of a compact 3-manifold [Sc]. Therefore, since $\ker\psi$ is finitely generated, by [St1] $\ker\psi$ is a *2-manifold group*, i.e. it is either free or the fundamental group of a closed surface. Hence $\ker\varphi$ is virtually a 2-manifold group.

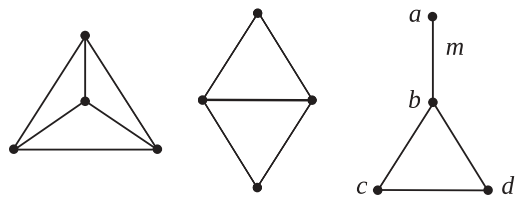
Suppose $A(2,2,m) \cong A(m) \times \mathbb{Z}$ is virtually a 3-manifold group. Then by the above discussion $A(m)$ has a subgroup B of finite index such that B is either free or the fundamental group of a closed orientable surface. Since $A(m)$ is the fundamental group of a compact, orientable, irreducible 3-manifold M whose boundary consists of tori, $\chi(A(m)) = \chi(M) = \frac{1}{2}\chi(\partial M) = 0$. Hence $\chi(B) = 0$, implying that B is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$. But if $m > 2$, this contradicts the fact that $A(m)$ contains a non-abelian free group.

Now let $A = A(2,3,5)$, and let $\varphi : A \to \mathbb{Z}$ be abelianization. Then $A' = \ker\varphi$ is finitely generated by [Me]. Suppose A is virtually a 3-manifold group. Then there is a 2-manifold subgroup B of A' of finite index. By a standard transfer argument, $H_2(B;\mathbb{Q}) \to H_2(A';\mathbb{Q})$ is surjective. But $\dim H_2(A';\mathbb{Q}) = 7$ [Sq2], whereas $\dim H_2(B;\mathbb{Q}) \leq 1$. $\square$

Proof of Theorem 1.1. Clearly (1) implies (2) and (3) implies (1); we must show that (2) implies (3).

A subgraph Γ_0 of Γ is *full* if every edge of Γ whose vertices are in Γ_0 is an edge of Γ_0. We recall the basic fact [Le] that if Γ_0 is a full subgraph of a labeled graph Γ then the homomorphism $A\Gamma_0 \to A\Gamma$ induced by the inclusion map $\Gamma_0 \subset \Gamma$ is injective.

Let Γ be a connected labeled graph, and suppose that $A\Gamma$ is virtually a 3-manifold group. By [Sc], 3-manifold groups, and hence virtual 3-manifold groups, are coherent. Also, a subgroup of a virtual 3-manifold group is clearly a virtual 3-manifold group. It follows from Lemma 2.1 that Γ is *chordal*, i.e. has no full subgraph that is a cycle of length ≥ 4. By Lemmas 2.4 and 2.5, any triangle in Γ has all labels equal to 2. If Γ is not a tree or a $(2,2,2)$ triangle, then Γ has a full subgraph Γ_0 of one of the forms shown in Figure 1 (where all unlabeled edges are understood to have label 2); see [D].



(i) (ii) (iii)

FIGURE 1

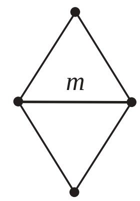


FIGURE 2

In cases (i) and (ii), $A\Gamma_0 \cong A \times \mathbb{Z}$, where in case (i) $A \cong \mathbb{Z}^3$, and in case (ii) A is the Artin group of a tree with two edges, each labeled 2. Since $A\Gamma_0$ is virtually a 3-manifold group by assumption, A has a subgroup B of finite index that is either free or the fundamental group of a closed orientable surface. Since $\chi(A) = 0$, we have $\chi(B) = 0$, and hence $B \cong \mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$, which is clearly impossible.

In case (iii), m odd, let $\varphi : A\Gamma_0 \to \mathbb{Z}$ be the epimorphism defined by $\varphi(a) = \varphi(b) = 1$, $\varphi(c) = \varphi(d) = 0$. By [Me], $\ker\varphi$ is finitely generated. Hence $\ker\varphi$ has a subgroup B of finite index that is either free or the fundamental group of a closed orientable surface. Since $\ker\varphi$ contains $\langle c, d\rangle \cong \mathbb{Z} \times \mathbb{Z}$, we must have $B \cong \mathbb{Z} \times \mathbb{Z}$. But this contradicts the fact that $\ker\varphi$ also contains the commutator subgroup of $A(m)$, which is a non-abelian free group.

In case (iii), m even, define $\varphi : A\Gamma_0 \to \mathbb{Z}$ by $\varphi(b) = 1$, $\varphi(a) = \varphi(c) = \varphi(d) = 0$, as in [HM]. Then $\ker\varphi$ is finitely generated and contains both $\mathbb{Z} \times \mathbb{Z}$ and a non-abelian free group, giving a contradiction as before. □

3. **Coherence.**

It is natural to ask which Artin groups $A\Gamma$ are coherent. For graph groups, i.e. when all edge labels are 2, this has been answered by Droms [D]: $A\Gamma$ is coherent if and only if Γ is chordal. For the general case, it is necessary to be able to answer the following question.

Question 3.1. *Is $A(2,3,5)$ coherent?*

If at most one of p, q, r is even, the homomorphism $\varphi : A(p,q,r) \to \mathbb{Z}$ in the proof of Lemma 2.3 is abelianization, so that proof shows that if (p,q,r) is of infinite type then the commutator subgroup of $A(p,q,r)$ is finitely generated but not finitely presented. However, as pointed out in [Sq2], the commutator subgroup of $A(2,3,5)$ is finitely presented (the same argument applies to $A(2,3,3)$).

If $A(2,3,5)$ is incoherent, one can show that an Artin group $A\Gamma$ is coherent if and only if Γ is chordal, every complete subgraph of Γ with 3 or 4 vertices has at most one edge label > 2, and Γ has no full subgraph of the form shown in Figure 2, where $m > 2$ and unlabeled edges are understood to have label 2. If $A(2,3,5)$ is coherent, the characterization would be more complicated.

Let F_n denote the free group of rank n. A popular way of showing that a group is incoherent is to show that it has a subgroup isomorphic to $F_2 \times F_2$, which is well-known to be incoherent; see e.g. [G]. For example, $\mathrm{Aut}(F_3)$ has such a subgroup [FP], and hence $\mathrm{Aut}(F_n)$ is incoherent for $n \geq 3$. For $n = 2$ we have

Theorem 3.2. (1) $\mathrm{Aut}(F_2)$ *is incoherent.*
(2) $F_2 \times F_2$ *does not embed in* $\mathrm{Aut}(F_2)$.

Let B_n denote the n-strand braid group. Note that B_n is coherent for $n \leq 3$. Since $A(2,3,3) \cong B_4$, we see by Lemma 2.4 that B_n is incoherent for $n \geq 4$. Also, since the center of $F_2 \times F_2$ is trivial, by Part (2) of Theorem 3.2 and the proof of Part (1) below we recover the result of Akimenkov [A] that $F_2 \times F_2$ does not embed in B_4. It follows (see the proof of Lemma 2.4) that the incoherent groups $A(2,3,4)$ and $A(3,3,3)$ also do not contain an $F_2 \times F_2$. (It is shown in [Ma] that $F_2 \times F_2$ does embed in B_n for $n \geq 5$.)

Proof of Theorem 3.2. (1) Let $Z(B_4)$ denote the center of B_4; then $B_4/Z(B_4)$ is isomorphic to an index 2 subgroup of $\mathrm{Aut}(F_2)$ [DFG]. Now $A(3,3,3)$ embeds in B_4 [KP], and since it is a free product with amalgamation of the form $F_4 *_{F_7} F_3$ [Sq1], it has trivial center, and hence embeds in $\mathrm{Aut}(F_2)$. Since $A(3,3,3)$ is incoherent by Lemma 2.3, $\mathrm{Aut}(F_2)$ is also incoherent.

(2) There is a short exact sequence

$$1 \to F_2 \xrightarrow{i} \mathrm{Aut}(F_2) \xrightarrow{\pi} GL(2,\mathbb{Z}) \to 1 \ ,$$

where $i(g)$ is conjugation by g. Mapping $GL(2,\mathbb{Z})$ onto $PSL(2,\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$ gives the related sequence

$$1 \to \ker\rho \to \mathrm{Aut}(F_2) \xrightarrow{\rho} \mathbb{Z}_2 * \mathbb{Z}_3 \to 1 \ ,$$

where $\ker\pi$ has index 4 in $\ker\rho$. In particular, $\ker\rho$ is virtually free.

Suppose $H < \mathrm{Aut}(F_2)$, where $H = \langle\alpha,\beta\rangle \times \langle\gamma,\delta\rangle = H_1 \times H_2 \cong F_2 \times F_2$. We claim that either $\rho(H_1) = 1$ or $\rho(H_2) = 1$. For, if not, then writing $\bar{\alpha} = \rho(\alpha)$ etc., we may assume that $\bar{\alpha} \neq 1 \neq \bar{\gamma}$. By the Kurosh Subgroup Theorem, any abelian subgroup of $\mathbb{Z}_2 * \mathbb{Z}_3$ is cyclic. Therefore $\langle\bar{\alpha},\bar{\gamma}\rangle = \langle x\rangle$, $\langle\bar{\alpha},\bar{\delta}\rangle = \langle y\rangle$, say. Then we have $1 \neq \bar{\alpha} = x^p = y^q$ for some $p,q \in \mathbb{Z}$. It follows that $\langle x,y\rangle$ has a non-trivial center, and hence, again by the Kurosh Subgroup Theorem, $\langle x,y\rangle = \langle z\rangle$, say. Therefore $\langle\bar{\gamma},\bar{\delta}\rangle < \langle z\rangle$ is cyclic, implying that $(\ker\rho) \cap H_2 \neq 1$. Similarly $(\ker\rho) \cap H_1 \neq 1$. This gives $\mathbb{Z} \times \mathbb{Z} < \ker\rho$, contradicting the fact that $\ker\rho$ is virtually free.

We may assume, then, without loss of generality, that $\rho(H_2) = 1$. Then $\rho|H_1$ is injective, otherwise we would have $\mathbb{Z} \times H_2 < \ker\rho$, again contradicting the fact that $\ker\rho$ is virtually free. It follows that $\pi|H_1$ is injective. Also, $H_2 < \ker\rho$, and therefore $(\ker\pi) \cap H_2$ has finite index in H_2. Hence $(\ker\pi) \cap H_2 = i(G)$, where $G < F_2$ has rank ≥ 2.

Note that since F_2 has trivial center, $\varphi \in \mathrm{Aut}(F_2)$ commutes with $i(g)$, $g \in F_2$, if and only if $\varphi(g) = g$. Hence if $\varphi \in H_1$, then $G < \mathrm{Fix}(\varphi)$. Since rank $G \geq 2$, it follows from the Scott Conjecture [BH] that $\mathrm{Fix}(\varphi)$ has rank 2. Furthermore, by [CT] there is a basis a,b of

F_2 such that $\varphi(a) = a$, $\varphi(b) = ba^n$. Hence $\pi(\varphi) \in GL(2, \mathbb{Z})$ has trace 2. But since $\pi(H_1)$ is a free group of rank 2, this is a contradiction. □

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