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ON THE REVERSIBILITY OF TWIST-SPUN KNOTS

CAMERON MCA. GORDON *

The University of Texas at Austin
Department of Mathematics, 1 University Station C1200, Austin, TX 78712-0257

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ABSTRACT

Litherland has shown that if a knot is (+)-amphicheiral then its m-twist-spin is reversible. We show that, for classical knots, in many cases the converse holds.

The irreversibility (sometimes called noninvertibility) of certain twist-spun knots has been established by Ruberman [10], using the Farber-Levine linking pairing and the Casson-Gordon invariants. More recently, alternative proofs of the irreversibility of the 2-twist-spun trefoil have been given by Carter, Jelsovsky, Kamada, Langford and Saito [4], and by Rourke and Sanderson [9], using quandle cohomology and the homotopy theory of racks, respectively. Similar methods have been used by Satoh [11] to prove the irreversibility of certain other twist-spun torus knots. In the present note we use more geometric methods to prove the following more general result.

Theorem 1. (1) *The 2-twist-spin of a rational knot κ is reversible if and only if κ is amphicheiral.*

(2) *If m, p, q are > 1 then the m-twist-spin of the (p, q) torus knot is irreversible.*

(3) *If $m \geq 3$ then the m-twist-spin of a hyperbolic knot κ is reversible if and only if κ is $(+)$-amphicheiral.*

Since a rational knot is either hyperbolic or a $(2, q)$ torus knot, we obtain the following corollary.

Corollary 2. *If $m \geq 2$ then the m-twist-spin of a rational knot κ is reversible if and only if κ is amphicheiral.*

The "if" directions in parts (1) and (3) of Theorem 1 are due to Litherland [5], who shows that the m-twist-spin of a $(+)$-amphicheiral knot is always reversible.

We work in the PL category. A *knot* κ (more precisely, an *n-knot*) is a locally flat oriented pair (S^{n+2}, K), where K is homeomorphic to S^n. (The knots in Theorem 1

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are 1-knots.) Two knots $\kappa_1 = (S^{n+2}, K_1)$ and $\kappa_2 = (S^{n+2}, K_2)$ are *equivalent* if there is an orientation preserving homeomorphism of pairs $h : (S^{n+2}, K_1) \to (S^{n+2}, K_2)$. A knot $\kappa = (S^{n+2}, K)$ is *reversible* if it is equivalent to $(S^{n+2}, -K)$; it is (ε)-*amphicheiral* if it is equivalent to $(-S^{n+2}, \varepsilon K)$, $\varepsilon = \pm$. Note that if κ is reversible then $(+)$- and $(-)$-amphicheirality coincide, and hence, since rational knots are reversible, we can unambiguously use the term *amphicheiral* in part (1) of Theorem 1 and in Corollary 2.

Let $\kappa = (S^{n+2}, K)$ be a knot. Then K has a regular neighborhood $N(K)$, where $(N(K), K) \cong (S^n \times D^2, S^n \times \{(0,0)\})$ [13], and the *exterior* of κ is $X = \overline{S^{n+2} - N(K)}$. Recall that κ is *fibered* if X is a fiber bundle over S^1; the fiber is then $M_0 = \overline{M - B}$, where M is a closed, connected, orientable $(n+1)$-manifold and B is an $(n+1)$-ball in M, and X is homeomorphic to the identification space $M_0 \times I/f = M_0 \times I/((x,0) \sim (f(x),1)$ for all $x \in M_0)$, for some orientation preserving homeomorphism $f : M_0 \to M_0$, the *monodromy* of the bundle.

The observation that lies behind Theorem 1 is the following, the first part of which is due to Ruberman [10].

Proposition 3. *Let κ be a fibered knot with fiber M_0 and monodromy f. If κ is reversible then M_0 and $-M_0$ are h-cobordant* rel ∂. *Moreover, the orientation reversing self-homotopy equivalence $g : M_0 \to M_0$ induced by the h-cobordism satisfies $fgf \simeq g$.*

Before giving the proof of Proposition 3, we show how it implies Theorem 1.

First note that attaching $B \times I$ to the h-cobordism between M_0 and $-M_0$ in the obvious way gives an h-cobordism between M and $-M$. Also, the corresponding extension of g to M and any extension of f to M still satisfy $fgf \simeq g$.

Next recall Zeeman's theorem on twist-spinning [14]: if κ is an n-knot, and m is a positive integer, then $\kappa^{(m)}$, the m-twist-spin of κ, is an $(n+1)$-knot which is fibered with fiber M_0 and monodromy f, where M is the m-fold branched cyclic covering of κ, and f is the restriction to M_0 of the canonical covering translation of M.

Proof of Theorem 1

(1) If $\kappa_{p/q}$ is the rational knot associated with the rational number p/q, then the 2-fold branched covering of $\kappa_{p/q}$ is the lens space $L(p,q)$. Hence Proposition 3 implies that if the 2-twist-spin $\kappa^{(2)}_{p/q}$ is reversible then $L(p,q)$ and $-L(p,q)$ are h-cobordant. By the G-signature theorem (see [1, p.479]), two lens spaces are h-cobordant if and only if they are homeomorphic as oriented manifolds. Hence if $\kappa^{(2)}_{p/q}$ is reversible then $q^2 \equiv -1 \pmod p$, which implies that $\kappa_{p/q}$ is amphicheiral [12]. (Since $\kappa_{p/q}$ is reversible, $(+)$- and $(-)$-amphicheirality coincide.) On the other hand, for any n-knot κ, if κ is $(+)$-amphicheiral then $\kappa^{(m)}$ is reversible [5].

(2) Let $\tau_{p,q}$ denote the (p,q) torus knot. The m-fold branched cyclic covering M of $\tau_{p,q}$ is a Seifert fiber space. Since $\tau_{2,q}$ is a non-amphicheiral rational knot, we

may assume by part (1) that either $m > 2$ or p and q are both > 2. Then M is not a lens space (including S^3 and $S^2 \times S^1$); see for example [8, Theorem 1]. Also, the Euler number $e(M) \neq 0$, by [7, Theorem 1.2] (see [8]). Hence by [7, Theorem 8.2], M admits no orientation reversing self-homotopy equivalence. It follows from Proposition 3 that $\tau_{p,q}^{(m)}$ is irreversible.

(3) If κ is the figure eight knot then κ is (+)-amphicheiral, and hence $\kappa^{(m)}$ is reversible [5], so the theorem holds in this case.

If κ is a hyperbolic knot other than the figure eight knot, and $m \geq 3$, then the m-fold branched cyclic covering M of κ is hyperbolic, and the canonical covering translation $f : M \to M$ is an isometry [2], [3]. Let $\tilde{K}$ be the (geodesic) fixed point set of f, and let N be a tubular neighborhood of $\tilde{K}$, consisting of all points of M within some sufficiently small distance of $\tilde{K}$. Note that N can be parametrized as $S^1 \times D$, where each meridian disk $\{x\} \times D$ is a geodesically embedded copy of the disk D of some radius centered at the origin $(0, 0)$ in the disk model of $\mathbb{H}^2$, and where $\tilde{K} = S^1 \times \{(0, 0)\}$. Then $f(N) = N$, and, taking polar co-ordinates on D, $f|N$ is given by $f(x, (r, \theta)) = (x, (r, \theta + \frac{2\pi}{m}))$.

Now suppose that $\kappa^{(m)}$ is reversible, and let $g : M \to M$ be the degree -1 homotopy equivalence given by Proposition 3. By [6], $g \simeq \gamma$, where γ is an isometry. Since $f\gamma f \simeq \gamma$, we have, again by [6], that $f\gamma f = \gamma$. In particular, $\gamma(\tilde{K}) = \tilde{K}$. There are two possibilities: (i) $\gamma|\tilde{K}$ is orientation preserving, and (ii) $\gamma|\tilde{K}$ is orientation reversing.

In case (ii), $\gamma|N$ is of the form $\gamma(x, d) = (\alpha(x), \beta_x(d))$, where $\beta_x : D \to D$ is some orientation preserving isometry. Hence β_x is given by $\beta_x(r, \theta) = (r, \theta + \theta_x)$, for some θ_x. Then $f\gamma f(x, (r, \theta)) = (\alpha(x), (r, \theta + \theta_x + \frac{4\pi}{m}))$, and hence, since $m \geq 3$, $f\gamma f \neq \gamma$, a contradiction.

It follows that case (i) must hold. Since $f\gamma f = \gamma$, γ induces an orientation reversing homeomorphism $h : S^3 \to S^3$, such that $h(K) = K$ and $h|K$ is orientation preserving. Thus κ is (+)-amphicheiral.

As noted above, the converse is proved in [5]. □

Proof of Proposition 3

The first part of the statement is due to Ruberman [10]; we include a proof for completeness. Let X be the exterior of κ, so we have $X \simeq M_0 \times I/f$. Suppose that κ is reversible. Then we have an orientation preserving homeomorphism $h : S^{n+2} \to S^{n+2}$ such that $h(K) = K$ and $h|K$ is orientation reversing. By an isotopy we may assume that $h(N(K)) = N(K)$, and that, under the homeomorphism $(N(K), K) \cong (S^n \times D^2, S^n \times \{(0, 0)\})$, $h|N(K) = \alpha \times \beta$, where $\alpha : S^n \to S^n$ is some orientation reversing homeomorphism, and $\beta : D^2 \to D^2$ is given by $\beta(r, \theta) = (r, -\theta)$. Lifting the restriction $h|X$ to the infinite cyclic covering of X, we get an orientation preserving homeomorphism $\tilde{h} : M_0 \times \mathbb{R} \to M_0 \times \mathbb{R}$, such that $\tilde{h}|S^n \times \mathbb{R} = \alpha \times \varepsilon$, where $\varepsilon : \mathbb{R} \to \mathbb{R}$ is given by $\varepsilon(t) = -t$.

Let $M_0' = \tilde{h}(M_0 \times \{0\})$, and choose $t \in \mathbb{R}$, $t > 0$, so that $M_0 \times \{t\}$ is disjoint from

M_0'. Let W be the compact submanifold of $M_0 \times \mathbb{R}$ cobounded by $M_0 \times \{t\}$ and M_0'. Orient M_0 and $\mathbb{R}$, and thereby $M_0 \times \mathbb{R}$, and orient M_0' so that the induced orientation on $\partial M_0' = S^n \times \{0\}$ is the same as that induced by $M_0 \times \{0\}$. Thus W is an oriented cobordism rel ∂ between $M_0 \times \{t\}$ and M_0'. Note that the homeomorphism $\tilde{h}|M_0 \times \{0\} : M_0 \times \{0\} \to M_0'$ is then orientation reversing, since $\tilde{h}|S^n \times \{0\} : S^n \times \{0\} \to S^n \times \{0\}$ is the orientation reversing homeomorphism α. Hence W is an oriented cobordism rel ∂ between M_0 and $-M_0$.

Now $\overline{M_0 \times \mathbb{R} - W} = U \amalg V$, where $U = M_0 \times [t, \infty)$ and $V = \tilde{h}(M_0 \times [0, \infty))$. Hence there is a strong deformation retraction $M_0 \times \mathbb{R} \to W$. Since the inclusions of $M_0 \times \{t\}$ and M_0' into $M_0 \times \mathbb{R}$ are homotopy equivalences, it follows that W is an h-cobordism.

Let $i_0 : M_0 \to M_0 \times \mathbb{R}$ be the inclusion map $i_0(x) = (x, 0)$, and let $p : M_0 \times \mathbb{R} \to M_0$ be projection onto the first factor. Then the orientation reversing self-homotopy equivalence $g : M_0 \to M_0$ induced by the h-cobordism W is given by $g = p\tilde{h}i_0$.

The group of covering translations of the infinite cyclic covering $M_0 \times \mathbb{R}$ of X is generated by $T : M_0 \times \mathbb{R} \to M_0 \times \mathbb{R}$, where $T(x,t) = (f(x), t+1)$. Note that $\tilde{h}$ is the lift of h that takes $S^n \times \{0\}$ to $S^n \times \{0\}$, and $\tilde{h}T$ is the lift of h that takes $S^n \times \{0\}$ to $S^n \times \{-1\}$. Hence $\tilde{h}T = T^{-1}\tilde{h}$, giving $T\tilde{h}T = \tilde{h}$. Let $S : M_0 \times \mathbb{R} \to M_0 \times \mathbb{R}$ be given by $S(x,t) = (x, t+1)$. Observe that $i_0 f = TS^{-1}i_0$, and that $fp = pT$. Then $fgf = fp\tilde{h}i_0 f = pT\tilde{h}TS^{-1}i_0 = p\tilde{h}S^{-1}i_0 \simeq p\tilde{h}i_0 = g$, since $S \simeq id$. □

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