Appendix: Summary of Prerequisits

In this appendix, we will recount for the reader's convenience some of the theory of ∞-categories, higher algebra, and spectral algebraic geometry, which we will use throughout the rest of this work. We do not strive to give a complete presentation; rather we do little more than fix notation and try to get across some intuition. In particular, we will abstain from giving proofs. For a detailed and exhaustive treatment, see HTT, HA, and SAG.

A.1. **Higher category theory.** The title of this subsection is somewhat misleading: we will be concerned with ∞-categories, or more precisely $(\infty, 1)$-categories, which are intuitively analogues of categories which possess objects, morphisms, morphisms between morphisms, and so on indefinitely, but all higher morphisms (morphisms between morphisms and further up) are required to be invertible. That is a very versatile context for homotopical reasoning and does subsume usual category theory, that is to say $(1,1)$-categories, but *does not* subsume what is classically called 2-categories, which might be more precisely termed $(2,2)$-categories, or higher n-categories for $n \geq 2$.

A.1.1. *Simplicial sets.* The formal backbone of ∞-category theory is the theory of simplicial sets. Let $\mathbf{\Delta}$ denote the *simplex category*, i.e. the category of non-empty finite linearly ordered sets with not-necessarily-strictly order-preserving maps. Equivalently, objects of $\mathbf{\Delta}$ may be set to consist of $[n] = \{0 < 1 < \cdots < n\}$ for all non-negative integers n. The *category of simplicial sets* $\mathcal{S}\mathrm{et}_\Delta$ is defined to be the presheaf category on $\mathbf{\Delta}$. That is to say, a simplicial set is a functor $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ and a morphism of simplicial sets is a natural transformation.

The representable objects in $\mathcal{S}\mathrm{et}_\Delta$ are called *standard simplices* and denoted $\Delta^n := \mathrm{Hom}_{\mathbf{\Delta}}(-, [n])$. For any simplicial set $X \in \mathcal{S}\mathrm{et}_\Delta$, we denote $X_n := X([n])$ and call it the *n-simplices of X*. Sometimes 0-simplices will be referred to as *vertices.* It follows from the Yoneda lemma that for any simplicial set X the set of its n-simplices X_n is in bijective correspondence with morphisms $\Delta^n \to X$ in $\mathcal{S}\mathrm{et}_\Delta$.

To specify a simplicial set X though, it doesn't suffice to merely specify its sets of simplices $\{X_n\}_{n\geq 0}$. By definition, a simplicial set is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$, and we must also specify how it behaves with respect to morphisms in $\mathbf{\Delta}$. Fortunately morphisms in $\mathbf{\Delta}$ are quite simple; they are non-strictly increasing functions $[n] \to [m]$ and an easy inductive argument shows that they can all be built by finite composition out of morphisms

$$\delta_i : [n-1] \to [n], \qquad \sigma_i : [n+1] \to [n],$$

where δ_i is the function which skips the element $i \in [n]$ and σ_i hits it twice, or explicitly

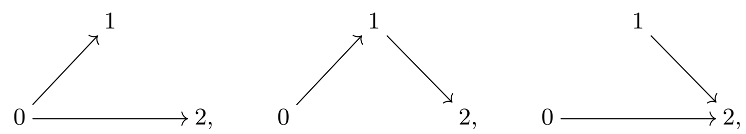
$$\delta_i(k) = \begin{cases} k & 0 \leq k < i, \\ k+1 & i \leq k \leq n, \end{cases} \qquad \sigma_i(k) = \begin{cases} k & 0 \leq k \leq i, \\ k-1 & i < k \leq n+1. \end{cases}$$

For any simplicial set X these induce functions $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$, called *faces* and *degeneracies* respectively. From the observation that any morphism in $\mathbf{\Delta}$ is a composition of various δ_i and σ_j, it follows that the simplicial set X can be completely recovered from its sets of n-simplices together with the collection of faces and degeneracies $\{d_i, s_i\}_{0\leq i\leq n}$ for every X_n. It is possible to write down an explicit list of identities that functions $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ for $1 \leq i \leq n$ must satisfy for them to define an $X \in \mathcal{S}\mathrm{et}_\Delta$, and this was indeed the original definition of simplicial sets.

Indispensable for defining ∞-categories in the next subsection will be the *i-horn of the standard simplex* Δ^n, a simplicial set denoted Λ^n_i with k-simplices

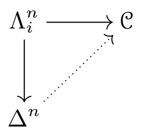
$$(\Lambda^n_i)_k = \{f \in \mathrm{Hom}_{\mathbf{\Delta}}([k],[n]) \mid [n] \smallsetminus \{i\} \notin f([k])\}.$$

Clearly $(\Lambda^i_n)_k \subseteq (\Delta^n)_k$ for every k and functoriality, i.e. simplicial set structure, of the horn is induced via this inclusion. Geometrically, the i-th horn Λ^n_i is obtained by removing the interior and the face opposite to the i-th simplex from Δ^n, so for insance the horns of Δ^2 may be depicted as



where these are Λ^2_0, Λ^2_1, and Λ^2_2 respectively.

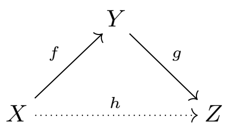
A.1.2. ∞-*categories.* An ∞-*category*[1] is a simplicial set $\mathcal{C} \in \mathcal{S}\mathrm{et}_\Delta$ which satisfies the *inner horn filling condition*: for every n and every $0 < i < n$, every solid diagram



in which the vertical arrow is the evident inclusion map, admits an extension to a commutative diagram in $\mathcal{S}\mathrm{et}_\Delta$ together with the dotted arrow. To get some feeling for how ∞-categories work, let us explain how to export several basic notions associated to ordinary categories to this context.

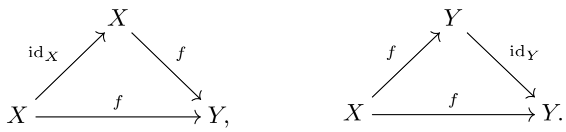
For an ∞-category $\mathcal{C}$, the set of vertices $\mathcal{C}_0$ is called *objects of* $\mathcal{C}$ and we often denote that X is an object of $\mathcal{C}$ by writing $X \in \mathcal{C}$. The 1-simplices of $\mathcal{C}$ are likewise *morphisms* and the face maps $d_0, d_1 : \mathcal{C}_1 \to \mathcal{C}_0$ are then called the *domain* and *codomain* or *source* and *target.* That is to say, for a morphism $f \in \mathcal{C}_1$, we often write $f : d_1(f) \to d_0(f)$ to designate between which pair of objects it goes. The image of the degeneracy $s_0 : \mathcal{C}_0 \to \mathcal{C}_1$ is called the *identity.* More precisely, for any object $X \in \mathcal{C}$ we call the morphism $s_0(X) : X \to X$ *the identity morphism on* X and denote it id_X.

The horn filling condition allows us to define composition: for a pair of morphisms $f : X \to Y$ and $g : Y \to X$, we may define a map of simplicial sets $\Lambda^2_1 \to \mathcal{C}$ which selects in $\mathcal{C}$ the solid diagram

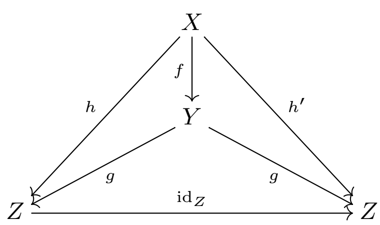


and the inner horn filling condition guarantees the existence of a morphism $g \circ f : X \to Z$ as denoted, together with a filling of the triangle $\Delta^2 \to \mathcal{C}$ which we say *exhibits* h as the composite of g and g. There is no guarantee as to uniqueness of h and indeed we can not expect it to be such - as we had seen, the arrow h is not *equal* to subsequently traversal of arrows f and g, but is instead connected to it by a 2-simplex. Intuitively, there is only a *homotopy* $h \simeq g \circ f$. This is an example of a common phenomenon in ∞-category theory: picking out a specific morphism, or object of some other sort, is most often impossible. The best we can do is pick out a homotopy class of such objects.

There are distinguished compositions of any morphism $f : X \to Y$ together with id_X or with id_Y. It is given by the respective degeneracy map $s_0, s_1 : \mathcal{C}_1 \to \mathcal{C}_2$ evaluated at f, and they give rise to 2-simplices in $\mathcal{C}$ of the form



Though the choice of a composition of two morphisms is not unique, it is *essentially unique*, or better unique up to a contractible space of choices. For example, suppose we are given a pair of 2-simplices $\sigma, \sigma' : \Delta^2 \to \mathcal{C}$ with $f = d_2(\sigma) = d_2(\sigma')$ and $g = d_0(\sigma) = d_0(\sigma')$, which therefore exhibit the morphism $h = d_1(\sigma)$ and $h' = d_1(\sigma')$ respectively as a composition on f and g in $\mathcal{C}$. Then the following collection of 2-simplices in $\mathcal{C}$



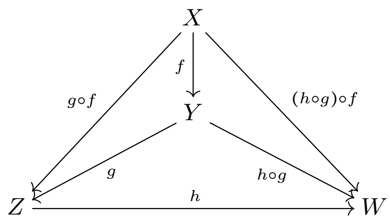
defines a map of simplicial sets $\Lambda^3_1 \to \mathcal{C}$ and by inner horn filling it extends uniquely to a 3-simplex in $\mathcal{C}$. Taking the 1-face of this 3-simplex, we obtain a 2-simplex that witnesses a homotopy between h and h'. We then write $h \simeq h'$ and say that the morphisms h and h' are *equivalent.*

We saw already that composition of a fixed pair of morphisms in an ∞-category exists, and is unique up to equivalence, however it is not unique and as such whenever we wish to use it, it does not suffice to merely *call upon its existence* to summon it, as we are used to doing in ordinary category theory, but

[1]Formally these are more correctly called *quasicategories.* There also exists a variety of other models of the intuitive idea of ∞-categories (or to be completely formal, $(\infty, 1)$-categories), and a significant literature on comparing them to each other and showing that they yield equivalent theories. Because of this, there is a drive in the ∞-category community to work model-independently, i.e. relying only on those features of higher category theory which should be present in all models. While we will mostly unconsciously abide by this policy, the quasicategorical model is, thanks to Lurie's seminal work, most highly developed, and since HTT and HA are our primary references, we will mostly refer to it and the technical achievements it boasts.

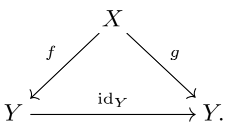
must instead *specify* a particular instance of a composite morphism. Actually, we must do more: we must specify a morphisms which is to be the composite of the two given morphisms, but then we must also give a 2-simplex which exhibits that this is the desired composite, i.e. the filling of the horn above. This paradigm of *conditions* becoming *additional data* is another hallmark of ∞-category theory.

Associativity may also be handled by horn filling: suppose we are given three morphisms $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$ in $\mathcal{C}$, together with a choice of compositions which we will abusively denote $g \circ f$, $h \circ h$ and $(h \circ h) \circ h$. This amounts to specifying three 2-simplices in $\mathcal{C}$ which together form a simplicial subset

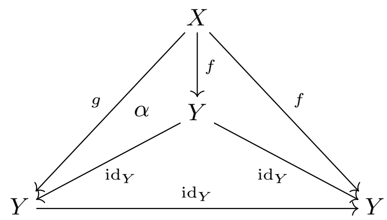


in $\mathcal{C}$. This can clearly be recognized as a map $\Lambda^3_1 \to \mathcal{C}$ which extends by horn filling to $\Delta^3 \to \mathcal{C}$. Its new face exhibits an equivalence $(h \circ g) \circ f \simeq h \circ (g \circ f)$. Therefore the associativity for the composition of three morphisms is exhibited by a 3-simplex in $\mathcal{C}$. An analogous argument shows that composition of strings of composable morphisms of length n in an ∞-category is associative for every n, and this associativity is witnessed by a particular n-simplex in $\mathcal{C}$. Therefore associativity of composition in an ∞-category holds only up to a coherent system of equivalences - in notation of homotopy theory, we might say that composition is associative in the $\mathbb{A}_\infty$ sense.

In the introduction to this subsection of the appendix, we mentioned that all higher morphisms in an ∞-category are invertible. Let us examine one incarnation of this, for 2-morphisms. Given a parallel pair of morphisms (1-morphisms, if you wish) $f, g : X \to Y$ in an ∞-category $\mathcal{C}$, we may define a 2-morphism $\alpha : f \to g$ to be a 2-simplex $\alpha : \Delta^2 \to \mathcal{C}$ of the form



Given such a 2-morphism, we may define a map $\Lambda^3_1 \to \mathcal{C}$ which may be depicted as the subsimplex



in $\mathcal{C}$. Inner horn filling condition asserts that it extends to a 3-simplex $\Delta^3 \to \mathcal{C}$ and the new 2-face of this simplex gives a 2-morphism $g \to f$ which is a (left) inverse to α in terms of the evident composition of 2-morphisms. Of course, analogous arguments show that n-morphisms in $\mathcal{C}$ are invertible for every $n \geq 2$. Therefore the notion of an ∞-category, as we have defined it via simplicial sets, realizes the heuristic idea of an $(\infty, 1)$-category.

A.1.3. *Nerve of an ordinary category.* We have defined ∞-categories as a particular sort of simplicial sets, but we would obviously desire the theory of ∞-categories to generalize usual category theory. Fortunately there is a canonical way of associating a simplicial set to an (ordinary) category.

The *nerve* of the category $\mathcal{C}$ is the simplicial set $\mathrm{N}\mathcal{C}$ defined by $\mathrm{N}\mathcal{C}_n = \mathrm{Hom}_{\mathcal{C}\mathrm{at}}([n], \mathcal{C})$, where the poset $[n]$ is identified with a category in the usual way, that is to say the category which may be represented as $0 \to 1 \to \cdots \to n$. The n-simplices of $\mathrm{N}\mathcal{C}$ are composable sequences of morphisms in $\mathcal{C}$ of length n. Face and degeneracy maps are given by composing or inserting the identity, which is to say

$$
\begin{aligned}
d_i\Big(C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} C_n\Big) &= C_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{i-1}} C_{i-1} \xrightarrow{f_{i+1} \circ f_i} C_{i+1} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_n} C_n, \\
s_i\Big(C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} C_n\Big) &= C_0 \xrightarrow{f_1} \ldots \xrightarrow{f_i} C_i \xrightarrow{\mathrm{id}_{C_i}} C_i \xrightarrow{f_{i+1}} \cdots \xrightarrow{f_n} C_n.
\end{aligned}
$$

It is easy to see that the nerve contains all the information about the category. In fact, the objects, morphisms, etc. of the category $\mathcal{C}$ may be inferred from its nerve $\mathrm{N}\mathcal{C}$ in precisely the same way in which we defined their analogues for an ∞-category in the previous subsection. It is not hard to see that $\mathrm{N}\mathcal{C}$ satisfies the inner horn filling property, where the filling of a horn is defined via composition of morphisms in $\mathcal{C}$.

The nerve of any category is hence an ∞-category. From the description of functors between ∞-categories in the next subsection, it easily follows that the nerve functor $\mathrm{N} : \mathcal{C}\mathrm{at} \to \mathcal{S}\mathrm{et}_\Delta$ is fully faithful. The nerve construction is also compatible with functors in the sense that

$$\mathrm{Fun}\,(\mathrm{N}\mathcal{C}, \mathrm{N}\mathcal{D}) \simeq \mathrm{N}\,\mathrm{Fun}\,(\mathcal{C}, \mathcal{D}),$$

where Fun on the left hand side is the ∞-category of functors between two ∞-categories and Fun on the right hand side is the ordinary functor category. Therefore there is no loss in identifying ordinary categories with the ∞-categories given by their nerves, and dropping N from notation.

A.1.4. *Functors.* Given two ∞-categories $\mathcal{C}$ and $\mathcal{D}$, we define a *functor* $F : \mathcal{C} \to \mathcal{D}$ to be any morphism of simplicial sets, or equivalently, natural transformation between the respective functors. This consists of a map $F_n : \mathcal{C}_n \to \mathcal{D}_n$ for every n, which may for $n = 0$ be identified with the object map of the functor and for $n = 1$ with the morphism map of the functor. From the preceding discussion it follows that an n-simplex of an ∞-category $\mathcal{C}$ may be identified with a string of composable morphisms in $\mathcal{C}$ of length n, together with the specification of all their possible compositions. Compatibility of the n-simplex map F_n with the face maps d_i means that it is compatible with this structure and in particular respects composition, and compatibility with degeneracies s_i mean that F preserves identity morphisms. Therefore the ∞-categorical notion of a functor carries the same intuition as functors classically do in ordinary category theory.

We may canonically upgrade functors between two ∞-categories to a simplicial set by setting[2]

$$\mathrm{Fun}\,(\mathcal{C}, \mathcal{D})_n := \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(\mathcal{C} \times \Delta^n, \mathcal{D})$$

and it turns out that the simplicial set $\mathrm{Fun}\,(\mathcal{C}, \mathcal{D})$ again satisfies the inner horn filling condition, so that $\mathrm{Fun}\,(\mathcal{C}, \mathcal{D})$ is itself an ∞-category. Furthermore the collection of all (small, but we will follow the good old practice in category theory of ignoring set theoretic issues) ∞-categories forms itself an ∞-category $\mathcal{C}\mathrm{at}_\infty$.

A.1.5. *The homotopy category.* By quotienting out the spaces of morphisms by the relation of equivalence, we may associate to an ∞-category $\mathcal{C}$ an ordinary category $\mathrm{h}\mathcal{C}$ or $\mathrm{Ho}(\mathcal{C})$, called the *homotopy category of* $\mathcal{C}$. Observe that

$$\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y) \simeq \pi_0\,\mathrm{Map}_{\mathcal{C}}(X, Y)$$

for any pair of objects $X, Y \in \mathcal{C}$. This gives rise to a functor $\mathrm{h} : \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}$ which is the left adjoint to the nerve functor $\mathrm{N} : \mathcal{C}\mathrm{at} \to \mathcal{C}\mathrm{at}_\infty$. That is to say, for any ∞-category $\mathcal{C}$ and ordinary category $\mathcal{D}$, there is a natural equivalence

$$\mathrm{Map}_{\mathcal{C}\mathrm{at}_\infty}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Map}_{\mathcal{C}\mathrm{at}}(\mathrm{h}\mathcal{C}, \mathcal{D})$$

(recall that we are omitting N from notation) arising from the canonical functor $\mathcal{C} \mapsto \mathrm{h}\mathcal{C}$. We often refer to this functor as *passing to homotopy.*

Given a morphism $f : X \to Y$ in an ∞-category $\mathcal{C}$, we say that is is an *equivalence* if it induces an isomorphism upon passage to the homotopy category $\mathrm{h}\mathcal{C}$. If there exists an equivalence between two objects X and Y of an ∞-category $\mathcal{C}$, we shall say that X and Y are *equivalent* and write $X \simeq Y$.

Setting $\mathcal{C}$ to $\mathcal{C}\mathrm{at}_\infty$, we obtain a notion of equivalence between ∞-categories. However this admits a more explicit description. A functor $F : \mathcal{C} \to \mathcal{D}$ is called *fully faithful* if the induced map $\mathrm{Map}_{\mathcal{C}}(X, Y) \to \mathrm{Map}_{\mathcal{D}}(F(X), F(Y))$ is a homotopy equivalence of spaces for every pair of objects $X, Y \in \mathcal{C}$, and it is called *essentially surjective* if the functor it induces on homotopy $\mathrm{h}F : \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$ is essentially surjective in the usual meaning of category theory, i.e. for every object $D \in \mathcal{D}$ there exists some $X \in \mathcal{C}$ such that $F(X) \simeq D$. Just like between ordinary categories, a functor between ∞-categories is an equivalence precisely when it is both fully faithful and essentially surjective.

Given an ∞-category $\mathcal{C}$, we call a *subcategory* and simplicial subset $\mathcal{D} \subseteq \mathcal{C}$ which is also itself an ∞-category and for which $\mathrm{h}\mathcal{C}$ is a subcategory of $\mathrm{h}\mathcal{D}$. The simplicial set inclusion then defines a functor of ∞-categories $\mathcal{D} \to \mathcal{C}$ and we say that $\mathcal{D}$ is a *full subcategory* if this functor is fully faithful, which is to say if $\mathrm{h}\mathcal{D}$ is a full subcategory of $\mathrm{h}\mathcal{C}$.

A.1.6. *Spaces are Kan complexes are ∞-groupoids.* We have already seen that all ordinary categories may be considered as ∞-categories via the nerve construction. Another crucial class of examples of ∞-categories is offered by spaces. This is where much of the motivation for ∞-category theory comes from: Grothendieck's realization that spaces may be equivalently thought as ∞-groupoids.

In simplicial approaches to homotopy theory, *space* is often taken to mean Kan complex, i.e. a simplicial set satisfying the horn filling condition. A Kan complex is clearly also an ∞-category, since these were defined to be simplicial sets satisfying horn filling only for inner horns.

The reader might object that the term space should be reserved for topological spaces (by which we shall always mean objects of some convenient category of topological spaces, e.g. compactly generated weakly Hausdorff spaces). We denote the category of spaces by $\mathcal{T}$, and remark that it actually carries more structure: it is a model category.

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[2]Observe that this is no cleaver trick, but rather a definition forced upon us by the fact that inner Hom, should it exist, has to be right adjoint to taking products. Since the Yoneda lemma gives a natural bijection between n-simplices of a simplicial set and simplicial set maps from Δ^n into it, we find ourselves before the given formula for $\mathrm{Fun}\,(\mathcal{C}, \mathcal{D})$.

foreshadowing that it arises as a homotopy category of an ∞-category of spaces.

Simplicial sets admit a natural functor to topological spaces. It comes from sending the standard n-simplex Δ^n to its geometric counterpart $|\Delta^n| := \{(x_0, \dots, x_n) \in [0,1]^{n+1} \mid x_0 + \dots + x_n \leq 1\}$ and then extending to an arbitrary simplicial set X by setting

$$|X| := \varinjlim_{\Delta^n \to X} |\Delta^n|.$$

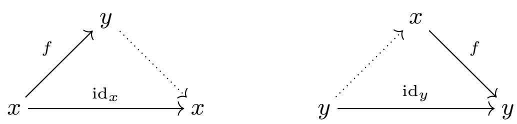
Recall that $\mathcal{S}\mathrm{et}_\Delta$ is a presheaf category in which Δ^n are the representable functors, which implies that $X \simeq \varinjlim_{\Delta^n \to X} \Delta^n$ in for any $X \in \mathcal{S}\mathrm{et}_\Delta$. Consequently the *geometric realization functor* $|-| : \mathcal{S}\mathrm{et}_\Delta \to \mathcal{T}$ commutes with colimits.

To any topological space X we may associate a Kan complex $\mathrm{Sing}^\bullet X$ defined as a functor by $[n] \mapsto \mathrm{Hom}_{\mathcal{T}}(|\Delta^n|, X)$. From this definition it follows quickly that for any simplicial set K and topological space X there is a natural equivalence

$$\mathrm{Hom}_{\mathcal{T}}(|K|, X) \simeq \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(K, \mathrm{Sing}^\bullet X),$$

showing that geometric realization and $\mathrm{Sing}^\bullet$ are adjoint functors. It is a classical theorem that this adjunction is a Quillen equivalence for certain model category structures on both side, which translates into an equivalence between the ∞-category of (convenient) topological spaces, obtained from the model category $\mathcal{T}$, and the ∞-category of Kan complexes, which comes from the standard model structure on $\mathcal{S}\mathrm{et}_\Delta$ in which Kan complexes are the fibrant objects. We will from now on not distinguish between these two ∞-categories, denoting both by $\mathcal{S}$ and calling their objects spaces.

In particular, the functor $\mathrm{Sing}^\bullet : \mathcal{S} \to \mathcal{C}\mathrm{at}_\infty$ is fully faithful and allows us to identify spaces with certain ∞-categories, namely with Kan complexes. Objects of $\mathrm{Sing}^\bullet(X)$ are precisely the points of X and a morphism $x \to y$ is a path between those points. From this it is evident that any morphism in the ∞-category $\mathrm{Sing}^\bullet(X)$ admits both a left and the right inverse, corresponding to traversing the same path in the opposite direction. This may also be witnessed on the level of the definition of a Kan complex. Given any morphism $f : x \to y$ in a Kan complex K, the solid diagrams



determine maps $\Lambda^2_0 \to K$ and $\Lambda^2_2 \to K$ (observe that these are the outer 2-horns, the ones excluded from the inner horn filling condition defining ∞-categories). The horn filling condition ensures that these extend to maps $\Delta^2 \to K$ and the new 1-simplex, denoted by the dotted arrow in the above diagrams, provides the let and right compositional inverse to f. We had seen how higher morphisms are invertible in any ∞-catgory in subsection **??**, thus this implies that in a Kan complex all morphisms are invertible. Conversely it is easy to see that this is nothing else than a restatement of the horn filling condition defining a Kan complex. Therefore spaces are to ∞-categories what groupoids are to categories, justifying Grothendieck's motto that spaces are ∞-groupoids.

An equivalent restatement of what it means for an ∞-category $\mathcal{C}$ to be an ∞-groupoid is to assert that the homotopy category $\mathrm{h}\mathcal{C}$ is a groupoid. For a topological space X, this groupoid may be recognized as the fundamental groupoid $\pi_{\leq 1}(X)$. This is the groupoid containing information about path-connected components of X and about homotopy classes of paths in X. Then by analogy the ∞-groupoid corresponding to X might be denoted $\pi_{\leq\infty}(X)$ and its n-morphisms encode homotopy classes of n-simplices in X for all $n \geq 0$. The assertion that Kan complexes are equivalent to topological spaces may then be states as saying that, unlike the fundamental groupoid $\pi_{\leq 1}(X)$, the "fundamental ∞-groupoid" $\pi_{\leq\infty}(X)$ loses no information about the homotopy type of the topological space X.

The geometric realization functor $|-| : \mathcal{C}\mathrm{at}_\infty \to \mathcal{S}$ is a left adjoint to the fully faithful inclusion $\mathcal{S} \to \mathcal{C}\mathrm{at}_\infty$ identifying spaces with ∞-groupoids, but said inclusion also admits a right adjoint $\mathcal{C} \mapsto \mathcal{C}^\simeq$. We call $\mathcal{C}^\simeq$ the *maximal subgroupoid* of $\mathcal{C}$ and its universal property may be restated as asserting a natural equivalence

$$\mathrm{Map}_{\mathcal{C}\mathrm{at}_\infty}(X, \mathcal{C}) \simeq \mathrm{Map}_{\mathcal{S}}(X, \mathcal{C}^\simeq)$$

for any space X. There is in particular a canonical (up to equivalence) functor $\mathcal{C}^\simeq \to \mathcal{C}$ corresponding to picking $X \simeq \mathcal{C}^\simeq$ and the identity map on the right of the above equivalence. This functor admits a more explicit description: the objects of the ∞-category $\mathcal{C}^\simeq$ are the same as those of $\mathcal{C}$, the space $\mathrm{Map}_{\mathcal{C}^\simeq}(C, D)$ consists for any $C, D \in \mathcal{C}$ of those components of the space $\mathrm{Map}_{\mathcal{C}}(C, D)$ which correspond to invertible morphisms in $\pi_0 \mathrm{Map}_{\mathcal{C}}(C, D) \simeq \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(C, D)$, and the functor $\mathcal{C} \to \mathcal{C}^\simeq$ is the identity on objects and the inclusion $\mathrm{Map}_{\mathcal{C}^\simeq}(C, D) \to \mathrm{Map}_{\mathcal{C}}(C, D)$ on morphisms.

For a pair of ∞-categories $\mathcal{C}$ and $\mathcal{D}$ the ∞-category of functors $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ may in general possess non-invertible morphisms. The ∞-category of all (small) ∞-categories $\mathcal{C}\mathrm{at}_\infty$ must on the other hand have $\mathrm{Map}_{\mathcal{C}\mathrm{at}_\infty}(\mathcal{C}, \mathcal{D})$ be a space, which is to say an ∞-groupoid. It follows that $\mathrm{Map}_{\mathcal{C}\mathrm{at}_\infty}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D})^\simeq$, hinting that, just like the category of all (small) categories $\mathcal{C}\mathrm{at}$ is actually a 1-categorical shadow of a 2-category, so is $\mathcal{C}\mathrm{at}_\infty$ an $(\infty,1)$-categorical shadow of an $(\infty,2)$-category.

A.1.7. *Initial and terminal objects.* The true utility of ∞-categories is that a vast chunk of ordinary category theory may be transported to the ∞-categorical setting with very little changes.

For instance an object X of an ∞-category $\mathcal{C}$ is *initial* (resp. *terminal*) if it is an initial (resp. terminal) object of the ordinary category $\mathrm{h}\mathcal{C}$. Equivalently, that means that the space $\mathrm{Map}_{\mathcal{C}}(X,Y)$ (resp. $\mathrm{Map}_{\mathcal{C}}(Y,X)$) is non-empty and contractible for all objects $Y \in \mathcal{C}$. Just like in ordinary category theory, initial (resp. terminal) objects need not exist, but if they do, they are essentially unique, which is to say unique up to a contractible ambiguity.

Of course we do not need to deal with dual notions such as initial and terminal objects separately, but may treat them simultaneously by use of the opposite category. Recall that the simplex category $\mathbf{\Delta}$ may be identified with the category of all non-empty finite linearly ordered sets. This category admits an involution $\mathbf{\Delta} \to \mathbf{\Delta}$ obtained by simply reversing each order. Given any simplicial set $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$, its *opposite* may be defined by pre-composing it with that involution. For any ∞-category $\mathcal{C}$ the opposite simplicial set $\mathcal{C}^{\mathrm{op}}$ is also an ∞-category and called the *opposite ∞-category* of $\mathcal{C}$. Explicitly the objects of $\mathcal{C}^{\mathrm{op}}$ are just the objects of $\mathcal{C}$, while for all pairs $X, Y \in \mathcal{C}$ we have

$$\mathrm{Map}_{\mathcal{C}^{\mathrm{op}}}(X,Y) \simeq \mathrm{Map}_{\mathcal{C}}(Y,X).$$

This recovers the usual notion of an opposite category upon passage to homotopy, in the sense that there is a natural equivalence of categories $\mathrm{h}(\mathcal{C}^{\mathrm{op}}) \simeq (\mathrm{h}\mathcal{C})^{\mathrm{op}}$. Clearly initial objects in $\mathcal{C}^{\mathrm{op}}$ are precisely terminal objects in $\mathcal{C}$ and vice versa.

A.1.8. *Overcategories and undercategories.* In order to be able to discuss overcategories, undercategories, limits and colimits in the ∞-categorical setting, it is useful to recall the *join* operation of simplicial sets. It is defined by setting $\Delta^i \star \Delta^j := \Delta^{i+j+1}$ and extending to arbitrary simplicial sets by requiring $\star : \mathcal{S}\mathrm{et}_\Delta \times \mathcal{S}\mathrm{et}_\Delta \to \mathcal{S}\mathrm{et}_\Delta$ to preserve colimits in each factor separately (since any simplicial set may be written as a limit of standard simplices). Under geometric realization this corresponds to the geometric join operation, classically constructed by embedding two complexes K and L into a pair of trivially intersecting hyperplanes inside some big enough euclidean space, and then setting $K \star L$ to be the union of all line segments connecting any pair of a point in K and a point in L. In particular if $\mathcal{C}$ is an ∞-category, the *left cone* $\mathcal{C}^{\triangleleft} := \mathcal{C} \star \Delta^0$ and *right cone* $\mathcal{C}^{\triangleright} := \mathcal{C} \star \Delta^0$ realize the idea of adding a disjoint initial and disjoint terminal object to $\mathcal{C}$.

Given a functor of ∞-categories $F : \mathcal{C} \to \mathcal{D}$, the *overcategory over F* $\mathcal{D}_{/F}$ is defined by the universal property that for every ∞-category $\mathcal{E}$ there is an equivalence

$$\mathrm{Fun}\,(\mathcal{E}, \mathcal{D}_{/F}) \simeq \mathrm{Fun}\,(\mathcal{E} \star \mathcal{C}, \mathcal{D}) \times_{\mathrm{Fun}\,(\mathcal{C},\mathcal{D})} \{F\}.$$

By selecting $\mathcal{E}$ to be Δ^n, we may read off the above universal property the n-simplices of $\mathcal{D}_{/F}$ and hence prove the existence of $\mathcal{D}_{/F}$ as a simplicial set. Intuitively an object of the overcategory $\mathcal{D}_{/F}$ consists of an object $D \in \mathcal{D}$ together with maps $D \to F(C)$ for every object $C \in \mathcal{C}$, which must be compatible with functoriality of F up to coherent homotopy. More formally, objects of $\mathcal{D}_{/F}$ are precisely functors $\mathcal{C}^{\triangleleft} \to \mathcal{D}$ such that their restriction under the canonical functor $\mathcal{C} \to \mathcal{C}^{\triangleleft}$ is equivalent to F. These are the analogues of what is classically often called *cones over F*.

An object C in an ∞-category $\mathcal{C}$ may be identified with a functor $C : \Delta^0 \to \mathcal{C}$ from the trivial ∞-category with one object and no non-identity morphisms. Then the overcategory $\mathcal{C}_{/C}$ consists object-wise of all morphisms with codomain C in $\mathcal{C}$, and e.g. the space of morphisms between two objects $X \to C$ and $Y \to C$ is precisely the space of 2-simplices in $\mathcal{C}$ exhibiting commutativity of the triangle

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow \quad \swarrow & \\ & C & \end{array}$$

in which the two diagonal arrows are the structure morphisms of the two objects of $\mathcal{C}_{/C}$.

A.1.9. *Limits and colimits.* For any functor $F : \mathcal{C} \to \mathcal{D}$, a terminal object of the ∞-category $\mathcal{D}_{/F}$ is called the *limit* of F and denoted $\varprojlim F$. We will also often abusively refer to the object in $\mathcal{D}$ obtained as the image of the distinguished initial vertex of $\mathcal{C}^{\triangleleft}$ under the functor $\varprojlim F : \mathcal{C}^{\triangleleft} \to \mathcal{D}$ as the limit of F. As a terminal object, the limit of any functor is essentially unique in case it exists.

The *undercategory* $\mathcal{D}_{F/}$ may either be defined by passing to opposite categories from the overcategory, or directly by the universal property

$$\mathrm{Fun}\,(\mathcal{E}, \mathcal{D}_{F/}) \simeq \mathrm{Fun}\,(\mathcal{C} \star \mathcal{E}, \mathcal{D}) \times_{\mathrm{Fun}\,(\mathcal{C},\mathcal{D})} \{F\}$$

for any ∞-category $\mathcal{E}$. Objects of the undercategory $\mathcal{D}_{F/}$ are functors $\mathcal{C}^{\triangleright} \to \mathcal{D}$ which restrict to F on $\mathcal{C}$, which might be called cocones over F. An initial object of $\mathcal{D}_{/F}$ is called a *colimit* of F and denoted $\varinjlim F$.

If $F : \mathcal{C} \to \mathcal{D}$ is a functor and $\mathcal{D}$ is (the nerve of) an ordinary category, then $\varprojlim F$ and $\varinjlim F$ recover their usual meanings from ordinary category theory as the limit and colimit of the functor between ordinary categories $\mathrm{h}F : \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D} \simeq \mathcal{D}$.

The formal theory of limits and colimits in ∞-category theory is quite similar to its counterpart in ordinary category theory. We also have special names for certain limits and colimits: a family of objects $\{C_i\}_{i\in I}$ in an ∞-category $\mathcal{C}$ indexed by some indexing set I, may be viewed as a functor $I \to \mathcal{C}$ from the discrete ordinary category I (the set of objects is I and there are no non-identity morphisms) and its limit is called the *product* of the family $\{C_i\}_{i\in I}$ and denoted $\prod_{i\in I} C_i$, while the colimit of this functor is called the *coproduct* of the family $\{C_i\}_{i\in I}$ and denoted $\coprod_{i\in I} C_i$. When the indexing set is empty, this recovers the terminal and initial object respectively. When I has two elements and the family of objects in question is $\{X, Y\}$, the product is denoted $X \times Y$ and the coproduct $X \coprod Y$.

Given an object C of an ∞-category $\mathcal{C}$, the product in the overcategory $\mathcal{C}_{/C}$ is called *fibered product* and for two objects $X \to C$ and $Y \to C$ in $\mathcal{C}_{/C}$ the underlying object of $\mathcal{C}$ of their product is denoted $X \times_C Y$. That is to say, there exists a commutative square in $\mathcal{C}$

$$\begin{array}{ccc} X \times_C Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & C \end{array}$$

in which the two arrows ending at C are the structure morphisms of the two objects of $\mathcal{C}_{/C}$, such that this square is terminal among all such commutative squares. We call such a diagram *pullback square*, and the dual notion, corresponding to a coproduct $X \coprod_C Y$ in an undercategory $\mathcal{C}_{C/}$, a *pushout square.*

It should be pointed out that limits and colimits of ∞-categories tend to not be preserved upon passage to homotopy. In particular, if $X \times Y$ is the product of objects X and Y in an ∞-category $\mathcal{C}$, then it most likely will not be the case that is is also a product of X and Y in the ordinary category $\mathrm{h}\mathcal{C}$. This has to do with the fact that in homotopical categories, say in topological spaces, the formation of products (and (co)limits in general) is usually not homotopy invariant. But the ∞-categorical world is only capable of discussing notions up to homotopy, so non-homotopy-invariant notions are not well defined. Instead ∞-categorical limits and colimits recover the notion of *homotopy limits and colimits*, the derived functors (in Quillen's model category approach to homotopical category theory) of the usual limit and colimit functors.

For instance, if $\Delta : X \to X \times X$ denotes the diagonal map of any given space X, the diagram

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}} & X \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \Delta} \\ X & \xrightarrow{\Delta} & X \times X \end{array}$$

is a pullback square in the usual category of spaces. On the other hand this is *not* a pullback square in the ∞-category of spaces $\mathcal{S}$, where the pullback is instead given by

$$\begin{array}{ccc} \mathcal{L}X & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \Delta} \\ X & \xrightarrow{\Delta} & X \times X \end{array}$$

in which $\mathcal{L}X \simeq \mathrm{Map}_{\mathcal{S}}(S^1, X)$ is the free loop space on X and the unlabelled maps $\mathcal{L}X \to X$ are evaluations at a point of the circle.

A.1.10. *Preseheaves and adjoint functors.* For any ∞-category $\mathcal{C}$ we define the *presheaf ∞-category on* $\mathcal{C}$ to be $\mathcal{P}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$. Limits and colimits in functor categories are calculated object-wise, and since $\mathcal{S}$ possesses all (small) limits and colimits, the same holds for $\mathcal{P}(\mathcal{C})$. One universal property of the presheaf ∞-category is that for any ∞-category $\mathcal{D}$ there is a natural equivalence of ∞-categories (note that this is really an $(\infty, 2)$-categorical universal property)

$$\mathrm{Fun}(\mathcal{D}, \mathcal{P}(\mathcal{C})) \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{D}, \mathcal{S}).$$

Setting $\mathcal{D} \simeq \mathcal{C}$ and selecting from the right hand side the canonical functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$ given object-wise by $(X, Y) \mapsto \mathrm{Map}_{\mathcal{C}}(X, Y)$, we obtain a functor $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ which we call the *Yoneda embedding.* As the name suggests, this functor is a fully faithful embedding by courtesy of the ∞-categorical Yoneda lemma. Just like in the classical case, the objects of the essential image of j are called *representable presheaves.*

The Yoneda embedding allows us to talk about adjoints. Given a pair of functors of ∞-categories $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, we say that they are *adjoint*, or more precisely that F is *left adjoint to* G and that G is *right adjoint to* F, if there is a functorial equivalence

$$\mathrm{Map}_{\mathcal{D}}(F(C), D) \simeq \mathrm{Map}_{\mathcal{C}}(C, G(D))$$

for all objects $C \in \mathcal{C}$ and $D \in \mathcal{D}$. More precisely, the two functors F and G define a pair of functors

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \xrightarrow{F^{\mathrm{op}} \times \mathrm{id}_{\mathcal{D}}} \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{S}, \qquad \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \xrightarrow{\mathrm{id}_{\mathcal{C}^{\mathrm{op}}} \times G} \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S},$$

where the unlabelled arrows are the pairings obtained from the Yoneda embedding, and F is left adjoint to G precisely when these functors are equivalent as objects of the functor ∞-category $\mathrm{Fun}\,(\mathcal{C}^{\mathrm{op}} \times \mathcal{D}, \mathcal{S})$. Whenever a functor admits a left or right adjoint, said adjoint is essentially unique.

A hallmark property of adjoint functors, carried over from ordinary category theory, is that left adjoints preserve colimits and right adjoints preserve limits. There is likely no single other fact that we use quite as often or as potently in this thesis as this one.

Under certain conditions on ∞-categories, the converse to this statement is also true. The condition in question is presentability[3], a very powerful smallness condition that makes ∞-categories especially amenable to study. Namely, an ∞-category $\mathcal{C}$ is *presentable* if it admits all small colimits, the morphisms spaces $\mathrm{Map}_{\mathcal{C}}(X, Y)$ are small for all objects $X, Y \in \mathcal{C}$, and there exists a small set of objects that generates all objects of $\mathcal{C}$ under colimits. While only the existence of (small) colimits is postulated in the definition of a presentable ∞-category, it turns out that all (small) limits exist also. The ∞-category $\mathcal{S}$ of (small) spaces is a chief example of a presentable ∞-category.

For a functor between presentable ∞-categories, the Adjoint Functor Theorem asserts that left adjointness is equivalent to preservation of colimits and right adjointness is equivalent to preservation of limits. The Adjoint Functor Theorem might best be appreciated as an existence theorem, making various functors between presentable ∞-categories appear out of thin air.

Given a pair of ∞-categories $\mathcal{C}$ and $\mathcal{D}$, let us introduce some more notation. Let $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ and $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}, \mathcal{D})$ denote the full subcategories of the functor ∞-category $\mathrm{Fun}\,(\mathcal{C}, \mathcal{D})$ spanned by all those functors which are left and right adjoint respectively. That is to say, $F : \mathcal{C} \to \mathcal{D}$ belongs to $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ precisely when it possesses a right adjoint. When $\mathcal{C}$ and $\mathcal{D}$ are both presentable, this is equivalent to F preserving colimits, and belonging to $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}, \mathcal{D})$ is equivalent to preserving limits. With this new notation, we may formulate another universal property for $\mathcal{P}(\mathcal{C})$: for any presentable ∞-category $\mathcal{D}$ there is a natural equivalence

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{P}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}\,(\mathcal{C}, \mathcal{D}).$$

This in particular implies the fact that every presheaf $P \in \mathcal{P}(\mathcal{C})$ may be written as a colimit of representables, in the form $P \simeq \varinjlim_{C \in \mathcal{C}_{/P}} j(C)$.

Presentable ∞-categories may be organized into ∞-categories $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ and $\mathcal{P}\mathrm{r}^{\mathrm{R}}$ in which the space of morphisms between two arbitrary ∞-categories $\mathcal{C}$ and $\mathcal{D}$ is given by $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ and $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}, \mathcal{D})$ respectively. Associating to a left adjoint functor its right adjoint and vica versa defines an equivalence of ∞-categories $\mathcal{P}\mathrm{r}^{\mathrm{L}} \simeq (\mathcal{P}\mathrm{r}^{\mathrm{R}})^{\mathrm{op}}$, which is the identity on objects.

A.1.11. *Localization.* A functor of ∞-categories is called a *localization* if it possesses a fully faithful right adjoint. That is to say, we without loss of generality consider a localization as a functor $L : \mathcal{C} \to \mathcal{C}$ such that, if $L\mathcal{C}$ denotes the essential image of L in $\mathcal{C}$, the codomain-restricted functor $L : \mathcal{C} \to L\mathcal{C}$ is a left adjoint to the inclusion $L\mathcal{C} \to \mathcal{C}$. Given such a localization $L : \mathcal{C} \to \mathcal{C}$, let S denote the collection of all morphisms f in $\mathcal{C}$ for which Lf is an equivalence. Then for any ∞-category $\mathcal{D}$ composition with L defines a fully faithful embedding

$$\mathrm{Fun}\,(L\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}\,(\mathcal{C}, \mathcal{D}),$$

the essential image of which consists of all those functors $F : \mathcal{C} \to \mathcal{D}$ which carry all the morphisms in S to equivalences.

In many cases it is also possible to go backwards, starting from the collection of morphisms which the localization takes to equivalences. Suppose we are given S any small[4] collection of morphisms in a presentable ∞-category $\mathcal{C}$. An object $Z \in \mathcal{C}$ is *S-local* if for every $f : X \to Y$ in S the induced map $\mathrm{Map}_{\mathcal{C}}(Y, Z) \to \mathrm{Map}_{\mathcal{C}}(X, Z)$ is an equivalence. Let $S^{-1}\mathcal{C}$ to be the full subcategory of $\mathcal{C}$ spanned by all S-local objects. Then $S^{-1}\mathcal{C}$ is the essential image of a localization $L : \mathcal{C} \to \mathcal{C}$ and furthermore every localization of $\mathcal{C}$ is of this for some S.

A.1.12. *∞-topoi.* Any presentable ∞-category may be realized as a special sort of localization of a presheaf category. If said localization functor $L : \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{C})$, exhibiting a presentable ∞-category $\mathcal{X} \simeq L\mathcal{P}(\mathcal{C})$, also preserves finite limits, then $\mathcal{X}$ is called an *∞-topos*[5]. This means that an ∞-topos is a full subcategory $\mathcal{X} \subseteq \mathcal{P}(\mathcal{C})$ of a presheaf ∞-category together with a localization $L : \mathcal{P}(\mathcal{C}) \to \mathcal{X}$, which plays the role of a sheafification functor.

[3]Presentability has another characterization: an ∞-category is *presentable* if and only if it arises from a combinatorial model category. It is a stroke of luck that the following motto therefore makes sense: an ∞-category is presentable if and only if it admits a model categorical "presentation".

[4]An important relaxation may be afforded here by instead only requiring S to contain a small subset which generates the same strongly saturated class of morphisms as S.

[5]Beware that an ∞-topos is an ∞-categorical generalization of a Grothendieck topos, and not of the more general notion of an elementary topos.

Indeed, suppose that $\mathcal{C}$ is a (small) ∞-category and let $\mathrm{h}\mathcal{C}$ carry a Grothendieck topology $\mathcal{J}$ (since $\mathrm{h}\mathcal{C}$ is an ordinary category, we mean here a Grothendieck topology in the usual sense). Let S denote the class of all monomorphisms $U \to j(C)$ in $\mathcal{P}(\mathcal{C})$ which correspond to covering sieves of objects C in $\mathcal{C}$ (or equivalently in $\mathrm{h}\mathcal{C}$) with respect to the Grothendieck topology $\mathcal{J}$. Then we define the *∞-category of sheaves on $\mathcal{C}$ with respect to $\mathcal{J}$* to be $\mathrm{Shv}(\mathcal{C}, \mathcal{J}) := S^{-1}\mathcal{P}(\mathcal{C})$. The condition on a presheaf $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ to be a sheaf is therefore the same as in classical topos theory: for every covering sieve $U \to j(C)$ with respect to the chosen Grothendieck topology, the canonical map

$$\mathscr{F}(C) \simeq \mathrm{Map}_{\mathcal{P}(\mathcal{C})}(j(C), \mathscr{F}) \to \mathrm{Map}_{\mathcal{P}(\mathcal{C})}(U, \mathscr{F}),$$

where we have used the Yoneda lemma on the left, must be an equivalence.

For use in spectral algebraic geometry, we will require also sheaves with values in various other ∞-categories than just $\mathcal{S}$, such as spectra and $\mathbb{E}_\infty$-rings. It is beneficial to be more general from the outset and define sheaves on an arbitrary ∞-topos $\mathcal{X}$ with values in an arbitrary ∞-category $\mathcal{C}$ with all small limits. A *$\mathcal{C}$-valued sheaf of $\mathcal{X}$* is a functor $\mathscr{F} : \mathcal{X}^{\mathrm{op}} \to \mathcal{C}$ which preserves small products. Of particular interest is the ∞-topos corresponding to a space X, which is to say $\mathcal{X} \simeq \mathrm{Shv}(\mathcal{U}(X))$ where $\mathcal{U}(X)$ is the poset of open subsets in X considered as a category with the usual Grothendieck topology. In this ∞-topos, a $\mathcal{C}$-valued sheaf is equivalent to a functor $\mathscr{F} : \mathcal{U}(X)^{\mathrm{op}} \to \mathcal{C}$ satisfying the familiar-looking *sheaf condition*: for any open set $U \subseteq X$ the canonical morphism

$$\mathscr{F}(U) \to \varprojlim_{V \subseteq U} \mathscr{F}(V),$$

ranging over all the open subsets of U, is an equivalence in $\mathcal{C}$.

The ∞-category of (unless stated otherwise always $\mathcal{S}$-valued) sheaves $\mathrm{Shv}(\mathcal{C}, \mathcal{J})$, which we may also denote $\mathrm{Shv}(\mathcal{C})$ when the Grothendieck topology is clear from the context, is an ∞-topos. However unlike the situation in ordinary topos theory, not every ∞-topos needs be of this form.

A.1.13. *Effective epimorphisms.* Just like ordinary (Grothendieck) topoi do among ordinary categories, ∞-topoi admit an elegant characterization among ∞-categories by the Giradu axioms. Let us mention only one, the analogue of which is in the classical topos theory is usually stated in the form that all equivalence relations in a topos are effective. To formulate its ∞-categorical analogue, we need a new notion.

A *groupoid object in $\mathcal{X}$* is a functor $U : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{X}$ such that for any non-negative integer n and any partition $[n] = S \cup S'$ into subsets for which the intersection $S \cap S' = \{s\}$ is a singleton, the induced diagram

$$\begin{array}{ccc} U([n]) & \longrightarrow & U(S') \\ \downarrow & & \downarrow \\ U(S) & \longrightarrow & U(\{s\}) \end{array}$$

is a pullback square in $\mathcal{X}$. Conversely given any morphism $U_0 \to U_{-1}$ in $\mathcal{X}$, we may define a groupoid object U in $\mathcal{X}$ inductively by setting $U([0]) := U_0$ and requiring

$$\begin{array}{ccc} U([n+1]) & \longrightarrow & U_0 \\ \downarrow & & \downarrow \\ U([n]) & \longrightarrow & U_{-1} \end{array}$$

to be a pullback square for every $n \geq 0$. This is compatible with the convenient convention of denoting $U([n])$ simply by U_n. In the described case we call U the *Čech nerve* of the morphism $U_0 \to U_{-1}$ and denote it $\check{\mathrm{C}}(U_0 \to U_{-1})$, so that we have

$$\check{\mathrm{C}}(U_0 \to U_{-1})_n \simeq \underbrace{U_0 \times_{U_{-1}} \cdots \times_{U_{-1}} U_0}_{n+1}$$

for all non-negative integers n. A groupoid object U is *effective* if it admits a limit $U_{-1} \simeq \varinjlim U \in \mathcal{X}$ such that $U \simeq \check{\mathrm{C}}(U_0 \to U_{-1})$. The ∞-categorical Giradu axiom in question now asserts that every groupoid object in an ∞-topos is effective.

A closely related notion is that of an *effective epimorphism*, which is such a morphism $f : X \to Y$ that the canonical map $\varinjlim \check{\mathrm{C}}(f) \to Y$ is an equivalence. The above stated Giradu axiom may be restated as saying that for any groupoid object U in an ∞-topos $\mathcal{X}$ the morphism $U_0 \to \varinjlim U$ is an effective epimorphism.

A.1.14. *Unstraightening.* This is a construction which is essential in setting up much of the fundamentals of algebra in the ∞-categorical setting, and we are reviewing it here primarily because of the repeated encounters with it in the subsection **??**.

The idea is that, for any given ∞-category $\mathcal{C}$, a functor $F : \mathcal{C} \to \mathcal{C}\mathrm{at}_\infty$ consists of a particular sort of collection of ∞-categories $F(X)$ ranging over objects X of $\mathcal{C}$, and it is possible to specify very explicitly precisely what sort of a collection that is. Namely, we may collect them together into one big ∞-category $\mathcal{D}$, equipped with a functor $p : \mathcal{D} \to \mathcal{C}$ such that its fibres are $\mathcal{D}_X = \mathcal{D} \times_{\mathcal{C}} \{X\} \simeq F(X)$. Then the *unstraightening construction* exhibits an equivalence between $\mathrm{Fun}(\mathcal{C}, \mathcal{C}\mathrm{at}_\infty)$ and the ∞-category of coCartesian fibrations[6] $p : \mathcal{D} \to \mathcal{C}$ with morphisms consisting of those morphisms in the overcategory $\mathcal{C}\mathrm{at}_{\infty/\mathcal{C}}$ which preserve coCartesian lifts of morphisms in $\mathcal{C}$.

To illustrate one use of unstraightening, let X be a space, which we may identify with an ∞-groupoid. It can be shown that the requirement on a functor $\mathcal{C} \to X$ to be a coCartesian fibration, implies that $\mathcal{C}$ is also an ∞-groupoid. Furthermore the requirement that a relative map of spaces $Y \to Z$ over X preserve coCartesian lifts is always automatically satisfied, and so the unstraightening construction yields an equivalence $\mathrm{Fun}(X, \mathcal{S}) \simeq \mathcal{S}_{/X}$ between the ∞-category of space-valued functors on X and the overcategory of spaces over X. We will regularly employ make heavy use of this equivalence in subsequent chapters.

The real utility of unstraightening lies in the fact that, given two functors $F, G : \mathcal{C} \to \mathcal{C}\mathrm{at}_\infty$ with corresponding coCartesian fibrations $p : \mathcal{D} \to \mathcal{C}$ and $q : \mathcal{E} \to \mathcal{C}$, then natural transformations $F \to G$ correspond to just those morphisms $\mathcal{D} \to \mathcal{E}$ in $\mathcal{C}\mathrm{at}_{\infty/\mathcal{C}}$ which preserve coCartesian lifts. By relaxing this requirement, we may therefore obtain various extensions of the notion of functors between morphisms, which can be used to "cheat" into the theory of ∞-category, which is to say $(\infty, 1)$-categories, several notions which might more organically belong to the $(\infty, 2)$-world. We will encounter many examples of this in the next subsection.

A.2. **Higher categorical algebra.** Out of the sections comprising the Appendix, this one is probably the most cluttered and technical and the least readable. One the one hand, it introduces several of the key notions that we will spend the rest of the thesis studying, so its importance should not be underestimated. But on the other hand the formidable technical intricacies involved in setting up the theory, will have afterwards served their role, and become very efficiently swept under the rug of the notions they helped define. For instance, while ∞-operads feature quite prominently in this subsection, they will not appear anywhere else in the rest of this thesis.

A.2.1. *Symmetric monoidal ∞-categories.* Let $\mathcal{F}\mathrm{in}_*$ denote the (nerve of the) category whose objects are the sets $\langle n \rangle = \Delta^0 \cup \{1, 2, \dots, n\}$ for all non-negative integers n and whose morphisms are maps $\alpha : \langle n \rangle \to \langle m \rangle$ such that $\alpha(*) = *$. For a fixed n, we define the morphisms $\rho^i : \langle n \rangle \to \langle 1 \rangle$ in $\mathcal{F}\mathrm{in}_*$ for all $1 \le i \le n$ by sending i to 1 and all the other elements of $\langle n \rangle$ to $*$. A map $f : \langle m \rangle \to \langle n \rangle$ is called *inert*, if it arises by choosing a certain subset of $\langle m \rangle$, sending it to the base point $*$, and doing nothing else. In particular, the maps ρ^i are inert for all i, since they send all elements in $\langle n \rangle$ aside from i to the base-point.

A *symmetric monoidal ∞-category* is a functor $\mathcal{C}^\otimes : \mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty$ such that $\mathcal{C}(\Delta^0) \simeq \Delta^0$, and for every positive integer n, the morphisms

$$\mathcal{C}^\otimes(\langle n \rangle) \to \prod_{1 \le i \le n} \mathcal{C}^\otimes(\langle 1 \rangle),$$

defined by the maps ρ^i for $1 \le i \le n$, is an equivalence. The ∞-category $\mathcal{C}^\otimes(\langle 1 \rangle)$ is called the *underlying ∞-category of* $\mathcal{C}^\otimes$ and is denoted $\mathcal{C}$.

A map $\alpha : \langle n \rangle \to \langle m \rangle$ in $\mathcal{F}\mathrm{in}_*$ may be interpreted as a specification of which variables in a product are to move to which place (i should move to place $\alpha(i)$), which to multiply together (multiply $0 \le i \le n$ and $0 \le j \le n$ if $\alpha(i) = \alpha(j)$, i.e. if α assigns them to the same place), on which places to include the unit (those $1 \le j \le m$ which are not in the image of α), and which to forget (those $1 \le i \le n$ for which $\alpha(i) = *$, i.e. those to which α does not assign a place). A symmetric ∞-category therefore consists of an ∞-category $\mathcal{C}$ equipped with a functorial association of a functor $\mathcal{C}^n \to \mathcal{C}^m$ to every such specification α, such that the functors associated to ρ^i is the i-th coordinate projection $\mathrm{pr}_i : \mathcal{C}^n \to \mathcal{C}$. In line with this interpretation, we define the functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ to be the one corresponding to the morphism $\alpha : \langle 2 \rangle \to \langle 1 \rangle$ in $\mathcal{F}\mathrm{in}_*$ with $\alpha(1) = \alpha(2) = 1$. Likewise is $\mathbf{1} : \Delta^0 \to \mathcal{C}$, or equivalently an object $\mathbf{1} \in \mathcal{C}$, defined to correspond to the map $\Delta^0 \to \langle 1 \rangle$. Sometimes, when we will wish to highlight the dependence on the symmetric monoidal ∞-category, we will denote these also by $\otimes_{\mathcal{C}}$ and $\mathbf{1}_{\mathcal{C}}$.

The intuition behind the notion of an *∞-operad*[7] is similar: an ∞-operad $\mathcal{O}^\otimes$ should consist of an underlying ∞-category $\mathcal{O}$ together with just about any collections of homotopically compatible operations on it. The formal implementation of this idea is trickier. Taking a cue from the unstraightening construction,

[6]Do not be distraught by all the coCartesian business. It is a simple enough technical lifting condition, the details of which can be found in HTT, which underlies much of the unstraightening story, but which we will never in any way use, and therefore hope we will be excused for omitting from our already cumbersome review of ∞-category theory.

[7]The terminology here is perhaps a bit misleading: ∞-operads are the ∞-categorical generalizations of colored operads, also known as multicategories, a notion already more general than usual (i.e. monochromatic) operads.

a symmetric monoidal ∞-category may be equated with $\mathcal{C}^{\otimes}$ a coCartesian fibration $p : \mathcal{C}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ for some abusively denoted ∞-category $\mathcal{C}^{\otimes}$ with fibers $\mathcal{C}^{\otimes}_{\langle n \rangle} \simeq \mathcal{C}^{\otimes}(\langle n \rangle)$. Then ∞-operads are defined as certain types of functors $p : \mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$. Because we will only use ∞-operads as tools to talk about other objects of interest to us, let us indicate only very rougly what kind of conditions need to be imposed on a functor to make it an $\otimes$-operad. First of all, p-coCartesian lifts of all inert morphisms in $\mathcal{F}\mathrm{in}_*$ should exists. This implies that any inert map $f : \langle m \rangle \to \langle n \rangle$ gives rise to a functor $f_! : \mathcal{O}^{\otimes}_{\langle m \rangle} \to \mathcal{O}^{\otimes}_{\langle n \rangle}$ Secondly, in analogy with with symmetric monoidal ∞-categories, a choice of p-coCartesian lifts $\rho^i_! : \mathcal{O}^{\otimes}_{\langle n \rangle} \to \mathcal{O}^{\otimes}_{\langle 1 \rangle}$ for $1 \le i \le n$ together define an equivalence

$$\mathcal{O}^{\otimes}_{\langle n \rangle} \simeq \prod_{1 \le i \le n} \mathcal{O}^{\otimes}_{\langle 1 \rangle}.$$

Thirdly, a similar condition should also hold on the level of morphism spaces of the ∞-category $\mathcal{O}^{\otimes}$. By the way the definition of an ∞-operad is set up, an ∞-operad with the structure functor p a coCartesian fibration is by unstraightening equivalent to a symmetric monoidal ∞-category. Our interest will lie primarily with symmetric monoidal ∞-categories, but the language of ∞-operads, which is at this point a harmless additional level of generality, will prove very useful in discussing various structures related to them.

The *∞-category of symmetric monoidal ∞-categories* is defined to be the full subcategory[8] $\mathcal{C}\mathrm{at}^{\otimes}_{\infty}$ of the functor ∞-category $\mathrm{Fun}(\mathcal{F}\mathrm{in}_*, \mathcal{C}\mathrm{at}_{\infty})$. Its morphisms are called *symmetric monoidal functors.* While symmetric monoidal functors are for many purposes the correct class of functors to consider between symmetric ∞-categories, we shall specify another weaker class of morphisms in subsection **??**. It turns out that it is the latter instead of the former that is the better notion to consider for general ∞-operads.

A.2.2. *Examples of symmetric monoidal ∞-categories.* Let us list some examples of symmetric monoidal ∞-categories.

(i) The first example is $\mathrm{Comm}^{\otimes}$, the *commutative ∞-operad* (even though it is actually a symmetric monoidal ∞-category). It is defined as the composite $\mathcal{F}\mathrm{in}_* \to \Delta^0 \to \mathcal{C}\mathrm{at}_{\infty}$ of the terminal functor to the one-object ∞-category Δ^0, with the inclusion of the ∞-category Δ^0 into the ∞-category of all ∞-categories. Thus $\mathrm{Comm}^{\otimes}_n \simeq \Delta^0$ for every n, and $\mathrm{Comm}^{\otimes}$ is by construction the terminal symmetric monoidal ∞-category and more generally the terminal ∞-operad.

(ii) Now let $\mathcal{C}$ be any ∞-category with finite products. For a given positive integer n, let P denote the inclusion-ordered power set of $\{1, 2, \dots, n\} \subset \langle n \rangle$. Let $\mathcal{C}^{\times}(\langle n \rangle)$ denote the ∞-category of all functors $f : \mathrm{N}(P)^{\mathrm{op}} \to \mathcal{C}$ for which, given any subset $S \subseteq \{1, 2, \dots, n\}$, the morphisms $f(S) \to f(\{j\})$ induced by the element inclusions $\{j\} \subseteq S$ exhibit an equivalence $f(S) \simeq \prod_{j \in S} f(\{j\})$. This extends to a functor $\mathcal{C}^{\times} : \mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_{\infty}$ which we call the *Cartesian symmetric monoidal structure on* $\mathcal{C}$. Clearly the underlying ∞-category of $\mathcal{C}^{\times}$ is $\mathcal{C}$ and the multiplication operation $\mathcal{C}^n \to \mathcal{C}$ with respect to this symmetric monoidal ∞-category is just the product functor $(X_1, \dots, X_n) \mapsto X_1 \times \cdots \times X_n$.

(iii) Observe that given any symmetric monoidal ∞-category $\mathcal{C}^{\otimes} : \mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_{\infty}$, we may produce a new one by composing it with the self-equivalence $\mathcal{C}\mathrm{at}_{\infty}$ given by $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}}$. If $\mathcal{C}$ is the underlying ∞-category of $\mathcal{C}^{\otimes}$, then its opposite $\mathcal{C}^{\mathrm{op}}$ is the underlying ∞-category of $\mathrm{op} \circ \mathcal{C}^{\otimes}$. We may apply this construction in the case when $\mathcal{C}$ is an ∞-category closed under finite coproducts. Then $\mathcal{C}^{\mathrm{op}}$ has finite products, so it admits a Cartesian symmetric monoidal structure $(\mathcal{C}^{\mathrm{op}})^{\times}$. Then the symmetric monoidal ∞-category $\mathcal{C}^{\amalg} := \mathrm{op} \circ (\mathcal{C}^{\mathrm{op}})^{\times}$ is called *coCartesian symmetric monoidal ∞-structure on* $\mathcal{C}$. Its underlying ∞-category is $\mathcal{C}$ and its operation $\mathcal{C}^n \to \mathcal{C}$ is given by the coproduct functor $(X_1, \dots, X_n) \mapsto X_1 \coprod \cdots \coprod X_n$.

(iv) On the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of presentable ∞-categories with left adjoints for morphisms, there exists a distinguished symmetric monoidal ∞-category structure given by

$$\mathcal{C} \otimes \mathcal{D} := \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}).$$

The unit object for this symmetric monoidal structure is $\mathcal{S}$. The functor $\otimes : \mathcal{P}\mathrm{r}^{\mathrm{L}} \times \mathcal{P}\mathrm{r}^{\mathrm{L}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ preserves colimits in each variable. Furthermore, we have $\mathcal{C} \otimes \mathcal{S}_* \simeq \mathcal{C}_*$ and $\mathcal{C} \otimes \tau_{\le n}\mathcal{S} \simeq \tau_{\le n}\mathcal{C}$, showing that pointification and truncation are special cases of tensoring presentable ∞-categories.

A.2.3. *Lax symmetric monoidal functors.* Given a pair of ∞-operads $p : \mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ and $q : \mathcal{O}'^{\otimes} \to \mathcal{F}\mathrm{in}_*$, we define an *$\infty$-operad map from $\mathcal{O}^{\otimes}$ to $\mathcal{O}'^{\otimes}$* to be a functor $F : \mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ in the overcategory $\mathcal{C}\mathrm{at}_{\infty/\mathcal{F}\mathrm{in}_*}$, which preserves coCartesian lifts of all inert morphism in $\mathcal{F}\mathrm{in}_*$. Let $\mathrm{Alg}_{\mathcal{O}}(\mathcal{O}')$ denote the ∞-category of ∞-operad maps from $\mathcal{O}^{\otimes}$ to $\mathcal{O}'^{\otimes}$. In slightly greater generality, which will be very useful in subsection **??**,

[8]This is rather unfortunate notation since we are also using the notation $\mathcal{C}^{\otimes}$ for a symmetric monoidal ∞-category with underlying ∞-category $\mathcal{C}$ and monoidal operation $\otimes$. Therefore it might seems like $\mathcal{C}\mathrm{at}^{\otimes}_{\infty}$, instead of being the ∞-category of all symmetric monoidal ∞-categories, is some sort of symmetric monoidal structure on $\mathcal{C}\mathrm{at}_{\infty}$. However since we will not be using any symmetric monoidal structures on $\mathcal{C}\mathrm{at}_{\infty}$ other than the Cartesian one, we doubt this will be the cause of much confusion.

given two ∞-operad maps $p : \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ and $q : \mathcal{O}''^{\otimes} \to \mathcal{O}^{\otimes}$ we may also define an ∞-category $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{O}'')$ of *relative ∞-operad maps* to consist of commuting triangles

$$\begin{array}{ccc} \mathcal{O}'^{\otimes} & \longrightarrow & \mathcal{O}''^{\otimes} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & \mathcal{O}^{\otimes} & \end{array}$$

of $\otimes$-operad maps. When $\mathcal{O}^{\otimes} = \mathrm{Comm}^{\otimes}$, the terminal ∞-operad, we recover $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{O}'') \simeq \mathrm{Alg}_{\mathcal{O}'}(\mathcal{O}'')$.

Though ∞-operad maps are important in their own right, we will primarily be interested in some particular cases of them. When $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ are symmetric monoidal ∞-categories, the objects of $\mathrm{Alg}_{\mathcal{C}}(\mathcal{D})$ are called *lax symmetric monoidal functor from $\mathcal{C}^{\otimes}$ to $\mathcal{D}^{\otimes}$*. The property of a functor $F : \mathcal{C} \to \mathcal{D}$ between the underlying ∞-categories of two symmetric monoidal ∞-categories $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ being symmetric monoidal, is exhibited by some family of compatibility diagrams which must be homotopy coherent, in the sense that they possess mutually compatible fillings. For example, these fillings include equivalences

$$\mathbf{1}_{\mathcal{D}} \simeq F(\mathbf{1}_{\mathcal{C}}), \qquad F(X) \otimes_{\mathcal{D}} F(Y) \simeq F(X \otimes_{\mathcal{C}} Y)$$

for all objects X and Y in $\mathcal{C}$. The property of a functor F as above being lax symmetric monoidal is exhibited by the same compatibility diagrams, only that they are not required to have fillings, only compatible systems of morphisms. E.g. there are compatible morphisms

$$\mathbf{1}_{\mathcal{D}} \to F(\mathbf{1}_{\mathcal{C}}), \qquad F(X) \otimes_{\mathcal{D}} F(Y) \to F(X \otimes_{\mathcal{C}} Y),$$

but they are not required to be equivalences. To formalize this idea as we have set it up here, we would require there to be non-invertible 2-cells in our ∞-category $\mathcal{C}$. However since we are working with $(\infty, 1)$-categories, this is not possible. The unstraightening construction provides a way around that, by encoding what would more naturally be $(\infty, 2)$-categorical data inside ∞-category theory[9]. Indeed, if we required coCartesian lifts of all morphisms in $\mathcal{F}\mathrm{in}_*$ to be preserved by F in the above definition, we would have rediscovered symmetric monoidal functors.

A.2.4. *Commutative algebras and commutative monoids.* Let $\mathcal{C}^{\otimes}$ be a symmetric ∞-category. The *∞-category of commutative algebra objects in $\mathcal{C}^{\otimes}$* is $\mathrm{CAlg}(\mathcal{C}) := \mathrm{Alg}_{\mathrm{Comm}}(\mathcal{C})$. Informally, a commutative algebra object $A \in \mathrm{CAlg}(\mathcal{C})$ consists of an *underlying object* denoted, by standard abuse of notation, also $A \in \mathcal{C}$, together with a map $\mathbf{1}_{\mathcal{C}} \to A$ in $\mathcal{C}$, which is the unit of A, a morphism $A \otimes A \to A$ in $\mathcal{C}$ which exhibits the operation on A, and these morphisms together satisfy all the possible compatibility relations up to coherent homotopies, encoded by lax symmetric monoidal functoriality with respect to $\mathrm{Comm}^{\otimes}$.

Commutative algebra objects with respect to a Cartesian symmetric monoidal structure admit a more explicit description. A *commutative monoid object* in an ∞-category with finite products $\mathcal{C}$ is a functor $X : \mathcal{F}\mathrm{in}_* \to \mathcal{C}$ such that for every positive integer n, the morphism $X(\langle n \rangle) \to X(\langle 1 \rangle)^n$ in $\mathcal{C}$, defined by the maps $\rho^i : \langle n \rangle \to \langle 1 \rangle$ in $\mathcal{F}\mathrm{in}_*$ for $1 \leq i \leq n$, is an equivalence. Let $\mathrm{CMon}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{F}\mathrm{in}_*, \mathcal{C})$ spanned by commutative monoid objects. When $\mathcal{C} = \mathcal{C}\mathrm{at}_\infty$ is the ∞-category of ∞-categories, then clearly $\mathrm{CMon}(\mathcal{C}) \simeq \mathcal{C}\mathrm{at}_\infty^{\otimes}$. That is to say, symmetric monoidal ∞-categories are the commutative monoid objects in $\mathcal{C}\mathrm{at}_\infty$. More generally, equipping $\mathcal{C}$ with the symmetric monoidal ∞-structure $\mathcal{C}^{\times}$, there is an equivalence

$$\mathrm{CAlg}(\mathcal{C}) \simeq \mathrm{CMon}(\mathcal{C}),$$

allowing us to express commutative algebra objects in a Cartesian symmetric monoidal ∞-category entirely internal to the underlying ∞-category $\mathcal{C}$. By the same consideration as in **??** for symmetric monoidal ∞-categories, we may see that the functoriality of a commutative monoid object X equips its underlying object $X := X(\langle 1 \rangle) \in \mathcal{C}$ with an operation $X \times X \to X$ which is commutative, associative, and unital, up to coherent homotopy.

Unlike in the Cartesian case, commutative algebra objects in a coCartesian symmetric monoidal ∞-category $\mathcal{C}^{\amalg}$ may be identified by the equivalence

$$\mathrm{CAlg}(\mathcal{C}) \simeq \mathcal{C}.$$

This is a sophisticated incarnation of a well-known phenomenon: for any space $X \in \mathcal{S}$, the diagonal map $\Delta : X \to X \times X$ exhibits the structure of a commutative coalgebra on X. That is the same as saying that it exhibits $X \in \mathrm{CAlg}(\mathcal{S}^{\mathrm{op}})$, where the symmetric monoidal structure on $\mathcal{S}^{\mathrm{op}}$ is given by the product of spaces in $\mathcal{S}$, and is therefore the coCartesian structure.

For any symmetric monoidal ∞-category $\mathcal{C}$, the ∞-category of commutative algebra objects $\mathrm{CAlg}(\mathcal{C})$ inherits a canonical symmetric monoidal structure from $\mathcal{C}$, and this structure is coCartesian. This is analogous to the classical fact that the tensor product is the coproduct in the category of commutative

[9]This is also the historic way the Grothendieck construction, the analogue of the (un)straightening construction for ordinary categories, first arose. Studying moduli spaces, Grothendieck encountered algebraic stacks, an inherently 2-categorical structure. The Grothendieck construction was his solution, designed as a way to package discussion of stacks inside the 1-categorical language of fibered categories.

rings. This means that coproducts, and colimits more generally, in $\mathrm{CAlg}(\mathcal{C})$ tend to differ greatly from the colimits of the underlying objects in $\mathcal{C}$. On the other hand, the forgetful functor $\mathrm{CAlg}(\mathcal{C}) \to \mathcal{C}$ does not only preserve limits, it even creates them, in the sense that $K^{\triangleleft} \to \mathrm{CAlg}(\mathcal{C})$ is a limit diagram if and only if its composite with the forgetful functor is.

A.2.5. *$\mathbb{E}_\infty$-spaces and infinite loop spaces.* Let $\mathrm{CMon} \coloneqq \mathrm{CMon}(\mathcal{S})$ be the *∞-category of $\mathbb{E}_\infty$-spaces*. An $\mathbb{E}_\infty$-space intuitively consists of a space X together with a map, which we will mostly write additively, $+ : X \times X \to X$ such that $(x+y)+z \simeq x+(y+z)$ for all $x, y, z \in X$ and all higher associativity and unitality laws hold up to coherent homotopy. As our choice of notion for their ∞-category suggests, $\mathbb{E}_\infty$-spaces are the homotopical analogues of commutative monoids from ordinary algebra.

The functor $\mathcal{S} \to \mathcal{C}\mathrm{at}_\infty$, which identifies spaces with ∞-groupoids, is the right adjoint of the maximal underlying space or core functor $\mathcal{C} \mapsto \mathcal{C}^{\simeq}$. In particular, it preserves products and as such extends to a symmetric monoidal functor between the Cartesian structures $\mathcal{S}^{\times} \to \mathcal{C}\mathrm{at}_\infty^{\times}$. It hence also induces a functor between commutative monoid objects $\mathrm{CMon} \to \mathcal{C}\mathrm{at}_\infty^{\otimes}$, since $\mathrm{CMon}(\mathcal{C}\mathrm{at}_\infty) \simeq \mathcal{C}\mathrm{at}_\infty^{\otimes}$. This means that, given a space X, specifying an $\mathbb{E}_\infty$-structure on it is equivalent to specifying a symmetric monoidal ∞-structure on it, if we view it as an ∞-groupoid.

A.2.6. *Monoids and monoidal ∞-categories.* Though our main interest resides with commutative phenomena, we will also consider their non-commutative analogues. As we have seen that the $\mathcal{F}\mathrm{in}_*$ controls commutative structures, so does the simplex category $\mathbf{\Delta}^{\mathrm{op}}$ control associative ones. Just like symmetric monoidal ∞-categories are equivalent to commutative monoid objects in the ∞-category $\mathcal{C}\mathrm{at}_\infty$, so can all monoidal ∞-categories be expressed in terms of monoid objects. It will turn out beneficial to consider a more general notion than that of a monoid.

Let $\mathcal{C}$ be an ∞-category with all finite products. A functor $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ is called a *simplicial object in* $\mathcal{C}$ and its n-simplices are usually denoted by $X_n \coloneqq X([n])$. A *category object in* $\mathcal{C}$ is a simplicial object $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ for which the maps $\rho^i : [1] \to [n]$ with $\rho^i(0) = i$, $\rho^i(1) = i-1$ for all $0 \leq i \leq n-1$ together exhibit an equivalence

$$X_n \simeq \underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{n}.$$

This requirement, called the Segal condition, is expressing the familiar property that every n-simplex in X is equivalent to a composition of a string of n composable 1-simplices.

Observe that groupoid objects treated in **??** are examples of category objects.

A category object X in $\mathcal{C}$, for which X_0 is a terminal object, is called a *monoid object in* $\mathcal{C}$. In that case we call $X_1 \in \mathcal{C}$ the *underlying object of the monoid object* and the Segal condition becomes $X_n \simeq X_1^n$, in clear analogy with the definition of a commutative monoid object in **??**.Let $\mathrm{Mon}(\mathcal{C})$ denote the *∞-category of monoid object in* $\mathcal{C}$, a full subcategory of $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{C})$. A monoid object in $\mathcal{C}\mathrm{at}_\infty$ is called a *monoidal ∞-category.* As the name suggests, the intuition behind monoidal categories is just like the one behind symmetric monoidal ∞-category, only without any commutativity assumptions.

There exists a canonical functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{F}\mathrm{in}_*$ defined by sending $[n] \mapsto \langle n \rangle$, the precise details of which will not matter for our purposes, more than to say that it induces a functor $\mathrm{CMon}(\mathcal{C}) \to \mathrm{Mon}(\mathcal{C})$ which associates the "underlying monoid object" to a commutative monoid object in any given ∞-category $\mathcal{C}$. Applying this to $\mathcal{C} = \mathcal{C}\mathrm{at}_\infty$, we can extract a monoidal ∞-category from a symmetric ∞-category $\mathcal{C}^{\otimes}$, which we will also denote by $\mathcal{C}^{\otimes}$. Conversely, given any monoidal ∞-category $\mathcal{C}^{\otimes}$, it corresponds by unstraightening to a certain coCartesian fibration $\mathcal{C}^{\otimes} \to \mathbf{\Delta}^{\mathrm{op}}$. Composing it with the functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{F}\mathrm{in}_*$, we obtain an ∞-operad. This allows us to identify a monoidal ∞-category $\mathcal{C}^{\otimes}$ with a certain ∞-operad, also denoted $\mathcal{C}^{\otimes}$ by the standard abuse of notation, but it will in general not be a symmetric monoidal ∞-category.

A.2.7. *Associative algebras.* In the previous subsection, we defined monoidal ∞-categories as particular kinds of monoid objects. In the discussion of commutative algebras, we saw that they only coincide with commutative algebra objects when considering Cartesian symmetric monoidal structure. To get the right notion of an associative algebra object in a monoidal category, we should encode the structure we are after operadically.

Recall the ordinary associativity operad, i.e. the multicategory **Ass** with one object $*$, for every finite set I the set of multimorphisms $\mathrm{Mul}_{\mathbf{Ass}}(\{*\}_{i\in I}, *)$ consisting of linear orderings on I, and composition defined by the evident merging of linear orderings. By the canonical way of identifying ordinary (possibly colored) operads with ∞-operads[10], we obtain the *associative ∞-operad* $\mathcal{A}\mathrm{ss}^{\otimes}$.

Let $\mathcal{C}^{\otimes}$ be a monoidal ∞-category. Viewing it as an ∞-operad as described at the end of the previous subsection, it admits a canonical ∞-operad map to $\mathcal{A}\mathrm{ss}^{\otimes}$. Then the ∞-category of relative ∞-operad maps $\mathrm{Alg}_{\mathcal{A}\mathrm{ss}/\mathcal{A}\mathrm{ss}}(\mathcal{C})$ is called the *∞-category of associative algebra objects in* $\mathcal{C}^{\otimes}$ and denoted simply by $\mathrm{Alg}(\mathcal{C})$.

[10]This is called the *operadic nerve* and the details concerning it may be found in HA. In particular, the commutative ∞-operad $\mathrm{Comm}^{\otimes}$ may also be identified with the operadic nerve of the classical commutative operad, the one-object multicategory with no non-identity multimorphisms.

An associative algebra object $A \in \mathrm{Alg}(\mathcal{C})$ intuitively consists of an underlying object $A \in \mathcal{C}$ together with a morphism $A \otimes A \to A$ in $\mathcal{C}$ which is unital and associative up to coherent homotopy.

Analogously to the commutative case treated in subsection **??**, when $\mathcal{C}^\times$ is a Cartesian (symmetric) monoidal structure, there is an equivalence

$$\mathrm{Alg}(\mathcal{C}) \simeq \mathrm{Mon}(\mathcal{C}).$$

In general for an symmetric monoidal ∞-category, there is a "underlying associative algebra object" functor $\mathrm{CAlg}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})$, which reduces to the underlying monoid object functor $\mathrm{CMon}(\mathcal{C}) \to \mathrm{Mon}(\mathcal{C})$ from the previous subsection.

A.2.8. *Left modules.* In a similar vein to $\mathcal{A}\mathrm{ss}^\otimes$, we may extract another ∞-operad $\mathcal{LM}^\otimes$, the *left module ∞-operad*, from an ordinary one. In particular, the colored operad $\mathbf{LM}$ is question has two objects $\mathfrak{a}$ and $\mathfrak{m}$ and, for any finite set I, the set of multimorphisms $\mathrm{Mul}_{\mathbf{LM}}(\{X_i\}_{i\in I}, Y)$ consists of

(1) linear orderings of I if $Y = \mathfrak{a}$ and $X_i = \mathfrak{a}$ for all $i \in I$.
(2) the empty set if $Y = \mathfrak{a}$ and $X_i \neq \mathfrak{a}$ for some $i \in I$.
(3) linear orderings $\{i_1 < \cdots < i_n\}$ of I such that $X_{i_n} = \mathfrak{m}$ and $X_j = \mathfrak{a}$ for all $j < i_n$, if $Y = \mathfrak{m}$.

Composition is given by the evident merging of linear orders, just like in $\mathbf{Ass}$. Clearly $\mathcal{LM}^\otimes$ is defined to capture the idea of a left action of an associative algebra object, with the color $\mathfrak{a}$ standing for the algebra and $\mathfrak{m}$ for the module. Restricting to the full suboperad generated by $\mathfrak{a}$ in the description above of $\mathbf{LM}$, which is equivalent to $\mathbf{Ass}$, gives rise to an ∞-operad map $\mathcal{A}\mathrm{ss}^\otimes \to \mathcal{LM}^\otimes$ and in turn a functor $\mathrm{Alg}_{\mathcal{LM}/\mathcal{A}\mathrm{ss}}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})$ for any monoidal ∞-category $\mathcal{C}$. Given an associative algebra object $A \in \mathrm{Alg}(\mathcal{C})$ in a monoidal ∞-category $\mathcal{C}^\otimes$, the *∞-category of (left) modules over A* is defined to be the fiber

$$\mathrm{LMod}_A(\mathcal{C}) \coloneqq \mathrm{Alg}_{\mathcal{LM}/\mathcal{A}\mathrm{ss}}(\mathcal{C}) \times_{\mathrm{Alg}(\mathcal{C})} \{A\}.$$

Restriction of $\mathbf{LM}$ to the full subcategory spanned by the color (i.e. object) $\mathfrak{m}$ likewise induces a functor $\mathrm{Alg}_{\mathcal{LM}/\mathcal{A}\mathrm{ss}}(\mathcal{C}) \to \mathcal{C}$, called the *underlying object of the module*. The data of a left A-module intuitively corresponds to an object $M \in \mathcal{C}$ together with morphisms $A \otimes M \to M$ which is appropriately compatible with respect to the associative algebra structure on A and itself associative up to coherent homotopy.

Viewing a monoidal ∞-category $\mathcal{C}^\otimes$ as an associative algebra object in $\mathcal{C}\mathrm{at}_\infty^\times$, an module $\mathcal{M} \in \mathrm{LMod}_{\mathcal{C}^\otimes}(\mathcal{C}\mathrm{at}_\infty)$ is called an *∞-category left tensored over* $\mathcal{C}^\otimes$. Informally, this consists of an underlying ∞-category $\mathcal{M}$ together with a functor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$, which is unital and associative up to coherent homotopy.Furthermore, this tensoring functor must be compatible with the monoidal structure $\mathcal{C}^\otimes$, in the sense that there is a natural equivalence

$$\mathrm{Map}_{\mathcal{M}}\big((C' \otimes C) \otimes M, N\big) \simeq \mathrm{Map}_{\mathcal{M}}\big(C' \otimes (C \otimes M), N\big)$$

which is also compatible with all the diagrams exhibiting higher associativity relations. For any associative algebra object $A \in \mathrm{Alg}(\mathcal{C})$, a modification of the above construction of left modules produces an ∞-category $\mathrm{LMod}_A(\mathcal{M})$, allowing A to also act upon objects of $\mathcal{M}$. This comes with an underlying-object functor $\mathrm{LMod}_A(\mathcal{M}) \to \mathcal{M}$ which always possesses a left adjoint, which we say associates to an object $M \in \mathcal{M}$ the *free A-module generated by M*. The underlying object of such a free A-module is $A \otimes M \in \mathcal{M}$.

When $\mathcal{C}^\otimes$ is a symmetric monoidal ∞-category and $A \in \mathrm{CAlg}(\mathcal{C})$ a commutative algebra object, we will commonly denote its ∞-category of modules by $\mathrm{Mod}_A(\mathcal{C})$ and it possesses a canonical symmetric monoidal structure. The unit object $\mathbf{1} \in \mathcal{C}$ always admits a commutative algebra structure, more specifically as the initial object in $\mathrm{CAlg}(\mathcal{C})$, and the forgetful functor exhibits the equivalence $\mathrm{Mod}_{\mathbf{1}}(\mathcal{C}) \simeq \mathcal{C}$.

Given any ∞-category with colimits $\mathcal{C}$, it is naturally tensored over $\mathcal{S}^\times$, in such a way that the tensoring functor $\otimes : \mathcal{S} \times \mathcal{C} \to \mathcal{C}$ preserves colimits in each variable separately. For a space X and object $C \in \mathcal{C}$, this is obtained by setting

$$X \otimes C \simeq (\varinjlim_{x\in X}\{x\}) \otimes C \coloneqq \varinjlim_{x \in X} C.$$

Similarly a pointed ∞-category $\mathcal{C}$ is canonically tensored over pointed spaces $\mathcal{S}_*$ by an analogous formula. When $\mathcal{C}$ only possesses finite colimits, $\mathcal{C}$ is tensored over *the ∞-category of finite spaces* $\mathcal{S}^{\mathrm{fin}}$, , i.e. the full subcategory of $\mathcal{S}$ generated by a point under finite colimits, or, in case $\mathcal{C}$ is pointed, over $\mathcal{S}^{\mathrm{fin}}_*$.

A.2.9. *The ∞-categorical Barr-Beck theorem.* For any ∞-category $\mathcal{C}$, the endofunctor ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ may be promoted to a monoidal ∞-category with the monoidal operation given by composition of functors and the identity functor being the monoidal unit. The algebra objects $T \in \mathrm{Alg}(\mathrm{Fun}(\mathcal{C}, \mathcal{C}))$ are called *monads on* $\mathcal{C}$, and particularly important examples of monads arise from adjunctions. Given a functor $F : \mathcal{C} \to \mathcal{D}$ with a right adjoint $G : \mathcal{D} \to \mathcal{C}$, there is a canonical monad T on $\mathcal{C}$ with the underlying functor $G \circ F$. The unit map of the adjunction $\varepsilon : \mathrm{id}_{\mathcal{C}} \to G \circ F \simeq T$ doubles as the unit of the associative algebra, and the counit of the adjunction $\eta : G \circ F \to \mathrm{id}_{\mathcal{D}}$ gives rise to the "operation" on the algebra object T as

$$T \circ T \simeq (G \circ F) \circ (G \circ F) \simeq G \circ (F \circ G) \circ F \xrightarrow{\eta} G \circ F \simeq T.$$

The homotopy coherent nature of composition in ∞-categories ensures that this extends to exhibit an associative algebra structure on T. The ∞-category $\mathcal{C}$ is naturally tensored over the monoidal ∞-category

Fun$(\mathcal{C},\mathcal{C})^{\otimes}$ with the tensoring functor $\otimes : \mathrm{Fun}(\mathcal{C},\mathcal{C})\otimes\mathcal{C}\to\mathcal{C}$ given by assigning $(F,X)\mapsto F(X)$. Therefore it makes sense for a monad T on $\mathcal{C}$ to talk about modules in $\mathcal{C}$ over it. Lurie's ∞-categorical generalization of the Barr-Beck theorem is a powerful criterion for recognizing when a functor is equivalent to the underlying object projection $\mathrm{LMod}_T(\mathcal{C})\to\mathcal{C}$ from such a module ∞-category. Before we can state it, we need to familiarize ourselves with some simplicial terminology.

Let $\mathbf{\Delta}_{-\infty}$ denote the (nerve of the) ordinary category of non-empty linearly ordered finite sets for objects and monotonic maps which preserve the minimal element of the ordering for morphisms. The assignment $[n]\mapsto[n]\cup\{-\infty\}$, where the element $-\infty$ is defined to be smaller than all numbers, defines an embedding $\mathbf{\Delta}\to\mathbf{\Delta}_{-\infty}$. A simplicial object $\mathbf{\Delta}^{\mathrm{op}}\to\mathcal{C}$ in an ∞-category $\mathcal{C}$ is called *split* if it extends along the described embedding to a functor $(\mathbf{\Delta}_{-\infty})^{\mathrm{op}}\to\mathcal{C}$. Similarly, given a functor $F:\mathcal{C}\to\mathcal{D}$, a simplicial object X in $\mathcal{C}$ is *F-split* if the simplicial object $F\circ X$ is split in $\mathcal{D}$, that is to say, if it fits into a commutative diagram of ∞-categories

If a simplicial object is split, then it is also F-split for every functor F. For any simplicial object $X:\mathbf{\Delta}^{\mathrm{op}}\to\mathcal{C}$, we say that the colimit $\varinjlim X\in\mathcal{C}$, if it exists, is the *geometric realization of* X and denote it $|X|$. Every split simplicial object has a geometric realization .

Theorem A.2.1 (Bar-Beck, Theorem HA.4.7.4.5)**.** *Let $G:\mathcal{D}\to\mathcal{C}$ be a functor between ∞-categories. Then there exists a monad T on $\mathcal{C}$ such that $\mathcal{D}\simeq\mathrm{LMod}_T(\mathcal{C})$ and p is equivalent to the canonical functor $\mathrm{LMod}_T(\mathcal{C})\to\mathcal{C}$ precisely when the following conditions are satisfied:*

(1) The functor G admits a left adjoint F.
(2) The functor G is conservative, i.e. if $G(f)$ is an equivalence in $\mathcal{C}$ for some morphism f in $\mathcal{D}$, then f is an equivalence in $\mathcal{C}$.
(3) The ∞-category $\mathcal{C}$ possesses geometric realizations of all G-split simplicial objects, and G preserves such geometric realizations.

In that case, T may be identified with the monad of the adjunction between F and G.

In the statement of the Barr-Beck theorem, we could have replaces the condition that $\mathcal{D}\simeq\mathrm{LMod}_T(\mathcal{C})$ for some monad T on $\mathcal{C}$ with the seemingly more general condition that there exists some monoidal ∞-category $\mathcal{A}^{\otimes}$, a tensored ∞-category over $\mathcal{A}^{\otimes}$ with the underlying ∞-category $\mathcal{C}$, and an algebra object $A\in\mathrm{Alg}(\mathcal{A})$ such that $\mathcal{D}\simeq\mathrm{LMod}_A(\mathcal{C})$. This is because, in the described situation, the functor $T(X):=A\otimes X$ for $X\in\mathcal{C}$ defines a functor $\mathcal{C}\to\mathcal{C}$, and the algebra structure on A assures that T is also an algebra object, i.e. a monad. Then $\mathrm{LMod}_A(\mathcal{C})\simeq\mathrm{LMod}_T(\mathcal{C})$, the left hand side referring to tensoring over $\mathcal{A}^{\otimes}$ and the right referring to tensoring over Fun$(\mathcal{C},\mathcal{C})^{\otimes}$.

A.3. **Higher linear and commutative algebra.**

A.3.1. *Stable ∞-categories.* An ∞-category $\mathcal{C}$ is *stable* if it is pointed with a zero (i.e. simultaneously initial and terminal) object 0, possesses finite limits and colimits, and a diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

in $\mathcal{C}$ is a pullback square if and only if it is a pushout square. That is to say, in a stable ∞-category fiber and cofiber sequences coincide. The condition that an ∞-category is stable admits a variety of equivalent restatements, for instance that $\mathcal{C}$ is pointed, closed under finite limits and the functor $\Omega X:=0\times_X 0$ is a self-equivalence $\Omega:\mathcal{C}\xrightarrow{\simeq}\mathcal{C}$. The suspension functor $\Sigma X:=0\coprod_X 0$ is its inverse. Finite products and coproducts in a stable ∞-category are equivalent, and so we may use the biproduct notation $\oplus$ for either.

If $X,Y\in\mathcal{C}$ are objects in a stable ∞-category, then since the functor $\mathrm{Map}_{\mathcal{C}}(-,Y)$ takes colimits to limits, we have

$$\mathrm{Map}_{\mathcal{C}}(X,Y)\simeq\mathrm{Map}_{\mathcal{C}}(\Sigma^k\Omega^k X,Y)\simeq\Omega^k\,\mathrm{Map}_{\mathcal{C}}(\Omega^k X,Y).$$

That is to say, the functor $\mathrm{Map}_{\mathcal{C}}:\mathcal{C}^{\mathrm{op}}\times\mathcal{C}\to\mathcal{S}$ canonically factors through the forgetful functor $\mathrm{CMon}^{\mathrm{gp}}\to\mathcal{S}$ that sends an infinite loop space to its underlying space. Thus we may view mapping spaces of stable ∞-categories as taking values in the ∞-category $\mathrm{CMon}^{\mathrm{gp}}$ of infinite loop spaces. In particular, $\pi_0\,\mathrm{Map}_{\mathcal{C}}(X,Y)\simeq\pi_2\,\mathrm{Map}_{\mathcal{C}}(\Omega^2X,Y)$ is an abelian group for all objects X and Y, so the homotopy category of $\mathcal{C}$ is additive.

A.3.2. *Comparison with abelian and triangulated categories.* The notion of a stable ∞-category, while also encompassing stability as usually understood in homotopy theory, is formally most closely analogous to the 1-categorical notion of an abelian category. Fibers and cofibers correspond to kernels and cokernels, and the condition that fiber and cofiber squares coincide is equivalent to the condition that $\ker(\operatorname{coker} f)\simeq\operatorname{coker}(\ker f)$ for any morphism f in an abelian category. This justifies that a stable ∞-category is a good context for homological algebra.

But it is a well-known fact in classical algebra that for most homological algebra, such as formation of derived functors, the correct context is not an abelian category $\mathcal{A}$, but rather its derived category of chain complexes $\mathcal{D}(\mathcal{A})$. Indeed, stable ∞-categories are formally closer to these latter objects than to abelian categories. In particular, the homotopy category of any stable ∞-category is naturally triangulated. The shift functor $[1]$ of the triangulated structure comes from the suspension functor Σ. Thus stable ∞-categories and triangulated categories may be viewed as a competing generalization of abelian categories. Though not every triangulated category appears as the homotopy category of a stable ∞-category, most "encountered in nature" do. While stable ∞-categories are hence in some sense less general, they also do not suffer certain deficits of triangulated categories, such as the clumsy definition (the octahedral axiom in particular) and the lack of functorial (co)kernels.

A.3.3. *Stabilization.* Given any ∞-category with finite limits $\mathcal{C}$, its *stabilization* is a stable ∞-category $\mathrm{Sp}(\mathcal{C})$ together with a functor $\Omega^\infty : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ which induces an equivalence

$$\mathrm{Fun}^{\mathrm{R}}(\mathcal{D}, \mathrm{Sp}(\mathcal{C})) \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{D}, \mathcal{C})$$

between right adjoint functors for any stable ∞-category $\mathcal{D}$. Since right adjoint functors preserve limits, it follows from this universal property that an ∞-category $\mathcal{C}$ with a terminal object $*$ and its *pointification* $\mathcal{C}_* \simeq \mathcal{C}_{*/}$ have equivalent stabilizations. Stabilization may be explicitly described as the limit

$$\mathrm{Sp}(\mathcal{C}) \simeq \varprojlim \Big(\cdots \to \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \Big). \tag{1}$$

Spectra are classically viewed as a convenient enlargement of the category of (co)homology theories, and this point of view can be pursued to explain stabilization of ∞-categories in general. Specifically, there is an equivalence $\mathrm{Sp}(\mathcal{C}) \simeq \mathrm{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})$ between the stabilization of an ∞-category $\mathcal{C}$ and the ∞-category of pointed excisive functors from the ∞-category of finite spaces into $\mathcal{C}$, where a functor is pointed if it preserves terminal objects and excisive if it takes pushouts to pullbacks. The condition of excisiveness is clearly analogous to the Mayer-Vietoris or excision axiom in the usual definition of a homology theory. The functor $\Omega^\infty : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ corresponds to evaluation of an excisive functor on S^0.

When $\mathcal{C}$ is a presentable ∞-category, the functor Ω^∞ admits a left adjoint $\Sigma^\infty_+ : \mathcal{C} \to \mathrm{Sp}(\mathcal{C})$ by the adjoint functor theorem. It follows from the anti-equivalence $(\mathcal{P}\mathrm{r}^{\mathrm{L}})^{\mathrm{op}} \simeq \mathcal{P}\mathrm{r}^{\mathrm{R}}$ that this satisfies the dual universal property: for any presentable stable ∞-category $\mathcal{D}$, the functor Σ^∞_+ induces an equivalence $\mathrm{Fun}^{\mathrm{L}}(\mathrm{Sp}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$.

Though this all holds for a presentable ∞-category $\mathcal{C}$ whether it is pointed or not, let us switch the notation from Σ^∞_+ to Σ^∞ when $\mathcal{C}$ is pointed. This is so that, if $+ : \mathcal{C} \to \mathcal{C}_*$ denotes the left adjoint to the inclusion $\mathcal{C}_* \to \mathcal{C}$, explicitly given by $C \mapsto C \coprod *$, we obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\;+\;} & \mathcal{C}_* \\ & {\scriptstyle \Sigma^\infty_+} \searrow \quad \swarrow {\scriptstyle \Sigma^\infty} & \\ & \mathrm{Sp}(\mathcal{C}). & \end{array}$$

The functor $\Omega^\infty : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ likewise factors through the forgetful functor $\mathcal{C}_* \to \mathcal{C}$, but we shall denote it Ω^∞ both times and make clear from the context which one we are referring to.

Beware that the stabilization construction is in general not functorial[11]. Nevertheless, when when a functor $F : \mathcal{C} \to \mathcal{D}$ preserves finite limits, for instance if it has a left adjoint, then it canonically induces a functor $\tilde{F} : \mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{D})$, defined by sending a pointed excisive functor $E \in \mathrm{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}) \simeq \mathrm{Sp}(\mathcal{C})$ to the left composite $F \circ E : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{D}$ which will again be pointed and excisive.

A.3.4. *Spectra.* A particularly important example of a stable ∞-category is the ∞*-category of spectra* Sp. It may be obtained as the stabilization of the ∞-category of spaces or equivalently, seeing how $\mathcal{S}$ is generated by a point under colimits, as the free stable ∞-category generated under colimits by a single object. This object is the *sphere spectrum* $S := \Sigma^\infty S^0$, where $\Sigma^\infty : \mathcal{S}_* \to \mathrm{Sp}(\mathcal{S}_*) \simeq \mathrm{Sp}$ is the suspension spectrum functor. In the description of spectra as pointed excisive functors, $S : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}$ is the evident inclusion of subcategory. Of course, Sp is equivalent to the ∞-category obtained from the model category of any of the standard constructions of spectra, e.g. symmetric spectra, orthogonal spectra, S-modules, etc.

Since the ∞-category Sp is stable, the suspension functor $\Sigma : \mathrm{Sp} \to \mathrm{Sp}$ is an equivalence with inverse $\Sigma^{-1} \simeq \Omega$. We may therefore define the *k-th homotopy group* of a spectrum $E \in \mathrm{Sp}$ for any $k \in \mathbf{Z}$ to be

$$\pi_k E := \pi_0(\Omega^\infty \Sigma^k E) \simeq \pi_0 \,\mathrm{Map}_{\mathrm{Sp}}(\Sigma^k S, E),$$

where the second equivalence follows from the adjunction between Σ^∞ and Ω^∞.

[11]This failure of functoriality is the subject of the entire field of Goodwillie calculus, as presented in HA Chapter 6.

If $\pi_k E \simeq 0$ for all $n < k$, we say that E is *n-connective.* A 0-connective spectrum is just called *connective*, while a spectrum which is connective for some unspecified n is called *eventually connective.* Let $\mathrm{Sp}^{\mathrm{cn}}$ denote the full subcategory of connective spectra in Sp. The functor $\Omega^\infty : \mathrm{Sp} \to \mathcal{S}$ restricts to an equivalence

$$\mathrm{Sp}^{\mathrm{cn}} \simeq \mathrm{CMon}^{\mathrm{gp}}$$

between connective spectra and grouplike $\mathbb{E}_\infty$-spaces, or equivalently, infinite loop spaces.

If $\pi_k E \simeq 0$ for all $k \neq 0$, then the spectrum E is called *discrete* and the full subcategory of those is denoted $\mathrm{Sp}^\heartsuit$. The functor $E \mapsto \pi_0 E$ defines an equivalence $\mathrm{Sp}^\heartsuit \simeq \mathcal{A}b$ between discrete spectra and the (nerve of the) ordinary category of abelian groups. The inverse of this equivalence can be identified with the composite of the inclusion $\mathcal{A}b \to \mathcal{S}_*$ of abelian groups into discrete spaces pointed at the unit element, and the suspension spectrum functor $\Sigma^\infty : \mathcal{S}_* \to \mathrm{Sp}$. The resulting spectrum for an abelian group $A \in \mathcal{A}b$ is classically denoted HA and called the *Eilenberg-MacLane spectrum*, but we shall instead identify $\mathcal{A}b$ with $\mathrm{Sp}^\heartsuit$ and not notationally distinguish between an abelian group and the spectrum it corresponds to.

A.3.5. *Smash product and $\mathbb{E}_\infty$-rings.* The ∞-category of spectra admits a symmetric monoidal structure $\otimes$, called the *smash product*, which is essentially unique in satisfying the conditions that $\otimes : \mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$ preserves colimits in each variable, and that the unit object for $\otimes$ is the sphere spectrum S. An analogous result holds for pointed spaces, namely there exists a unique symmetric monoidal structure $\wedge$, also called the smash product, such that $\wedge : \mathcal{S}_* \times \mathcal{S}_* \to \mathcal{S}_*$ preserves colimits in each variable, and the unit object for $\wedge$ is S^0. It follows from the universal properties of $\wedge$ and $\otimes$ that the functors $+ : \mathcal{S} \to \mathcal{S}_*$, $\Sigma^\infty : \mathcal{S}_* \to \mathrm{Sp}$ and $\Sigma^\infty_+ : \mathcal{S} \to \mathrm{Sp}$ are all symmetric monoidal, where $\mathcal{S}$ is equipped with the Cartesian symmetric monoidal structure.

We shall denote the ∞-category of commutative algebra objects in spectra for $\otimes$ by $\mathrm{CAlg} := \mathrm{CAlg}(\mathrm{Sp})$ and refer to it as the *∞-category of $\mathbb{E}_\infty$-rings*[12] This is the ∞-categorical incarnation of what is more classically called highly structured ring spectra or $\mathbb{E}_\infty$-ring spectra. For an $\mathbb{E}_\infty$-ring R, the set $\pi_0 R$ comes naturally equipped with a commutative ring structure. In fact, the subcategory $\mathrm{CAlg}^\heartsuit \subset \mathrm{CAlg}$ may be canonically identified with the (nerve of the) ordinary category of commutative rings. As with abelian groups, we will not distinguish between a commutative ring and the corresponding $\mathbb{E}_\infty$-ring.

We have seen that all ordinary commutative rings are examples of $\mathbb{E}_\infty$-rings. Another large class of examples of $\mathbb{E}_\infty$-rings may be obtained by observing that the functor Σ^∞_+ being monoidal implies that it induces a functor $\Sigma^\infty_+ : \mathrm{CMon} \to \mathrm{CAlg}$. Hence for any $\mathbb{E}_\infty$-space X, the suspension spectrum $\Sigma^\infty_+ X$ is an $\mathbb{E}_\infty$-ring. When we take X to be a point, we recover the fact that the sphere spectrum S is an $\mathbb{E}_\infty$-ring, which is also evident from the fact that it is the unit for the smash product.

A.3.6. *$\mathbb{E}_\infty$-rings vs homotopy commutative ring spectra.* Though the intuition behind them is similar, $\mathbb{E}_\infty$-rings should not be confused with the weaker notion of a homotopy commutative ring spectrum. The latter is an object $R \in \mathrm{Sp}$ for which $\pi_0 R$ is a commutative ring, i.e. there exists a ring structure on R on the level of the homotopy category. The notion of an $\mathbb{E}_\infty$-ring is much more restrictive, requiring instead the ring structure to exist even before passing to homotopy. More precisely, an $\mathbb{E}_\infty$-ring R consists of a spectrum R together with a multiplication map $\mu : R \otimes R \to R$ and a unit map $1 : S \to R$, such that the axioms for a commutative ring, stated diagramatically, hold up to a coherent system of homotopies. Note that this is not actually a condition to be imposed, but rather additional data to be specified: for every compatibility diagram which we could write down, we must specify a map that exhibits it commuting, and further compatibility maps will then depend on the ones chosen previously.

One major appeal of the ∞-categorical approach to stable homotopy theory is that it efficiently organizes these immense collections of coherence data, which could prove quite unmanageable if approached directly[13], in such a way that reasoning about them almost entirely analogously as with the corresponding classical objects is logically valid, as opposed to just a useful heuristic, which is often the case when phrasing things in terms of model categories.

A.3.7. *Modules and $\mathbb{E}_\infty$-R-algebras.* Given an $\mathbb{E}_\infty$-ring R, the *∞-category of modules over R* is defined to be $\mathrm{Mod}_R := \mathrm{Mod}_R(\mathrm{Sp})$. It inherits a symmetric monoidal operation $\otimes_R$ from Sp, which may again be specified uniquely up to equivalence by the requirements that $\otimes_R : \mathrm{Mod}_R \times \mathrm{Mod}_R \to \mathrm{Mod}_R$ preserves colimits in each variable, and that R is its unit. With respect to this symmetric monoidal structure, there is

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[12]As the notation suggests, there exists an entire hierarchy of $\mathbb{E}_n$-rings for every possibly unbounded positive integer n. More generally, for any symmetric monoidal ∞-category $\mathcal{C}$, its *$\mathbb{E}_n$-objects* may be defined either as algebras over the little n-disc operad, or recursively via the relation $\mathrm{Alg}_{\mathbb{E}_{n+1}}(\mathcal{C}) \simeq \mathrm{Alg}(\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}))$. In the initial and limiting case we recover $\mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C}) \simeq \mathrm{Alg}(\mathcal{C})$ and $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{C}) \simeq \mathrm{CAlg}(\mathcal{C})$.

[13]This might indeed be one of the reasons why it took quite long, until the publication of EKMM, that is [?], in the the 1990s, to develop a good point-set theory of highly structured ring spectra. From the ∞-categorical point of view, it is not surprising that the various strictly commutative models for $\mathbb{E}_\infty$-rings, such as S-modules, symmetric and orthogonal spectra etc. are nuanced and subtle objects whose construction requires some sophistication; they are strictifications of an $\mathbb{E}_\infty$-structure, and the existence of such strictifications, let alone an explicit construction, is most often a highly non-trivial matter.

an equivalence $\mathrm{CAlg}(\mathrm{Mod}_R) \simeq \mathrm{CAlg}_{R/}$ between two candidate notions for the *∞-category of $\mathbb{E}_\infty$-R-algebras* CAlg_R. Observe that for $R = S$, we recover $\mathrm{Mod}_S \simeq \mathrm{Sp}$, $\mathrm{CAlg}_S \simeq \mathrm{CAlg}$ and $\otimes_S \simeq \otimes$.

For any $\mathbb{E}_\infty$-ring R, there is a forgetful functor $\mathrm{Mod}_R \to \mathrm{Sp}$ which is symmetric monoidal and therefore comes with an associated forgetful functor $\mathrm{CAlg}_R \to \mathrm{CAlg}$. These functors admit left adjoints which may be described as sending $E \mapsto R \otimes E$. Composing this left adjoint $R \otimes - : \mathrm{CAlg} \to \mathrm{CAlg}_R$ with the already-discussed functor $\Sigma^\infty_+ : \mathrm{CMon} \to \mathrm{CAlg}$, we obtain a symmetric monoidal functor $\mathrm{CMon} \to \mathrm{CAlg}_R$, the image of an $\mathbb{E}_\infty$-space X under which we shall denote $R[X]$. It follows from the discussion that this functor satisfies the universal property that for any $\mathbb{E}_\infty$-R-algebra A there is an equivalence

$$\mathrm{Map}_{\mathrm{CAlg}_R}(R[X], A) \simeq \mathrm{Map}_{\mathrm{CMon}}(X, \Omega^\infty A).$$

In particular when R is the sphere spectrum, then $S[X]$ is just an alternative notation for the suspension $\mathbb{E}_\infty$-ring $\Sigma^\infty_+ X$. To highlight the analogy with classical algebra, we will mostly prefer to use the former notation.

A.3.8. *Recognition principle for modules.* Module ∞-categories are closed under small limits and colimits, and in fact this almost suffices to characterize them. We will state a theorem of Lurie which makes this idea precise. The theorem is closely related to a theorem of Schwede and Shipley, but is more specific and less general in the sense that it also takes into account the monoidal structure.

Let $\mathcal{C}$ be any stable ∞-category. As observed before, invertibility of the functor $\Omega : \mathcal{C} \to \mathcal{C}$ implies that mapping spaces in Sp possess the structure of infinite loop spaces. We can say more; using the equivalence of Sp with the limit of the tower

$$\cdots \to \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*,$$

we may associate to every pair of objects $X, Y \in \mathcal{C}$ a *mapping spectrum* $\mathrm{Map}_{\mathcal{C}}(X, Y) \in \mathrm{Sp}$. There is some room for confusion due to the fact that we are using the same notation for the mapping space and the mapping spectrum of a stable ∞-category, so we will have to make it evident from the context which will be meant.

Proposition A.3.1 (Proposition HA.7.1.2.7)**.** *Let $\mathcal{C}^\otimes$ be a symmetric monoidal ∞-category. Then there exists an $\mathbb{E}_\infty$-ring $R \in \mathrm{CAlg}$ such that $\mathcal{C}^\otimes \simeq \mathrm{Mod}_R^\otimes$ if and only if the following conditions are satisfied:*

(1) The ∞-category $\mathcal{C}$ is stable and presentable, and the functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separably in each variable.

(2) The unit object $\mathbf{1} \in \mathcal{C}$ is a compact, and generates $\mathcal{C}$ in the sense that, for any object $X \in \mathcal{C}$, if $\pi_k \mathrm{Map}_{\mathcal{C}}(\mathbf{1}, X) \simeq 0$ for all $k \in \mathbf{Z}$, then $X \simeq 0$.

In that case, there is an equivalence $R \simeq \mathrm{Map}_{\mathcal{C}}(\mathbf{1}, \mathbf{1})$.

A.3.9. *Modules over a commutative ring.* Let $R \in \mathrm{CAlg}^\heartsuit$ be a commutative ring. Then Mod_R is equivalent to the ∞-categorical derived category of chain complexes of ordinary R-modules by a version of the Dold-Kan correspondence. The homotopy group functor $\pi_k : \mathrm{Mod}_R \to \mathcal{A}b$ corresponds to the functor $M_\bullet \mapsto H_k(M_\bullet)$ taking a complex to its k-th homology group (or its $(-k)$-th cohomology group, if we were using the cohomological grading convention). In particular, the subcategory of discrete objects $\mathrm{Mod}_R^\heartsuit$ may be identified with the (nerve of the) ordinary category of R-modules.

Note that the in spite of our choice of notation, the smash product $\otimes_R$ on commutative rings $\mathrm{Mod}_R^\heartsuit$ *does not* coincide with the ordinary smash product over R. Indeed, the subcategory $\mathrm{Mod}_R^\heartsuit \subset \mathrm{Mod}_R$ is not closed under the smash product. This should not be very surprising in light of the above discussion, since Mod_R corresponds to the derived category of ordinary R-modules, and so we have for any $M, N \in \mathrm{Mod}_R^\heartsuit$

$$\pi_k(M \otimes_R N) \simeq \mathrm{Tor}_k^R(M, N).$$

Stated differently, the smash product $\otimes_R$ corresponds to what is in more classical accounts of homological algebra usually denoted $\otimes_R^{\mathrm{L}}$, the derived tensor product. Similarly we also have the dual equivalence

$$\pi_k \mathrm{Map}_{\mathrm{Mod}_R}(M, N) \simeq \mathrm{Ext}_R^{-k}(M, N),$$

thus the mapping spectrum $\mathrm{Map}_{\mathrm{Mod}_R}(M, N)$ corresponds to the object denoted in the more classical literature by $\mathrm{RHom}_R(M, N)$.

The functor $R[-] : \mathrm{CMon} \to \mathrm{CAlg}_R$ restricts, since R is discrete, to a functor between discrete objects $R[-] : \mathrm{CMon}^\heartsuit \to \mathrm{CAlg}_R^\heartsuit$. The left and right ∞-categories may be identified with the (nerves of the) ordinary categories of commutative monoids,, and commutative R-algebras respectively. Given an commutative monoid $G \in \mathrm{CMon}^\heartsuit$, the commutative R-algebra $R[G]$ resumes its familiar meaning from representation theory as the monoid R-algebra. In particular if G is an abelian group, then $\mathbf{Z}[G]$ is the usual group ring of G. Thus the analogy suggests that the suspension spectrum $S[X] \simeq \Sigma^\infty_+ X$ of an infinite loop space $X \in \mathrm{CMon}^{\mathrm{gp}}$ should be the analogue of the group ring, i.e. it should contain information about representations of X over the sphere spectrum.

A.3.10. *The cotangent complex.* The formalism of the cotangent complex, more classically known under the name (topological) André-Quillen homology, is the brave new algebra analogue of the algebraic theory of Kähler differentials. One way to approach it is through stabilization, starting from the equivalence

$$\mathrm{Sp}(\mathrm{CAlg}_R) \simeq \mathrm{Mod}_R$$

for any $\mathbb{E}_\infty$-ring R. This gives functor $\Sigma^\infty_+ : \mathrm{CAlg}_R \to \mathrm{Mod}_R$ and we may define *absolute cotangent complex of* R to be $L_R := \Sigma^\infty_+ R$. Using the description of $\mathbb{E}_\infty$-R-algebras $\mathrm{CAlg}_R \simeq \mathrm{CAlg}_{R/}$, we may identify the pointification $(\mathrm{CAlg}_{R/})_*$ with the ∞-category $\mathrm{CAlg}_{R//R}$ the objects of which are $\mathbb{E}_\infty$-ring maps $A \to R$ such that the composition $R \to A \to R$ is equivalent to the identity on R. The zeroth-space functor $\Omega^\infty : \mathrm{Mod}_R \to (\mathrm{CAlg}_R)_* \simeq \mathrm{CAlg}_{R//R}$ admits an explicit description as associating to an R-module M its *trivial square-zero extension* $R \oplus M \in \mathrm{CAlg}_{R//R}$. Informally, this augmented $\mathbb{E}_\infty$-R-algebra consists of the underlying module $R \oplus M$ equipped with the multiplication $(a,x)(b,y) := (ab, ay + bx)$. Of course, that only makes sense as stated when both R and M are discrete, but a similar formula holds with no reservations on the level of homotopy. That is, for $a \in \pi_i(R)$, $b \in \pi_j(R)$, $x \in \pi_k(M)$ and $y \in \pi_l(M)$

$$(a,x)(b,y) = (ab, ay + (-1)^{jk} bx)$$

Putting everything together, we may for every $\mathbb{E}_\infty$-R-algebra A and an R-module M rewrite the adjunction between Σ^∞_+ and Ω^∞ in the form

$$\mathrm{Map}_{\mathrm{Mod}_R}(L_R, M) \simeq \mathrm{Map}_{\mathrm{CAlg}_{R//R}}(R, R \oplus M).$$

The right hand side may be identified with R-linear derivations $R \to M$ (in this context, this may be taken as the definition of derivations), exhibiting that the cotangent complex satisfies the analogous universal property with respect to derivations that the module of Kähler derivations do in the classical context.

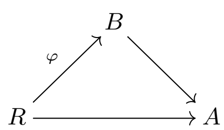
For a map of $\mathbb{E}_\infty$-rings $A \to B$, we define its *relative cotangent complex* $L_{B/A}$ by a cofiber sequence

$$L_A \otimes_A B \to L_B \to L_{B/A}$$

in the ∞-category Mod_B. One appearance of the relative cotangent complex is the result that a morphism of connective $\mathbb{E}_\infty$-rings $A \to B$ is an equivalence if and only if $\pi_0 A \to \pi_0 B$ is an isomorphism and $L_{B/A} \simeq 0$.

With the cotangent complex at our disposal, we may define the analogues of various classical notions from commutative algebra. An $\mathbb{E}_\infty$-R algebra A is *formally smooth* if $L_{A/R}$ is perfect, *formally étale* if $L_{A/R} \simeq 0$, and *differentially smooth* or *étale* if it is almost finitely presented and formally smooth or formally étale respectively.

Perhaps the most important property of étale $\mathbb{E}_\infty$-algebras is the fact that they are completely determined on the level of homotopy. Formally, that means that for any map of $\mathbb{E}_\infty$-rings $R \to A$, if $\mathrm{CAlg}^{\text{ét}}_{R//A}$ denotes the subcategory of $\mathrm{CAlg}_{R//A}$ consisting of all commutative triangles of $\mathbb{E}_\infty$-rings



for which φ is étale, and $\mathrm{CAlg}^{\heartsuit,\text{ét}}_{\pi_0 R//\pi_0 A}$ denotes the analogously defined ordinary category, then the map $B \mapsto \pi_0 B$ defines an equivalence of ∞-categories (or as it follows from this result, of ordinary categories)

$$\mathrm{CAlg}^{\text{ét}}_{R//A} \simeq \mathrm{CAlg}^{\heartsuit,\text{ét}}_{\pi_0 R//\pi_0 A}.$$

A.4. **Spectral algebraic geometry.** The idea of spectral algebraic geometry is straightforward: if modern (i.e. post-Grothendieck) algebraic geometry is the study of spaces that are geometrically glued-together out of commutative rings, then spectral algebraic geometry is the study of geometrically glued-together out of (perhaps connective) $\mathbb{E}_\infty$-rings.

A.4.1. *Spectral schemes.* Recall that if R is a commutative ring, its *spectrum* is the collection of prime ideals in R equipped with the *Zariski topology.* That is to say, closed sets in the Zariski topology are of the form

$$\mathrm{V}(I) := \{\mathfrak{p} \in \operatorname{Spec} R : I \subseteq \mathfrak{p}\},$$

where I may be any ideal in R, playing the role of the "variety cut out by the ideal I". A basis for the topology is given by the open sets

$$D(x) := \operatorname{Spec} R \smallsetminus \mathrm{V}((x)) = \{\mathfrak{p} \in \operatorname{Spec} R : x \notin \mathfrak{p}\}$$

for elements $x \in R$. The spectrum of R is made into a locally ringed space by specifying its *structure sheaf* on the covering $\{D(x)\}_{x \in R}$ by setting

$$\mathscr{O}_{\operatorname{Spec} R}(D(x)) := R[x^{-1}]$$

and letting the canonical localization morphisms to be the restriction maps of the sheaf. This makes $\mathscr{O}_{\operatorname{Spec} R}$ into a sheaf of commutative rings, and the *spectrum of* R is formally the locally ringed space $(\operatorname{Spec} R, \mathscr{O}_{\operatorname{Spec} R})$.

Given a connective $\mathbb{E}_\infty$-ring R, we may define its *spectrum* $\operatorname{Spec} R$ to be $\operatorname{Spec} \pi_0 R$, the space of prime ideals of the underlying commutative ring $\pi_0 R$ equipped the Zariski topology, together with a sheaf of $\mathbb{E}_\infty$-rings $\mathscr{O}_{\operatorname{Spec} R}$ satisfying the following:

(1) There is an equivalence of $\mathbb{E}_\infty$-rings $R \simeq \mathscr{O}_{\operatorname{Spec} R}(\operatorname{Spec} \pi_0 R)$
(2) For any $x \in \pi_0 R$ the compositum
$$R \simeq \mathscr{O}_{\operatorname{Spec} R}(\operatorname{Spec} \pi_0 R) \to \mathscr{O}_{\operatorname{Spec} R}(D(x))$$
of this equivalence with the restriction map, exhibits an equivalence
$$\mathscr{O}_{\operatorname{Spec} R}(D(x)) \simeq R[x^{-1}]$$
between the sections of the sheaf $\mathscr{O}_{\operatorname{Spec} R}$ over the open subset $D(x)$ and the $\mathbb{E}_\infty$-ring localization of R at the element x.

These requirements determine the spectrally ringed space $(\operatorname{Spec} R, \mathscr{O}_{\operatorname{Spec} R})$ essentially uniquely. A *spectral scheme* is then defined to be a spectrally ringed space locally equivalent to spectra of $\mathbb{E}_\infty$-rings. That is to say, a spectral scheme consists of a pair $(X, \mathscr{O}_X)$ of a space and a sheaf of $\mathbb{E}_\infty$-rings on it, such that every point $x \in X$ possesses an open neighbourhood U for which $(U, \mathscr{O}_X|_U)$ is equivalent to $\operatorname{Spec} R$ for some connective $\mathbb{E}_\infty$-ring R. By relaxing the requirement that R be connective, we arrive at the notion of a *non-connective spectral scheme.*

Non-connective spectral schemes admit an alternative description as locally spectrally ringed spaces $(X, \mathscr{O}_X)$ such that:

(1) The locally ringed space $(X, \pi_0 \mathscr{O}_X)$ is a scheme.
(2) For every $i \in \mathbf{Z}$ the homotopy sheaf $\pi_i \mathscr{O}_X$ is quasi-coherent (as a sheaf of modules on the scheme $(X, \pi_0 \mathscr{O}_X)$.
(3) For every $i \in \mathbf{Z}$ and every open affine subscheme $U \subseteq X$ the natural map
$$\pi_i(\mathscr{O}_X(U)) \to (\pi_i \mathscr{O}_X)(U)$$
is an isomorphism.

If furthermore the homotopy sheaves $\pi_i \mathscr{O}_X$ are trivial for all $i < 0$, then $(X, \mathscr{O}_X)$ is a spectral scheme. This second characterization captures how a (non-connective) spectral scheme is essentially a classical scheme together with homotopy groups.

The collection of all (non-connective) spectral schemes forms a full subcategory SpSch (resp. $\mathrm{SpSch}^{\mathrm{nc}}$) inside the ∞-category of locally spectrally ringed spaces. The full subcategory Aff spanned by all affine spectral schemes, which means spectral schemes equivalent to $\operatorname{Spec} R$ for any connective $\mathbb{E}_\infty$-ring R, is naturally anti-equivalent to the ∞-category $\mathrm{CAlg}^{\mathrm{cn}}$ of connective $\mathbb{E}_\infty$-rings. The anti-equivalence $\mathrm{CAlg}^{\mathrm{cn}} \simeq \mathrm{Aff}^{\mathrm{op}}$ is given by $R \mapsto \operatorname{Spec} R$ in one direction and $X \mapsto \mathscr{O}(X)$ in the other.

A.4.2. *Spectral Deligne-Mumford stakcs.* By replacing in the definition of a spectral scheme topological spaces with the more general ∞-topoi and the Zariski toplogy with the finer étale topology, we obtain the definition of a spectral Deligne-Mumford stack.

The small étale site of $\operatorname{Spec} R$ for an $\mathbb{E}_\infty$-ring R is equivalent to the ∞-category $\mathrm{CAlg}_R^{\mathrm{et}}$ of étale R-algebras with the Grothendieck cotopology defined by setting jointly faithfully flat finite families to be coverings. By the principle that a scheme should be identified with its étale ∞-topos, we will also denote the sheaf ∞-topos $\mathrm{Shv}(\mathrm{CAlg}_R^{\mathrm{et}})$ by[14] $\operatorname{Spec} R$. A sheaf of $\mathbb{E}_\infty$-rings on the ∞-topos $\operatorname{Spec} R$ is given by the forgetful functor $\mathrm{CAlg}_R^{\mathrm{et}} \to \mathrm{CAlg}_R$, which plays the role of $\mathscr{O}_{\operatorname{Spec} R}$ and makes $\operatorname{Spec} R$ into a spectrally ringed[15] ∞-topos.

A *spectral Deligne-Mumford stack* is a spectrally ringed ∞-topos $(\mathcal{X}, \mathscr{O})$, such that $\mathcal{X}$ admits a family of objects $\{U_\alpha\}$ such that the coproduct $\coprod U_\alpha$ is 0-connective, and there exists for every α a connective $\mathbb{E}_\infty$-ring R_α such that the spectrally ringed ∞-topos[16] $(\mathcal{X}_{/U_\alpha}, \mathscr{O}|_{U_\alpha})$ is equivalent to $\operatorname{Spec} R_\alpha$. By omitting the connectivity requirement on the $\mathbb{E}_\infty$-rings R_α, we obtain the notion of a *nonconnective spectral Deligne-Mumford stack.*

[14]In SAG the notation $\operatorname{Spét} R$ is used for this ∞-topos, but since we do not wish to notationally distinguish between spectral schemes and their corresponding spectral Deligne-Mumford stacks, it seems to make sense to stick to $\operatorname{Spec} R$.

[15]A spectrally ringed topos is a pair $(\mathcal{X}, \mathscr{O})$ of an ∞-topos $\mathcal{X}$ and a sheaf (i.e. small limit preserving functor) $\mathscr{O} : \mathcal{X}^{\mathrm{op}} \to \mathrm{CAlg}$ of $\mathbb{E}_\infty$-rings on $\mathcal{X}$.

[16]Here the restriction $\mathscr{O}|_U$ of the sheaf $\mathscr{O}$ on $\mathcal{X}$ to the overcategory $\mathcal{X}_{/U}$ is obtained as the composition
$$(\mathcal{X}_{/U})^{\mathrm{op}} \to \mathcal{X}^{\mathrm{op}} \xrightarrow{\mathscr{O}} \mathrm{CAlg},$$
where the first arrow is the forgetful functor sending an object $X \to U$ in $\mathcal{X}_{/U}$ to the object $X \in \mathcal{X}$.

The ∞-category of (non-connective) spectral Deligne-Mumford stacks SpDM (resp. $\mathrm{SpDM}^{\mathrm{nc}}$) is defined as the full subcategory of the ∞-category of Henslian[17] spectrally ringed ∞-topoi. spanned by spectral Deligne-Mumford stacks. By associating its étale topos to a (nonconnective) spectral scheme, we obtain a fully faithful embedding SpSch $\to$ SpDM (resp. $\mathrm{SpSch}^{\mathrm{nc}} \to \mathrm{SpDM}^{\mathrm{nc}}$), and we will generally not distinguish between a (nonconnective) spectral scheme and the corresponding (nonconnective) spectral Deligne-Mumford stack.

[17]Thought we shall not need to know the details, this is a locality condition analogous to the requirement that schemes be locally ringed topoi, but local with respect to the étale instead of the Zariski topology.