DAVID'S OPENING WORDS

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Dislaimer. This is a transcription of the notes I took of David Ben-Zvi's inaugural talk in the 2019 Spring semester *BZ(R) Seminar* at UT Austin, on February 8, 2019.

1. Motivating shifted symplectic and Poisson geometry

In this seminar we will primarily be concerned with shifted symplectic and shifted Poisson geometry. David likes the latter a bit more, so this talk may be slanted in that direction.

1.1. **Ordinary symplectic manifolds.** Symplectic manifolds are good for many things. Crucially, they are phase spaces of classical mechanical systems. The primary example is $M = T^*X$, where X is the configuration space of the mechanical system. Coordinates q on X encode *position*, while the corresponding coordinates p in the fiber directions of T^*X encode *momenta*. The ring of functions $\mathcal{O}(M)$ is the algebra of *observables* of the system, and comes equipped with a Poisson bracket $\{\cdot,\cdot\}$. Fixing an observable H to be the *Hamiltonian*, time evolution is given in terms of the Poisson bracket as $\frac{\partial}{\partial t} = \{H,\cdot\}$. This is the setting in which usual Hamiltonian mechanics takes place.

1.2. **Symplectic structures on moduli spaces.** Despite being a generalization of symplectic geometry, one important application of shifted symplectic geometry is that it can be used to produce several interesting examples of symplectic manifolds (presuming we already care about those). Let us take a look at two examples of interesting symplectic spaces, the existence of which we will later be able to describe via shifted symplectic methods.

Example 1.2.1. Let Σ be an oriented topological surface. That is to say, a Riemann surface, but we won't need its complex structure. The claim is that $\mathrm{Loc}_G(\Sigma)$ carries a symplectic structure. Here $\mathrm{Loc}_G(\Sigma)$ is the moduli space of G-local systems on Σ, e.g. locally constant sheaves with G-action, for G a fixed compact or complex reductive group. Slighly more precisely, we may write

$$\mathrm{Loc}_G(\Sigma) = \mathrm{Map}_{\mathrm{loc\ const}}(\Sigma, BG),$$

where on the right we are viewing Σ as a homotopy type. The points of $\mathrm{Loc}_G(\Sigma)$ can be identified with flat G-bundles on Σ, the latter being viewed as a smooth manifold. That is to say, we have $(\mathcal{P},\nabla) \in \mathrm{Loc}_G(\Sigma)$ with $\mathcal{P}$ a G-bundle and ∇ a flat connection on $\mathcal{P}$. We may identify the tangent space of this moduli space as

$$T_{(\mathcal{P},\nabla)}\mathrm{Loc}(\Sigma, BG) = \mathrm{H}^1(\Sigma; \mathrm{ad}(\mathcal{P},\nabla)).$$

For simplicilty, let us now assume that $G = \mathrm{GL}_n$, so the GL_n-bundle $\mathcal{P}$ corresponds to a vector rank n bundle E on Σ. In this case we may refine the above formula to

$$T_{(\mathcal{P},\nabla)}\mathrm{Loc}(\Sigma, BG) = \mathrm{H}^1_{\mathrm{dR}}(\Sigma; \mathrm{End}\, E).$$

On the right side, we really have (the cohomology of) the twisted de Rham complex $(\mathrm{End}(E)\otimes\Omega^\bullet,\nabla)$. Using the trace form on $\mathrm{End}\, E$ and integrating it over Σ, we obtain the symplectic structure on $\mathrm{Loc}_G\,\Sigma$.

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For an arbitrary group G, we do not have access to the trace form. To play its role, it suffices to choose an arbitrary non-degenerate invariant form on $\mathfrak{g}$, and then procede as before.

Example 1.2.2. In a similar spirit, let S be a holomorphic surface (4 real dimensions) which is Calabi-Yau, so that $\omega_S \simeq \mathcal{O}_S$. The funkiest example is when S is a K3 surface. But if that doesn't get you excited on its own, you may prefer the easier example of $S = T^*\Sigma$ for a Riemann surface Σ (or you could compactiy this cotangent space, ...).

Consider the moduli space $\mathrm{Bun}_G S$ of *holomorphic* G-bundles on S, that is, the mapping stack $\mathrm{Map}(S, BG)$. As before, this also turns out to carry a symplectic structure. These moduli spaces are very important in physics, for instance Hitchin systems arise this way. E.g., a special case is the Higgs moduli space

$$\mathrm{Bun}_G T^*\Sigma \overset{\mathrm{def}}{=} \mathrm{Higgs}_G \Sigma$$

whose points are G-budnles $\mathcal{P}$ on Σ together with a choice of $\eta \in \mathrm{H}^0(\mathcal{P}; \mathrm{ad}\,\mathcal{P} \otimes \Omega^1)$.

Remark 1.2.3. While Example 1.2.1 is certainly true as stated (up to properly defining everything in sight), there is some subtlety about how Example 1.2.2 has to be interpred. In particular, it might be better to consider it up to Fourier transform, by replacing $\mathrm{Bun}_G *S$ with an appropriate moduli space $\mathrm{Shv}_G S$ of sheaves with 1-dimensional support. That said, we shall not currently concern ourselves much with technicalities.

1.3. **Unifying principle, using shifted symplectic geometry.** The two examples are closely related, but different, both for their applications in physics, but also because the first time we considered local systems, whereas the second time we considered all holomorphic bundles. Likewise the base space was a Riemann surface in the first, while an Calabi-Yau algebraic surface in the second.

On the other hand, the key similarity between the two examples is that both consider a stack of the form

$$\mathrm{Map}(?, BG).$$

In light of this, both examples are special cases of the following paradigm:

Principle 1.3.1. *The classifying stack BG admits a **2-shifted** symplectic structure. This shifted structure induces an (ordinary) symplectic structure on* $\mathrm{Map}(?, BG)$*.*

Remark 1.3.1. Please note that this principle is not meant to be precise. In particular, we are purpusefully avoiding specifying what structure we must assume ? to come equipped with. We shall return to this point later, once we know at least vaguely what shifted symplectic structure really is.

1.4. **Quantum mechanics.** We exaplained why you might care about symplectic manifolds if you care about classical mechanics. But even if you are only interested in quantum mechanics, you probably care about them. That is because of *quantization*, the process of obtaining a quantum system out of a classical one.

As we explained in Section 1.1, the setting for classical mechanics is a symplectic manifold (M, ω). In fact, it suffices to extract the Poisson algebra of observables $(\mathcal{O}(M), \{\cdot,\cdot\})$. Its quantization should be an aprrropriate non-commutative algebra $\mathcal{O}_\hbar(M)$, depending on a parameter $\hbar$, and reducing to the classical case when $\hbar \mapsto 0$.

So now you are surely convinced that you care about symplectic manifolds, and by Section 1.3 thus also about shifted symplectic ones!

1.5. **Time for a definition.** Shifted symplectic manifolds should come in several types, depending on how shifted they are. The most basic example of a shifted symplectic maniold should be a symplectic manifold, which we will re-christen a 0-symplectic manifold. We have also encountered a 2-shifted symplectic space in the form of BG. Though it seems natural to continue trying to find examples of n-shifted symplectic spaces for increasing

n, it turns out that a lot of the subject is motivated by the other direction: as we will discuss more later (see Section 1.10), it turns out that (-1)-shifted symplectic structures are in some sense the most fundamental.

In any case, as the title of this section indicates, it is time for a definition. Recall first the one you presumably already know:

Definition 1.5.1. A *symplectic structure* is a closed 2-form $\omega \in \Omega_M^{2,\mathrm{cl}}$ such that the map it induces $T_M \xrightarrow{\omega} T_M^*$ is an isomorphism.

To arrive at the shifted analogue, we merely shift everyhting in sight witht the chain complex shift-by-k operation $[k]$.

Definition 1.5.2. A *k-shifted symplectic structure* is a k-shifted closed 2-form $\omega \in \Omega_M^{2,\mathrm{cl}}[k]$ such that the map it induces $T_M \xrightarrow{\omega} T_M^*[k]$ is an isomorphism.

For this to of course even be possible, we need to adopt the context of derived algebraic geometry. This means that we can take Spec of things such as CDGAs[1] giving us access to shifts. The *shifted cotangent bundle* may be defined as

$$T^*M[k] = \mathrm{Spec}_M \,\mathrm{Sym}\, T_M[-k],$$

the relative Spec of the CDGA $\mathrm{Sym}\, T_M[k]$, with T_M being the tangent bundle, viewed as a chain complex in degree 0. When M is not smooth, we need T_M to intead mean the tangent complex[2], but nothing else changes. The only thing that is a bit hard, as we shall see later in this seminar, is defining what it means for $\omega \in \Omega_M^2[k]$ to be closed.

1.6. **Shifted Poisson algebras.** One of the main upshots from having a shifted symplectic structure is that it allows to define shifted Poisson algebras. David finds those somewhat more fundamental and easier to remember.

Given M a k-shifted symplectic space, then $\mathcal{O}_M$ comes with the structure of a $\mathbb{P}_{k+1}$-*algebra*. That means that it has two pieces of data:

- a commutative product $\cdot$,
- a Poisson bracket $\{\cdot,\cdot\}$ of degree $1-(k+1) = -k$,

and, as any Poisson bracket, $\{\cdot,\cdot\}$ is required to be a Lie bracket, and a biderivation with respect to the product.

Remark 1.6.1. The numbers in $\mathbb{P}_{k+1}$ are a little annoying and non-intuitive, and occasionally require some light mental gymnastics such as $1-(k+1) = -k$ to make them seem plausible. This is due to conflicting with the standard $\mathbb{E}_n$-notation, because people who first studied shifted Poisson structures did not want to rename classical Poisson algebras as 1-Poisson, wanting them to be 0-Poisson instead.

Thus a functional definition of a shifted symplectic structure is: *what is necessary for the ring of functions to be shifted Poisson.*

But why would you care about shifted Poisson algebras? Well, if you care about homotopy theory, then you alread care. That is because it turns out that

$$\mathbb{P}_{k+1} = \mathrm{H}^*(\mathbb{E}_k).$$

Hence (shifted) Poisson structures may be viewed as some sort of linearizations or shadows of (higher homotopy) commutative ones.

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[1]CDGA stands for commutative differential graded algebra, and is a rather standard notion in homological algebra and related areas of algebra and geometry.

[2]This is the dual of the slightly more famous *cotangent complex*. David denotes it T_M^* and doesn't like that people often use L_M instead. Sorry, David! :)

1.7. **Examples of shifted symplectic structures.** The easiest example of a k-shifted symplectic space is the k-shifted cotangent complex $T^*X[k]$. That is very simple and non-surprising, but is already a source of genuinely interesting consequences.

Example 1.7.1. Consider the classifying space $BG = \bullet/G$. A standard DAG computation shows that $T_{\bullet/G} = \mathfrak{g}[1]$, and so $T^*_{\bullet/G} = \mathfrak{g}^*[-1]$, from which we can find that the 1-shifted cotangent bundle of BG is

$$T^*(\bullet/G)[1] = \mathfrak{g}^*/G.$$

Thus the coadjoint quotient $\mathfrak{g}^*/G$ is an example of a 1-shifted symplectic manifold.

An application of this shifted symplectic structure is that *Hamiltonian reduction* from ordinary symplextic geometry admits a nice description. Namely, there is an equivalence between Hamiltonian G-actions, which may be encoded as G-equivariant moment maps $M \xrightarrow{\mu} \mathfrak{g}^*$, and between 1-shifted Lagrangians[3] $M/G \to \mathfrak{g}^*/G$. Pavel Safranov has found striking applications of this in representation theory, repackaging some results about quantum groups to avoid some ugly formulas.

Example 1.7.2. The other example of a shifted symplectic space that we already encountered is the classifying space BG itself. We claimed it was 2-shifted symplectic, and indeed we may identify $\Omega^{2,\mathrm{cl}}_{BG}[2] = (\mathrm{Sym}^2\,\mathfrak{g}^*)^G$ with the space of bilinear invariant forms of $\mathfrak{g}$. For instance, we may pick the form to be the Killing form, i.e. the trace pairing. Furthermore the condition that the (2/shifted) 2-form is symplectic is equivalent to requiring the bilinear form to be nondegenerate.

There are various variants of this example. Here is one:

Example 1.7.3. Identifying $B\mathrm{GL}_n$ with the classifying stack of rank n vector bundles, we could also consider the classifying stack of perfect complexes, which we will denote $B\mathrm{Perf}$. It may be defined by functor of points as $\mathrm{Map}(X, B\mathrm{Perf}) = \mathrm{Perf}(X)$, the perfect chain complexes of sheaves on X. Then just like $B\mathrm{GL}_n$ has a 2-shifted symplectic structure, so does $B\mathrm{Perf}$.

1.8. **The general principle.** Time to return to the principle encountered in Section 1.3, and outline it in more detail.

Theorem 1.8.1 (Main Theorem of PTVV; reformulation of the (BV)AKSZ construction)**.** *Let M be a k-shifted symplectic space, and let Σ be d-oriented. Then the moduli stack* $\mathrm{Map}(\Sigma, M)$ *is $(k-d)$-shifted symplectic.*

The notion of Σ being *d-orientabile* means rougly that either:

- Σ is an oriented d-fold (topological dimension d),
- Σ is a Calabi-Yau d-fold (topological dimension $2d$).

In the situation of the theorem, we have the diagram

$$\mathrm{Map}(\Sigma, M) \xleftarrow{\pi} \Sigma \times \mathrm{Map}(\Sigma, M) \xrightarrow{\mathrm{ev}} M$$

and the main idea is the following: pull the shifted symplectic form on M back along the evaluation map ev, and then integrate it over the fibers of π to produce a form on $\mathrm{Map}(\Sigma, M)$. It is in the second step that the assumption of d-orientability is used, and it is also this fiber integration which looses d dimensions, explaining the discrepancy in the shiftedness of the symplectic structures on M and $\mathrm{Map}(\Sigma, M)$.

Remark 1.8.2. There also exists a version of Theorem 1.8.1 for shifted Poisson structures, and is appropriately the main theorem of CPTVV, the sequel paper to PTVV.

[3]Of course we haven't mentioned what shifted Lagrangians are yet, and we will not say more about them here and now, other than that a good theory of them exists, as we will return to them in a few weeks.

The moduli we talked about in Section 1.2 were all special cases of the construction of this Theorem for $M = BG$. Of course, the original idea behind those examples, as well as the PTVV Theorem and particularly its predecesor AKSZ, has to do with a certain highly popular buzzword that often rears its head in the BZ(R) Seminar:

1.9. **Quantum Field Theory.** Indeed, the authors initialized in AKSZ are all physicists; of note are K for Kontsevich and S for Albert Schwarz, David's academic great great grandfather.

The idea is this: *the fields in our field theory should be maps from the spacetime* Σ *to some appropriate codomains.* For instance, in the examples of Section 1.2 the codomain was BG, which therefore has to do with gauge theory.

In general, we are trying to get symplectic manifolds as phase spaces of field theory. But there is something more elemental we can do: we can write down the Lagrangian, as opposed to Hamiltonians implicit in the previous discussion.

The difference is that before, in the Hamiltonian formalism, we were ignoring time. In the Lagrangian approach, we are considering all paths $x(t)$ in our chosen symplectic manifold M (in which the fields take values), and their parameters t plays the role of time. The *action* is defined as

$$a(\varphi) = \int_\Sigma \mathscr{L}(\varphi(x))\, dx,$$

where $\mathscr{L}$ is the Lagrangian (a classical observable, i.e. a function on M) and φ is a field (i.e. a map $\varphi : \Sigma \to M$). In classical mechanics, we are trying to extremize the action. That is to say, the path that a particle *actually follows* is one which minimizes the action. Summarizing:

Principle 1.9.1 (of Least Action)**.** *Classical mechanics is concerned only with* $\mathrm{Crit}(a)$*.*

Here $\mathrm{Crit}(a)$ denotes the critical locus of the action functional a. But the above principle is only about classical mechanics, and we really want to QFT.

1.10. **Perturbative QFT.** Suppose we wanted to instead of classical do *perturbative quantum mechanics* - roughly that means that the quantum parameter $\hbar$ is very small[4]. Classically we only needed the critical locus of the action, but here we need more: paths can explore the whole space, as each path that a particle might have travelled (even if such a path would violate the laws of classical mechanics) can contribute to the Feynmann integral[5]

$$\int e^{-\frac{i}{\hbar}a(\varphi)}\, D\varphi,$$

through which probability amplitudes and expectation values can all be expressed. But all is not so bleak:

Principle 1.10.1 (that David learned from Costello)**.** *Perturbative quantization only depends on* $\mathrm{dCrit}(a)$*.*

Here $\mathrm{Crit}(a)$ denotes the *derived critical locus* of the action. More precisely, we need $\mathrm{dCrit}(a)$, together with some extra structure on it. The cool thing now is: it turns out that $\mathrm{dCrit}(a)$ carries a (-1)-shifted symplectic structure, and this is precisely the data needed to define field theory!

Remark 1.10.1. Note that Principle 1.10.1, while pleasingly parallel to the Least Action Principle 1.9.1 governing classical mechanics, is quite surprising. It is valid despite the fact that quantum mechanics, unlike classical mechanics, can move away from $\mathrm{Crit}(a)$!

[4]Recall that a basic principle of quantization is that setting $\hbar \mapsto 0$ should reproduce the classical system which we are trying to quantize, while the actual physically relevant quantum system will be obtained by setting $\hbar$ to be the Planck constant, the usual value this symbol takes in physics.

[5]These physical gadgets are notoriously hard to define rigorously, but in principle, we should be integrating over the space of all fields, and with respect to a mystical measure $D\varphi$.

To see what this Principle affords us, let's recall our setup: we want $\mathrm{Map}(\Sigma, M)$ to be the space of fields for the QFT. What we need to specify additionally is various sorts of coordinates on this space, which is what people clasically did. Instead, in light of Principle 1.10.1, *we just need the (-1)-shifted symplectic structure, which is the fundamental object.* That is (part of) what was meant by the somewhat cryptic remark in Section 1.5.

1.11. **Derived critical locus.** So, seeing how we now care about it, what is $\mathrm{dCrit}(a)$ really?

Let us answer in slighlty greater generality, abandoning the specific case of the action functional for a general space Z and function on it $f \in \mathcal{O}(Z)$. Then $\mathrm{dCrit}(f)$ is just the critical locus of f, but done correctly, in a sense in which the usual critical locus $\mathrm{Crit}(f)$ is not always correct. Recall that the latter critical locus may be defined as the intersection

$$\mathrm{Crit}(f) = \Gamma(df) \cap 0 \subset T^*Z$$

of the "graph" $\Gamma(df) \subset T^*Z$ of the differential df with the zero section $0 \subset T^*Z$ of the cotangent bundle. Conversely, the derived critical locus may also be written in the form

$$\mathrm{dCrit}(f) = \Gamma(df) \cap 0 \subset T^*Z,$$

but the intersection must be *taken in the derived sense.*

Remark 1.11.1. We briefly sketch how derived intersections work. Recall that an ordinary intersection $X \cap Y$ is given by a pair of points, one in each subspace $X, Y \subset Z$ we are trying to intersect, such that those two points are equal as points in Z. Similarly the derived intersection, which we will here denote $X \overset{\mathrm{R}}{\cap} Y$ to notationally distinguish it from classical intersection $X \cap Y$, is given by such a pair of points, one in each subspace X and Y that we are trying to intersect, but instead of requiring the two points to be equal, we require there to be a path inside the whole space Z connecting them[6]. Since a path is contractible and therefore from the point of view of homotopy theory just as good as a point, this is indeed a homotopical way of identifying the two points.

Returning to the intersection in hand, observe that both $\Gamma(f)$ and the zero section 0 are Lagrangians with respect to the symplicial structure on the cotangent bundle T^*Z. So we are led to ask what structure is to be found on an intersection of two Lagrangians. The classical answer is that it is a discrete set. The fancy answer is that it carries a (-1)-shifted symplectic form. To understand how that works, we need to stoop down to performing

1.12. **A computation.** Explicitly, we are trying to understand

$$\Gamma(df) \overset{\mathrm{R}}{\cap} 0 = \mathrm{Spec}\left(\mathcal{O}_{\Gamma(df)} \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_{T^*Z}} \mathcal{O}_{\text{zero section}}\right), \tag{1}$$

where the derived intersection $\overset{\mathrm{R}}{\otimes}$ on the LHS turned into the derived tensor product $\overset{\mathrm{L}}{\otimes}$ on the RHS. To compute the latter, we can resolve $\mathcal{O}_0 = \mathcal{O}_{\text{zero section}}$, which fiber-wise amounts to resolving a point inside a vector space.

This is achieved by the *Koszul resolution*: since cotangent spaces are all smooth and affine, it suffices to consider $X = \mathrm{Spec}\, S$ for $S = \mathbf{C}[x_1, \ldots, x_n]$. The point $0 \in X$ is given by the S-algebra $\mathbf{C} = \mathbf{C}_0$, where we have set $x_i = 0$ for all i. The Koszul complex is then $(S \otimes \Lambda^* T_0, d_{\mathrm{Kosz}})$, where T_0 is the tangent space to X at the point 0, and the Koszul differential d_{Kosz} is given by contracting with df in a way perhaps familiar from the theory of differential forms.

Such a Koszul resolution is a fiber-level construction, but simply tensoring with the variables of the base space Z extends it to a global one over Z. Let us specialize to the extreme case when $f = 0$, so that all the differentials in the Koszul complex vanish. By

[6] A slight subtlety is that is does not suffice to merely assert the existence of a path connecting the two points; we need instead to specify such a path as part of the data of a derived intersection.

replacing $\mathcal{O}_0$ with the Koszul complex $\Lambda^* T$ (here $T = T_Z$ is the tangent sheaf on Z), the derived tensor procut $\overset{\mathrm{L}}{\otimes}$ in (1) becomes replaces with the ordinary tensor product $\otimes$, and so we find that

$$\mathcal{O}(0 \overset{\mathrm{R}}{\cap}_{T^*Z} 0) = \Lambda^* T.$$

That is to say, derived self-intersection of a point looks like the exterior algebra, which may itself be written as

$$\Lambda^* T = \operatorname{Sym} T[1] = \mathcal{O}(T^*[-1]).$$

Summarizing, we have learned that in the most extreme case of a zero function we have

$$\mathrm{dCrit}(0) = T^*[-1].$$

More excitingly, for an arbitrary function f things looks almost the same, only with a different differential. Indeed, just as before

$$\mathcal{O}(\Gamma(df) \overset{\mathrm{R}}{\cap}_{T^*Z} 0) = (\Lambda^* T = \operatorname{Sym} T[1] = \mathcal{O}(T^*[-1]), \lrcorner\, df),$$

where, as indicated, the differential is given roughly by contracting with df.

We can now deliver on the promise made at the end of Section 1.11 and exibit a shifted symplectic structure on the derived critical locus. Indeed, the exterior algebra of multivectorfields $\Lambda^* T$ carries the usual Schouten-Nijenhuis bracket $\{-,-\}$, a degree 1 Lie bracket known to be compatible with contractions such as $\lrcorner\, df$, thus it descends to $\mathcal{O}(\mathrm{dCrit}(a))$ and makes it into a $\mathbb{P}_0$-algebra. But we learned in Section 1.6 that this is equivalent to a (-1)-shifted symplectic structure on $\mathrm{dCrit}(a)$ itself, and our promise is fulfilled.