PARTY HATS AND EXPLOSIONS

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The subject matter of this note concerns, as the title suggests, party hats and explosions. More precisely, we will explain how to explode a part hat. In order to be able to do so, we must first explain what we mean by both party hats and by explosions. We will do so in two ways, equivalent in content, but divergent in flavor:

1. Via classical algebraic geometry

Throughout this note, we work over a fixed algebraically closed field k. Take $k = \mathbf{C}$ if it eases psychological distress, but note that no special properties of the complex numbers will be used. That is to say, we will be engaging in *algebraic* geometry, as opposed to utilizing the rich analytic or differentiable structure that would be available over $\mathbf{C}$, but not over an arbitrary base field k.

1.1. Affine geometry.

As such, we adopt the terminological conventions of algebraic geometry:

- The *n-dimensional affine space* $\mathbf{A}^n$ is the vector space k^n.
- The *multiplicative group* $\mathbf{G}_m$ is the group of units $k^\times$ under multiplication.
- An *affine variety* is a subset $Y \subseteq \mathbf{A}^n$, cut out by (i.e. given as the simultaneous vanishing locus of) a collection of polynomial equations.

Though the notation and terminology suggest one should view $\mathbf{A}^n$ as an *affine* space, i.e. without a distinguished point $0 \in \mathbf{A}^n$, we will nonetheless make that choice for convenience. Nonetheless, while we use the notation and terminology of "the origin", this merely amounts to *a choice* of a point in affine n-space, and any other chosen point would do just as well. That is implicitly done anyways when writing things in $\mathbf{A}^n$ in terms of coordinates[1].

1.2. Projective space.

The basic idea of projective geometry is that the *n-dimensional projective space* should be

$$\mathbf{P}^n = \{\text{lines } L \subseteq \mathbf{A}^{n+1} \text{ through the origin } 0\}.$$

Any point $P \in \mathbf{A}^{n+1} - \{0\}$ determines a unique line $L = \overline{P0} \subseteq \mathbf{A}^{n+1}$ that passes both through it and the origin. Scaling the point P (or more precisely, the vector from 0 to P) does not change the line L. This suggests the quotient description

$$\mathbf{P}^n = (\mathbf{A}^{n+1} - \{0\})/\mathbf{G}_m$$

for the scaling $\mathbf{G}_m$-action on the punctured n-space $\mathbf{A}^{n+1} - \{0\}$.

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[1]In fact, more choice is required in that case: that of the n coordinate axis, and of a point $(0, \ldots, 0, 1, 0, \ldots, 0)$ on each.

1.3. **Homogeneous coordinates.** Let us try to understand this in coordinates. The we set $P = (x_0, \dots, x_n) \in \mathbf{A}^{n+1} - \{0\}$, and denote its image under the quotient map $\mathbf{A}^{n+1} - \{0\} \to \mathbf{P}^n$ by $L = \overline{P0} = [x_0 : \dots : x_n] \in \mathbf{P}^n$.

The *homgoeneous coordinates* $[x_0 : \dots : x_n] \in \mathbf{P}^n$ thus makes sense whenever $x_i \neq 0$ holds for at least one $0 \le i \le n$. Since $\overline{P0} = \overline{(\lambda P)0}$ holds for any scalar $\lambda \in k^\times$, we have

$$[x_0 : \dots : x_n] = [\lambda x_0 : \dots : \lambda x_n].$$

In light of the quotient description of $\mathbf{P}^n$, this fully determines homogeneous coordinates.

1.4. **Projective varieties.** Let $X \subseteq \mathbf{P}^n$ be a *projective variety*, i.e. the vanishing locus

$$X = \{[x_0 : \dots : x_n] \in \mathbf{P}^n \mid f_i(x_0, \dots, x_n) = 0 \ \ \forall i\}$$

of some collection of homogeneous polynomials $f_i(x_0, \dots, x_n) \in k[x_0, \dots, x_n]$. Homogeneity of the defining polynomial equations is imposed in order that the property of their vanishing is independent of rescaling, and hence makes sense for homogeneous coordinates.

1.5. **Tautological bundle.** A point $L \in \mathbf{P}^n$ may be viewed as a line $L \subseteq \mathbf{A}^{n+1}$, and as such as a 1-dimensional vector space. This leads to the *tautological line bundle* $\mathcal{O}(-1)$ on $\mathbf{P}^n$, whose fiber over a point $L \in \mathbf{P}^n$ is given by

$$\mathcal{O}(-1)_L := L.$$

Given a projective variety $X \subseteq \mathbf{P}^n$, it also admits a tautological bundle $\mathcal{O}_X(-1)$, obtined either as pullback (i.e. restriction) of the tautological bundle $\mathcal{O}(-1) = \mathcal{O}_{\mathbf{P}^n}(-1)$ from the ambient projective space $\mathbf{P}^n$, or equivalently by the same fiber-wise description

$$\mathcal{O}_X(-1)_L := L$$

for all points $L \in X$.

Remark 1.1. The reason for the -1 in the notation for the tautological bundle $\mathcal{O}_X(-1)$ is that there is an easy modification of it that makes sense of the so-called *Serre twists* $\mathcal{O}_X(n)$ for any $n \in \mathbf{Z}$. If $n \ge 0$, we define fiber-wise over $L \in X \subseteq \mathbf{P}^n$

$$\mathcal{O}_X(-n)_L := L^{\otimes n}, \qquad \mathcal{O}_X(n) := (L^\vee)^{\otimes n}.$$

These are still 1-dimensional vector spaces, but the difference is that the tensor powers and/or duals change the transition maps (suppressed in our account) that give the bundle structure on $\mathcal{O}_X(n)$.

1.6. **Blowup of a plane at a point.** The *blowup* $\mathrm{Bl}_0(\mathbf{A}^{n+1})$ *of* $\mathbf{A}^{n+1}$ *at the origin* is supposed to leave $\mathbf{A}^{n+1} - \{0\}$ unchanged, but replace the point $0 \in \mathbf{A}^{n+1}$ with the collection of all the lines through it. The lines in $\mathbf{A}^{n+1}$ through the origin form, as discussed above in Section 1.2, the projective space $\mathbf{P}^n$. This suggests defining the blowup as

$$\mathrm{Bl}_0(\mathbf{A}^{n+1}) := \{(P, L) \in \mathbf{A}^{n+1} \times \mathbf{P}^n \mid P \in L\}.$$

Let us verify that this definition does the job. Sending $(P, L) \mapsto P$ gives a canonical map $\pi : \mathrm{Bl}_0(\mathbf{A}^{n+1}) \to \mathbf{A}^{n+1}$, the fibers of which are

$$\pi^{-1}(P) \cong \{L \in \mathbf{P}^n \mid P \in L\} = \begin{cases} \{L = \overline{P0}\} & \text{if } P \neq 0, \\ \mathbf{P}^n & \text{if } P = 0 \end{cases}$$

as desired.

1.7. **Blowup in coordinates.** Let us express the subset $\mathrm{Bl}_0(\mathbf{A}^{n+1}) \subseteq \mathbf{A}^{n+1} \times \mathbf{P}^n$ in terms of the standard and homogeneous coordinates on paffine and projective space respectively. Setting $P = (x_0, \dots, x_n)$ and $L = [y_0 : \dots : y_n]$, a pair $(P, L) \in \mathrm{Bl}_0(\mathbf{A}^{n+1})$ satisfies $P \in L$ if and only if

$$[x_0 : \dots : x_n] = [y_0 : \dots : y_n].$$

By definition of homogeneous coordinates, that is equivalent to asking that there exists a scalar $\lambda \neq 0$ such that

$$x_i = \lambda y_i \quad \forall i.$$

For those $1 \leq i \leq n$ for which $y_i = 0$, this means that it must also be that $x_i = 0$. If $y_i \neq 0$, we may on the other hand write

$$\lambda = \frac{x_i}{y_i}. \tag{1}$$

This holds for all such i, hence the defining equation can be rewritten as

$$\frac{x_i}{y_i} = \frac{x_j}{y_j}$$

for all indices $0 \leq i, j \leq n$ for which $y_i, y_j \neq 0$. Rewriting this in the still equivalent form

$$x_i y_j = x_j y_i,$$

this now holds for all i, j, and encodes both (1) and the vanishing condition. Hence we obtain the coordinate description of the blowup of affine $(n+1)$-space at a point as

$$\mathrm{Bl}_0(\mathbf{A}^{n+1}) = \{((x_0, \dots, x_n), [y_0 : \dots, y_n]) \in \mathbf{A}^{n+1} \times \mathbf{P}^n \mid x_i y_j - x_j y_i = 0 \ \ \forall 0 \leq i, j \leq n\}.$$

Since this exibits it as the zero locus of polynomial equations, this shows $\mathrm{Bl}_0(\mathbf{A}^{n+1})$ is a variety itself[2].

1.8. **Blowup of an affine variety.** Let $Y \subseteq \mathbf{A}^{n+1}$ be an affine variety containing the origin, i.e. such that $0 \in Y$. The *blowup of Y along the point* $0 \in Y$ is defined as the closure

$$\mathrm{Bl}_0(Y) = \overline{\pi^{-1}(Y - \{0\})} = \overline{\{(P, L) \in (Y - \{0\}) \times \mathbf{P}^n \mid P \in L\}},$$

where the closure has the effect of adding in those lines $L \subseteq \mathbf{A}^{n+1}$ through 0 which are tangential directions of approach[3] to 0 along Y.

Remark 1.2. We may (fancifully) concieve of the blowup in the following way, (partiially) justifying the name. Set some explosives at the chosen point of the variety, then right after you detonate them, freeze time. Everything that was located at the point before is now sent flying into all directions - all directions away from the point, that is. On the other hand, things far away from the detonation site remain uneffected (at least in the moment right after the explosion - the shock waves are yet to reach them). precisely the geometric situation en

1.9. **Key property of blowup.** Keeping the same notation as in the previous section, there exists a canonical map $\pi : \mathrm{Bl}_0(Y) \to Y$, inherited from the eponymous one from the ambient case $Y = \mathbf{A}^{n+1}$. Its characterizing properties are:

(1) $E := \pi^{-1}(\{0\}) \subseteq \mathrm{Bl}_0(Y)$ is a *divisor*, i.e. a codimension 1 subvariety.
(2) $\mathrm{Bl}_0(Y) - E = \pi^{-1}(Y - \{0\}) \xrightarrow{\cong} Y - \{0\}$.

[2]However it is neither an affine or a projective variety. It instead an instance of a more general class of *quasi-projective varieties*. These are, formally speaking, locally closed subsets in $\mathbf{P}^n$ for the Zariski topology, and subsume both affine and projective varieties.

[3]Though this might sound as describing the entire tangent space $T_0 Y$, it instead ends up being the smaller *tangent cone* $TC_0 Y \subseteq T_0 Y$. When 0 is a non-singular point in Y, the two agree, but at a singularity $0 \in Y$, the tangent cone is a more refined "linear" approximate of Y at point 0.

Note that under the canonical map $\pi : \mathrm{Bl}_0(\mathbf{A}^n) \to \mathbf{A}^n$, the preimage of an (irreducible) affine variety $Y \subseteq \mathbf{A}^n$ has two irreducible components

$$\pi^{-1}(Y) = \mathrm{Bl}_0(Y) \cup \pi^{-1}(0);$$

the blowup $\mathrm{Bl}_0(Y)$ (sometimes also called the *proper transform* of Y) and $\pi^{-1}(0) \cong \mathbf{P}^n$.

Also, as the picture of blowing up the nodal (or cuspidal) plane cubic $Y \subseteq \mathbf{A}^2$, e.g. given by $x_0^2 = x_1^2(x_1 - 1)$ (resp. $x_0^2 = x_1^3$) at the node (resp. cusp) shows, blowing up at a singular point often has the effect of resolving (or at least reducing) singularieties.

Remark 1.3. It fact, any algebraic curve may be desingularized by a finite sequence of blowups at points. A vast generalization of this is Hironaka's Theorem, asserting the analogous assertion is true for a variety of arbitrary dimension over a field of characteristic zero, so long as we allow ourselves to blow up along more general subvarieties.

1.10. **Cone over a projective variety.** With the "explosions" part of the title now clarified, we turn next to "party hats". Let $X \subseteq \mathbf{P}^n$ by a projective variety. The *(affine) cone over* X is the set

$$C := \bigcup_{L \in X} L \subseteq \mathbf{A}^{n+1}.$$

That is to say, C consists of all the points in $\mathbf{A}^{n+1}$ which lie on lines L that define elements $L \in X \subseteq \mathbf{P}^{n+1}$. In terms of defining homogeneous polynomial equations for the projective variety

$$X = \{[x_0 : \dots : x_n] \in \mathbf{P}^n \mid f_i(x_0, \dots, x_n) = 0 \ \ \forall i\},$$

the affine cone over X is cut out by the same equations, but interpreted in affine space

$$C = \{(x_0, \dots, x_n) \in \mathbf{A}^{n+1} \mid f_i(x_0, \dots, x_n) = 0 \ \ \forall i\}.$$

In particular, it is an affine variety. Since the origin $0 \in \mathbf{A}^{n+1}$ lies on every line in $L \in X$ (or even $L \in \mathbf{P}^n$), or equivalently, since the polynomials f_i are all homogeneous and as such must vanish at the origin, it follows that the cone always includes a distinguished point $0 \in C$, called its *vertex*.

1.11. **Blowup of a cone at its vertex.** Real-life experience with cones, e.g. party hats, pyramids, etc. suggests that the vertex $0 \in C$ is likely to be a singular point. We may wonder if blowing it up resolves it.

Theorem 1.4. *Let X be a projective variety, and C its affine cone with vertex $0 \in C$. There is a canonical isomorphism of varieties*

$$\mathrm{Bl}_0(C) \cong \mathbf{V}_X(\mathcal{O}_X(-1))$$

between the blowup of the affine cone along the vertex and the total space of tautological bundle $\mathcal{O}_X(-1)$ over X.

Visually, we can imagine that blowing up a cone at the vertex creates a cylinder. This is what Theorem 1.4 justifies and generalizes.

Proof. Recall from Section 1.5 that the tautological line bundle $\mathcal{O}_X(-1)$ is given fiber-wise over $L \in X$ by $\mathcal{O}_X(-1)_L = L$. It follows that its total space is

$$\mathbf{V}_X(\mathcal{O}_X(-1)) = \{(P, L) \in \mathbf{A}^{n+1} \times X \mid P \in L\}, \tag{2}$$

with the bundle projection map $\mathbf{V}_X(\mathcal{O}_X(-1)) \to X$ given by $(P, L) \mapsto L$. Indeed, the fibers of this map are precisely

$$\mathbf{V}_X(\mathcal{O}_X(-1)) \times_X \{L\} \cong \{P \in \mathbf{A}^{n+1} \mid P \in L\} = L.$$

When $X = \mathbf{P}^n$, the total space formula (2) recovers the definition of the blowup from Section 1.6, so that we have

$$\mathbf{V}_{\mathbf{P}^n}(\mathcal{O}(-1)) \cong \mathrm{Bl}_0(\mathbf{A}^{n+1}).$$

The vector bundle structure is exhibited by sending the pair $(P, L) \in \mathbf{A}^{n+1} \times \mathbf{P}^n$ with $P \in L$ to the line component $L \in \mathbf{P}^n$, while the blowup structure is instead given by sending it to the point component $P \in \mathbf{A}^{n+1}$. In particular, this proves the theorem in the special case when $X = \mathbf{P}^n$, since the cone over it is clearly the entire affine space $C = \mathbf{A}^{n+1}$.

For an arbitrary projective variety $X \subseteq \mathbf{P}^n$, we have

$$\mathbf{V}_X(\mathcal{O}_X(-1)) \subseteq \mathbf{V}_{\mathbf{P}^n}(\mathcal{O}(-1)) \cong \mathrm{Bl}_0(\mathbf{A}^{n+1}),$$

allowing us to view the total space of the tautolotical bundle $\mathcal{O}_X(-1)$ as a subvariety of $\mathrm{Bl}_0(\mathbf{A}^{n+1})$. Note that the projection $\mathbf{V}_X(\mathcal{O}_X(-1)) \to \mathbf{A}^{n+1}$, given by $(P, L) \mapsto P$, surjects onto the affine cone $C \subseteq \mathbf{A}^{n+1}$ over X. Indeed, since $P \in L$, we have $P \in C$ if and only if $L \in X$ by the definition of the affine cone. Taking the preimage of $\mathbf{V}_X(\mathcal{O}_X(-1)) \to C$ over the complement of the vertex of the cone is

$$\begin{aligned}
\mathbf{V}_X(\mathcal{O}_X(-1)) \times_C (C - \{0\}) &= \{(P, L) \in (C - \{0\}) \times X \mid P \in L\} \\
&= \{(P, L) \in (C - \{0\}) \times \mathbf{P}^n \mid P \in L\} \\
&= \mathrm{Bl}_0(C) \times_C (C - \{0\}).
\end{aligned}$$

In light of the closure definition of the blowup $\mathrm{Bl}_0(C) \subseteq \mathrm{Bl}_0(\mathbf{A}^{n+1})$ from Section 1.8, it suffices, in order to identify $\mathbf{V}_X(\mathcal{O}_X(-1))$ and $\mathrm{Bl}_0(C)$, to show that the fiber of the map $\mathbf{V}_X(\mathcal{O}_X(-1)) \to C$ over the vertex $0 \in C$ is a divisor (see Section 1.9). For that, note that

$$\mathbf{V}_X(\mathcal{O}_X(-1)) \times_C \{0\} \cong \{L \in X \mid 0 \in L\} = X$$

is the zero section of the vector bundle $\mathbf{V}_X(\mathcal{O}_X(-1)) \to X$, and as such a divisor indeed. $\square$

2. Via scheme theory

After a hopefully-friendly classical geometric discussion so far, we will now redo some of it, and in particular re-prove Theorem 1.4, with the modern algebro-geometric language and technology. Our pace here will be much more brisk, as time constraints prohibit us from the (substantial undertaking) of giving a gentle introduction to scheme theory. Instead we merely provide a summary of the main notions that will show up in the re-proof of Theorem 1.4.

2.1. **Affine schemes.** Given a commutative k-algebra A, there is[4] an associated *affine scheme* $\mathrm{Spec}(A)$.

- A basis for its Zariski topology is given by *basic opens*

  $$D(f) = \mathrm{Spec}(A[f^{-1}])$$

  for elements $f \in A$. Viewing $f \in A = \mathcal{O}(\mathrm{Spec}(A))$ as a function on $\mathrm{Spec}(A)$, the basic open $D(f)$ is the open locus $\{f \neq 0\}$.
- A closed subscheme $Z \subseteq \mathrm{Spec}(A)$ is uniquely characterized by an ideal $I \subseteq A$, and isomorphic to $Z \cong \mathrm{Spec}(A/I)$. Viewing elements $f \in I \subseteq \mathcal{O}(\mathrm{Spec}(A))$ as functions on $\mathrm{Spec}(A)$, then Z is to be viewed[5] as the closed locus $\{f = 0, \forall f \in I\}$.
- If $A = k[x_1, \dots, x_n]/I$, then the quotient homomorphism $k[x_1, \dots, x_n] \to A$ induces a closed immersion

  $$\mathrm{Spec}(A) \hookrightarrow \mathrm{Spec}(k[x_1, \dots, x_n]) = \mathbf{A}^n,$$

  exhibiting the affine scheme $\mathrm{Spec}(A)$ as a closed subscheme of affine space.

[4]It may be defined as a ringed space by its underlying space consisting of all prime ideals in A, or via the functor-of-points approach as a corepresentable functor. These nuances are unimportant to us here, and we will leave the choice to the zealous reader.

[5]Note however that, unlike a *subvariety*, a closed *subscheme* is not merely a locus, but may admit non-reduced structure. That is to say, the actual ideal that cuts the subscheme is kept track of, as opposed to merely its radical.

- In contrast to the previous point, the power of scheme theory lies in allowing us to view *any* commutative ring A as a geometric object $\mathrm{Spec}(A)$, regardless of any relationship to polynomial rings. In that sense, we can work more coordinate-free and intrinsically, and the reducion of an algebro-geometric question to a purely algebraic problem is often formal and transparent (our proof of Theorem 2.3 may be seen as an instance, but there are many more convincing ones).

2.2. **Projective schemes.** Let S be a commutative $\mathbf{Z}_{\geq 0}$-graded k-algebra, i.e. $S = \bigoplus_{n\geq 0} S_n$ with $S_n S_m \subseteq S_{n+m}$. There is[6] an associated scheme $\mathrm{Proj}(S)$.

- A basis for its Zariski topology is given by *basic opens*

$$D_+(f) = \mathrm{Spec}(S[f^{-1}]_0)$$

  for homogeneous elements $f \in S_d$, $d \geq 1$. Here the localization $S[f^{-1}]$ is graded in the standard way, i.e. for any $n \in \mathbf{Z}$ the n-th piece of the grading is

$$S[f^{-1}]_n = \bigoplus_{\substack{i,j\geq 0\\ i-dj=n}} S_i f^{-j}.$$

- If $S = k[x_0, \ldots, x_n]/I$ (with the usual grading where $|x_i| = 1$ for all i) for a homogeneous ideal I, then the quotient homomorphism $k[x_0, \ldots, x_n] \to S$ induces a closed immersion

$$\mathrm{Proj}(S) \hookrightarrow \mathrm{Spec}(k[x_0, \ldots, x_n]) = \mathbf{P}^n,$$

  exhibiting the affine scheme $\mathrm{Proj}(S)$ as a closed subscheme of projective space.
- For any $n \in \mathbf{Z}$, the *n-th Serre twisting sheaf* $\mathcal{O}_{\mathrm{Proj}(S)}(n)$ over $\mathrm{Proj}(S)$ is defined by setting its sections over the basic open $D_+(f) \subseteq \mathrm{Proj}(S)$ for $f \in S_d$, $d \geq 1$ to be

$$\Gamma(D_+(f); \mathcal{O}_{\mathrm{Proj}(S)}(n)) := S[f^{-1}]_n,$$

  i.e. the homogeneous elements of the localization $S[f^{-1}]$ in degree n.
- If S is generated (as a ring by) $S_1 \subseteq S$ (as is for instance true for all projective varieties by the previous point), then there is the addition formula

$$\mathcal{O}_{\mathrm{Proj}(S)}(n+m) = \mathcal{O}_{\mathrm{Proj}(S)}(n) \otimes \mathcal{O}_{\mathrm{Proj}(S)}(m)$$

  for all $n, m \in \mathbf{Z}$. In particular, the Serre twisating sheaves are in that case line bundles. That is not generally true without the assumption of generation in degree 1.
- Unlike the correspondence $A \mapsto \mathrm{Spec}(A)$ between rings and affine schemes, which induces an equivalence of categories and as such loses and gains no data, the correspondence $S \mapsto \mathrm{Proj}(S)$ is more complicated. Diferent graded rings $S \neq S'$ can bive rise to the same scheme $\mathrm{Proj}(S) \cong \mathrm{Proj}(S')$; not every map of schemes $\mathrm{Proj}(S') \to \mathrm{Proj}(S)$ necessarily arises from a graded ring map $S \to S'$, nor does any such map induce a map of schemes on Proj, etc.

2.3. **Blowup of an affine scheme.** Let $Y = \mathrm{Spec}(A)$ be an affine scheme and $Z \subseteq Y$ a closed subscheme determined by the ideal $I \subseteq A$. The *blowup of Y along Z* is

$$\mathrm{Bl}_Z(Y) := \mathrm{Proj}(\bigoplus_{n\geq 0} I^n)$$

The A-module structure on the ideal I and its powers I^n gives rise to the structure map $\mathrm{Bl}_Z(Y) \to \mathrm{Spec}(A) = Y$, called either the blowup projection or sometimes the "blowdown".

[6]For what it's worth, its underlying space consists of all such homogeneous prime ideals $\mathfrak{p} \subseteq S$, which do not fully contain the irrelevant ideal $S_+ \subseteq S$.

Remark 2.1. Though we will not use anything other than the Proj definition of the blowup in what follows, let us nonetheless take a little detour to describe the schematic blowup, and see that it is indeed much like the classical above-discussed construction that it generalizes. The preimage under the blow-down projection $\pi : \mathrm{Bl}_Z(Y) \to \mathrm{Spec}(A) = Y$ of the "center of the blowup" $Z \subseteq Y$ is the *exceptional divisor*

$$\mathrm{E}_Z(Y) = \mathrm{Bl}_Z(Y) \times_Y Z = \mathrm{Proj}(\bigoplus_{n\geq 0} I^n \otimes_A A/I) = \mathrm{Proj}(\bigoplus_{n\geq 0} I/I^n),$$

which is precisely the definition of the projective normal cone $\mathbf{C}_{Z/Y}$. On the other hand, the restriction to the open complement

$$\mathrm{Bl}_Z(Y) - \mathrm{E}_Z(Y) = \mathrm{Bl}_Z(Y) \times_Y (Y - Z) \xrightarrow{\pi} Y - Z$$

is an isomorphism. Compare this to Section 1.9.

2.4. **Cone over a projective scheme.** With the scheme theory background reviewed, we can re-cast the content Section 1.10 in this setting.

Let $S = \bigoplus_{n\geq 0} S_n$ be a commutative graded k-algebra with $S_0 = k$, and let $X = \mathrm{Proj}(S)$. The *(affine) cone over* X is defined to be $C := \mathrm{Spec}(S)$. The *vertex* of this cone $0 \in C$ is the closed schscheme determined by the irrelevant ideal $S_+ := \bigoplus_{g\geq 1} S_n \subseteq S$. Indeed, the vertex is as a scheme of the form

$$\{0\} \cong \mathrm{Spec}(S/S_+) = \mathrm{Spec}(S_0) = \mathrm{Spec}(k);$$

a point as promised.

Remark 2.2. If S is generated as an algebra by S_1, then the Proj construction can be obtained by a quotient construction

$$X \simeq (C - \{0\})/\mathbf{G}_m$$

akin to the quotient presentation of projective space from Section 1.2. Here the "scaling" $\mathbf{G}_m$-action on the affine scheme $C = \mathrm{Spec}(S)$ is equivalent to (and induced from) the $\mathbf{Z}$-grading on the ring S. The vertex $\{0\} \subseteq C$ is precisely the fixed-point locus of this action, which is why its complement also inherits a $\mathbf{G}_m$-action.

2.5. **Vector bundles over schemes.** There is one last notion to recall: if $\mathcal{E}$ is a locally free sheaf of $\mathcal{O}_X$-modules, equivalent to a vector bundle on a scheme X, the corresponding *total space of* $\mathcal{E}$ is given by

$$\mathbf{V}_X(\mathcal{E}) := \mathrm{Spec}_X(\mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{E}^\vee)).$$

Here the *relative spectrum* $\mathrm{Spec}_X(\mathcal{A}) \to X$ of a quasi-coherent sheaf of $\mathcal{O}_X$-algebras $\mathcal{A}$ is a relative version (i.e. working over the general base X instead of merely a point like in 2.1) of Spec. It is defined via gluing from its restrictins over affine open subschemes $U \subseteq X$, where it is set to be

$$\mathrm{Spec}_X(\mathcal{A}) \times_X U \simeq \mathrm{Spec}(\Gamma(U; \mathcal{A})),$$

where the sheafyness of $\mathcal{A}$ enables gluing over different affine opens $U \subseteq X$.

2.6. **Blowup of a cone at its vertex revisited.** At last we turn to a scheme-theoretic proof of the next analogue of Theorem 1.4. The proof we sumarize can be found in full, and in a more general context, in EGA II, Section 7.8.

Theorem 2.3. *Let S be a graded k-algebra with $S_0 = k$, generated by $S_1 \subseteq S$. For $X = \mathrm{Proj}(S)$, the cone $C = \mathrm{Spec}(S)$ and vertex $0 \in C$ as in Section 2.4, there is a canonical isomorphism of schemes*

$$\mathrm{Bl}_0(C) \cong \mathbf{V}_X(\mathcal{O}_X(-1))$$

between the blowup of the affine cone along the vertex, and the total space of line bundle $\mathcal{O}_X(-1)$ over X.

Proof. The proof-strategy is as follows: we will rephrase each side of the desired isomorphism, the right side in (ii) and the left side in (i), using the assumed degree-one generation. Then we will identify the two rephrased terms, which will not require the generation assumption, and will reduce to a direct comparison of graded rings.

(i) The total space of the Serre twisting sheaf $\mathcal{O}_X(-1)$ is

$$\mathbf{V}_X(\mathcal{O}_X(-1)) = \mathrm{Spec}_X(\mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{O}_X(1))),$$

as we saw in Section 2.5. Since S is assumed to be generated in degree 1, the sheaf $\mathcal{O}_X(1)$ is invertible, and $\mathcal{O}_X(1)^{\otimes n} \cong \mathcal{O}_X(n)$. It being an invertible sheaf implies that the symmetric group action on its tensor powers are trivial, hence the canonical map

$$\mathcal{O}_X(n) \cong \mathcal{O}_X(1)^{\otimes n} \to \mathcal{O}_X(1)^n/\Sigma_n = \mathrm{Sym}^n_{\mathcal{O}_X}(\mathcal{O}_X(1))$$

is an isomorphism of sheaves of $\mathcal{O}_X$-modules. By passing to the sheaves of algebras and the relative spectrum over X, we obtain a canonical isomorphism

$$\mathbf{V}_X(\mathcal{O}_X(-1)) \cong \mathrm{Spec}_X\big(\bigoplus_{n\geq 0}\mathcal{O}_X(n)\big) \tag{3}$$

(ii) Recall from Section 2.3 that the blowup of the cone C at the vertex $0 \in C$, which is cut out by the irrelevant ideal $S_+ \subseteq S$, is given by

$$\mathrm{Bl}_0(C) = \mathrm{Proj}\big(\bigoplus_{n\geq 0} S_+^n\big).$$

Thanks to the degree-one generation assumption, we have $S_n = S_1^n$ for all $n \geq 0$, hence

$$\begin{aligned} S_+ &= S_1 + S_2 + S_3 + S_4 + \cdots \\ &= S_1 + S_1^2 + S_1^3 + S_1^4 + \cdots \\ &= S_1(k + S_1 + S_1^2 + S_1^3 + \cdots) \\ &= S_1(S_0 + S_1 + S_2 + S_3 + \cdots) \\ &= S_1 S. \end{aligned}$$

That is to say, the irrelevant ideal S_+ coincides with the ideal S_1S generated by the subset $S_1 \subseteq S$. It follows that $S_+ = S_1^n S$ holds for every $n \geq 0$. Let us introduce a new graded ring $S^\sharp$ by

$$S^\sharp := \bigoplus_{n\geq 0} S_{\geq n}, \tag{4}$$

i.e. the graded ring whose n-th graded piece is $S^\sharp_n = S_{\geq n} = \bigoplus_{m\geq n} S_m$. Thanks to the degree-1-generation assumption, we have

$$\begin{aligned} S^\sharp_n &= S_n + S_{n+1} + S_{n+2} + \cdots \\ &= S_1^n + S_1^{n+1} + S_1^{n+2} + \cdots \\ &= S_1^n(k + S_1 + S_1^2 + \cdots) \\ &= S_1^n S. \end{aligned}$$

We conclude that $S^\sharp_n = S_+^n$ holds for all $n \geq 0$, and so

$$\mathrm{Bl}_0(C) = \mathrm{Proj}(S^\sharp). \tag{5}$$

In light of (3) and (5), the desired assertion is reduced to the subsequent Lemma. $\square$

Lemma 2.4. *For any graded k-algebra S and $X = \mathrm{Proj}(S)$, with $S^\sharp$ defined as (4), there is a canonical isomorphism of schemes*

$$\mathrm{Proj}(S^\sharp) \cong \mathrm{Spec}_X\big(\bigoplus_{n\geq 0}\mathcal{O}_X(n)\big).$$

Proof. There is a canonical map of graded rings $(-)^\sharp : S \to S^\sharp$, obtained by sending a homogeneous element $f \in S_n$ to the homogeneous element $f^\sharp \in S^\sharp_n = S_{\geq n}$, which is defined to be "just itself" as $f \in S_n \subseteq S_{\geq n}$. It is clear from its definition that the image of the irrelevant ideal $S_+ \subseteq S$ under this map generates the irrelevant ideal $S^\sharp_+ \subseteq S^\sharp$, hence it gives rise to a corresponding morphism of schemes $\mathrm{Proj}(S^\sharp) \to \mathrm{Proj}(S) = X$.

Both schemes in question therefore admit canonical maps into X, and we will show that they agree upon restriction to a basic open $D_+(f) \subseteq X$ for any $f \in S_d$, $d \geq 1$. Since such open subschemes cover X, the observation that everything we will do will be compatible on intersections, implies the agreement of the two schemes over the entirety of X. Over a basic open in X in question, we obtain

$$\mathrm{Proj}(S^\sharp) \times_X D_+(f) = D_+(f^\sharp) \cong \mathrm{Spec}(S^\sharp[(f^\sharp)^{-1}]_0).$$

On the other hand, the relative spectrum restricts over the basic affine to

$$\begin{aligned} \mathrm{Spec}_X\Big(\bigoplus_{n\geq 0} \mathcal{O}_X(n)\Big) \times_X D_+(f) &\cong \mathrm{Spec}\Big(\bigoplus_{n\geq 0} \Gamma(D_+(f); \mathcal{O}_X(n))\Big) \\ &= \mathrm{Spec}\Big(\bigoplus_{n\geq 0} S[f^{-1}]_n\Big) \\ &= \mathrm{Spec}(S[f^{-1}]_{\geq 0}). \end{aligned}$$

To identify the two affine schemes, we must (compatibly in $f \in S_d$, in order to ensure gluing to all of X) identify the two rings $S^\sharp[(f^\sharp)^{-1}]_0$ and $S[f^{-1}]_{\geq 0}$. From the grading on homogeneous localizations of graded rings, recalled in 2.2, we find that

$$S^\sharp[(f^\sharp)^{-1}]_0 = \bigoplus_{\substack{i,j\geq 0 \\ i-dj=0}} S^\sharp_i (f^\sharp)^{-j} = \bigoplus_{\substack{i,j\geq 0 \\ i-dj=0}} S_{\geq i} f^{-j} = \bigoplus_{\substack{i,j\geq 0 \\ i-dj\geq 0}} S_i f^{-j} = S[f^{-1}]_{\geq 0}$$

as desired. □

Remark 2.5. The scheme-theoretic proof of Theorem 2.3 is formal and simple, but it obscures the geometric ideas involved in the original approach to Theorem 1.4. On the other hand, it amounts to a straighforward algebraic manipulation, with no deep geometric insight required - the same aspect is both a power and curse. That said, using Remark 2.2 and the "functor of points" approach to algebraic geometry (and specifically the universal property of blowups), Theorem 2.3 could be proved almost precisely the same way as Theorem 1.4. We leave that as a fun exercise to the reader.

A. Appendix

The goal we set ourselves out to do is concluded: we have investigated how to explode a party hat, i.e. blow up a cone. In this appendix, we collect some after-thoughts that are not a necessary part of the main narative, but that we touched upon in it, and that are fun, self-contained, and short enough to merit inclusion.

A.1. **Examples: singular plane cubic curves.** To get a feeling for blowups, let us work through the examples of blowing up plane cubic curves. Since blowing up a smooth point will not change the curve, we will treat the two singular plane cubics: the *nodal and cuspidal cubic curves*

$$Y_{\mathrm{node}} := \{(x,y) \in \mathbf{A}^2 \mid y^2 = x^2(x+1)\}, \quad Y_{\mathrm{cusp}} := \{(x,y) \in \mathbf{A}^2 \mid y^2 = x^3\}.$$

The singular point of both is at the origin $0 \in Y_{\mathrm{node}} \cap Y_{\mathrm{cusp}} \subseteq \mathbf{A}^2$, which is the point we will blow up.

We already know that $\mathrm{Bl}_0(\mathbf{A}^2) \subseteq \mathbf{P}^1 \times \mathbf{A}^2$ is cut out in coordinates $([s:t],(x,y))$ by the equation $tx = sy$. To simplify, let us restrict to the open locus $\{s \neq 0\} = \mathbf{P}^1 - \{[0:1]\} \cong \mathbf{A}^1$. Setting $s = 1$ (which we can do since homogeneous coordinates are only unique up to rescaling), we obtain a *unique* point $[t : 1]$ for any $t \in \mathbf{A}^1$. The blowup equation then simplifies to $y = tx$. The reason we lose nothing by restricting to the locus of $\mathrm{Bl}_0(\mathbf{A}^2)$ over

$\{s \neq 0\} \subseteq \mathbf{P}^1$ is because the line $L = \{(x,y) \in \mathbf{A}^2 \mid x = 0\} \subseteq \mathbf{A}^2$, which corresponds to the point $[0:1] \in \mathbf{P}^1$ (i.e. the locus $s \neq 0$), intersects the curves $Y_{\text{node}}, Y_{\text{cusp}} \subseteq \mathbf{A}^2$ only at the origin, and even there transversally[7]. That implies, by the description of the blowup from Section 1.8, that the subvarieties $\mathrm{Bl}_0(Y_{\text{node}}), \mathrm{Bl}_0(Y_{\text{cusp}}) \subseteq \mathrm{Bl}_0(\mathbf{A}^2)$ lie fully over the open locus $\{s \neq 0\} \subseteq \mathbf{P}^1$. Hence nothing is lost by setting $s = 1$, and we do so from here on.

A.2. **Blowup of the nodal conic.** The preimage $\pi^{-1}(Y_{\text{node}}) \subseteq \mathrm{Bl}_0(\mathbf{A}^2)$, is now cut out from $(t,x,y) \in \mathbf{A}^3$ as the simultaneous vanishing locus of the polynomials $y - tx$ and $y^2 - x^2(x+1)$. The latter equation may be simplified in light of the former into

$$(tx)^2 - x^2(x+1) = x^2(t^2 - x - 1).$$

Geometrically, this means that $\pi^{-1}(Y_{\text{node}}) \subseteq \mathrm{Bl}_0(\mathbf{A}^2)$ splits into two irreducible components: one given by $x^2 = 0$ - this is nothing but the exceptional divisor, i.e. the line $\pi^{-1}(0) = \mathbf{P}^1$ - and the other given by $x = 1 + t^2$. By definition of the blowup $\mathrm{Bl}_0(Y_{\text{node}}) \subseteq \mathrm{Bl}_0(\mathbf{P}^1)$, this is the second component

$$\begin{aligned}\mathrm{Bl}_0(Y_{\text{node}}) &= \{(t,x,y) \in \mathbf{A}^3 \mid x = 1 + t^2, y = tx\} \\ &= \{(t, 1+t^2, t+t^3) \in \mathbf{A}^3 \mid t \in \mathbf{A}^1\}.\end{aligned}$$

This is a parametrized curve inside $\mathbf{A}^3$, with the map $t \mapsto (t, 1+t^2, t+t^3)$ defining the isomorphism $\mathbf{A}^1 \cong \mathrm{Bl}_0(Y_{\text{node}})$. In particular, this is a smooth curve, and so in particular the blow-down map $\mathrm{Bl}_0(Y_{\text{node}}) \to Y_{\text{node}}$ is a resolution of singularieties.

A.3. **Blowup of the cuspidal conic.** By the analogous reasoning as in the nodal case, the preimage $\pi^{-1}(Y_{\text{cusp}}) \subseteq \mathrm{Bl}_0(\mathbf{A}^2)$ is cut out by the equation

$$(tx)^2 - x^3 = x^2(t^2 - x).$$

The irreducible component $\mathrm{Bl}_0(Y_{\text{cusp}}) \subseteq \pi^{-1}(Y_{\text{cusp}})$ is therefore given by $x = t^2$, and as such may be expressed in the parametrized form

$$\mathrm{Bl}_0(Y_{\text{cusp}}) = \{(t, t^2, t^3) \in \mathbf{A}^3 \mid t \in \mathbf{A}^1\}.$$

Not only is this a non-singular curve, making $\mathrm{Bl}_0(Y_{\text{cusp}}) \to Y_{\text{cusp}}$ into a resolution of singularities of the cuspidal cubic, but it is a particularly famous algebraic space curve: it is the ever-friendly *twisted cubic*!

A.4. **Blowup of a "higher order cusp".** In the previous two cases, blowing up a curve singularity resolved it. That need not be always the case (though a finite sequence of blowups will always do the job).

Indeed, consider a higher-degree analogue of the cuspidal cubic

$$Y_{m,n} := \{(x,y) \in \mathbf{A}^2 \mid y^m = x^n\}$$

for arbitrary integers $m, n \geq 1$. To organize ourselves, suppose that $n > m$ (though by change of coordinates x, y on $\mathbf{A}^2$, this could be switched around). The preimage $\pi^{-1}(Y_{m,n}) \subseteq \mathrm{Bl}_0(\mathbf{A}^2)$ under the blowup projection $\pi : \mathrm{Bl}_0(\mathbf{A}^2) \to \mathbf{A}^2$ is cut out as before by

$$(tx)^m - x^n = x^m(t^m - x^{n-m}).$$

The irreducible component $\mathrm{Bl}_0(Y_{m,n}) \subseteq \pi^{-1}(Y_{m,n})$ is therefore given by $x^{n-m} = t^m$. But unless $n - m = 1$, this does not give us a parametrization of x (and consequently also $y = tx$) in terms of t, which exhibited the smoothness of the blowup in the cubic case above. Instead, we find that

$$\mathrm{Bl}_0(Y_{m,n}) = \{(t,x,y) \in \mathbf{A}^3 : x^{n-m} = t^m,\ y = tx\}. \tag{6}$$

[7] The reader familiar with Bezout's Theorem will conclude that the "remaining two points of intersection" must occur at infinity. Indeed, the line in question is precisely the tangent line to the projective closures of Y_{node} and Y_{cusp} inside $\mathbf{P}^2$ at infinity $\mathbf{P}^2 - \mathbf{A}^2$.

To test whether this algebraic variety is non-singular or not, we must consider the derivatives of its defining equations. If $f(t,x,y) = x^{n-m} - t^m$ and $g(t,x,y) = y - tx$, so that the blow-up in question is the locus $\{f = g = 0\} \subseteq \mathbf{A}^3$, then its singulareness are controlled by the Jacobian matrix

$$\mathbf{J}_{f,g}(t,x,y) = \frac{\partial(f,g)}{\partial(t,x,y)} = \begin{pmatrix} \frac{\partial f}{\partial t} & \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial t} & \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}.$$

Indeed, the *Jacobian criterion for smothness* asserts that a point $(t,x,y) \in \{f = g = 0\}$ is singular if and only if all the minors of this matrix vanish[8]. The Jacobian matrix in question is

$$\mathbf{J}_{f,g}(t,x,y) = \begin{pmatrix} -mt^{m-1} & (n-m)x^{n-m-1} & 0 \\ -x & -t & 1 \end{pmatrix},$$

and its three minors are

$$\begin{aligned} \mathbf{J}_{f,g}(t,x,y)_1 &= (n-m)x^{n-m-1}, \\ \mathbf{J}_{f,g}(t,x,y)_2 &= mt^{m-1}, \\ \mathbf{J}_{f,g}(t,x,y)_3 &= mt^m + (n-m)x^{n-m}. \end{aligned}$$

The point $0 \in \mathrm{Bl}_0(Y_{n,m}) \subseteq \mathbf{A}^3$ will be a common zero of all these three minors, and as such a singular point of $\mathrm{Bl}_0(Y_{n,m})$, unless either

(a) $n - m = 1$ - such as was the case for the cuspidal conic, or
(b) $m = 1$ - in which case, the curve $Y_{m,n} \subseteq \mathbf{A}^2$ is already smooth.

Thus for $n \gg m$, blow-up at the singular point does not desingularize the curve $Y_{n,m}$. But some imporvement has been made - in the equations (6), the highest exponent that appears is $n - m$, which is a definite improvement on n. By blowing up sufficiently many times, it thus might not be too outrageous to hope that the singularity might be improved to the point of being no longer singular. Indeed, this turns out to be the case, but the proof requires more technology than we are willing to expand here.

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[8]A more common phrasing of the Jacobian criterion is the opposite: a point (t,x,y) is smooth i.e. nonsingular if and only if the matrix $\mathbf{J}_{f,g}(t,x,y)$ has maximal rank, which is equivalent to saying that it has a non-vanishing minor.