THE CRYSTALLINE SPACE AND DIVIDED POWER COMPLETION

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Overview. The contemporary theory of D-modules in derived algebraic geometry, as studied for instance in [8], rests on the observation by Carlos Simpson in [16] that D-modules may be understood as quasi-coherent sheaves on the de Rham space. The de Rham space is also useful because it provides one approach to derived formal completion.

In this note we introduce the crystalline space, an analogue of the de Rham space, such that the sheaves on it are crystals in the sense of Grothendieck. Its relationship with the crystalline topos is intended to mirror the relationship of de Rham space with the infinitesimal topos. We use the crystalline space to study a derived form of pd-completion, and discuss the relationship of our work with Drinfeld's stacky approach to crystals in [5].

Setting. We work in the setting of derived algebraic geometry, with liberal use of ∞-categories. Some of the most common notation we use is $\mathcal{S}$ for the ∞-category of spaces, DSch for the ∞-category of derived schemes, Sch for the category of ordinary schemes, $\mathrm{CAlg}^{\heartsuit}$ for the category of commutative rings, and sCAlg for the ∞-category of simplicial commutative rings. We choose to use the latter as the setting[1] for derived algebraic geometry, and mostly pursue a functor of points philosophy (schemes and stacks are identified with functors they represent, etc.). We cite [8] as GR, [12] as HA, [14] as SAG, and [4] as SP.

1. Crystalline space

Definition 1.1. The *crystalline space* of a functor $X : \mathrm{sCAlg} \to \mathcal{S}$ is the functor $X_{\mathrm{crys}} : \mathrm{sCAlg} \to \mathcal{S}$ defined by

$$X_{\mathrm{crys}}(R) = \varinjlim_{(I,\gamma)} X(\pi_0(R)/I),$$

where the colimit ranges over the poset of nilpotent ideals I in $\pi_0(R)$ and pd-structures γ on I.

Remark 1.2 (Crystalline vs de Rham space, I)**.** The definition of the crystalline space is analogous to defining the de Rham space of a functor $X : \mathrm{sCAlg} \to \mathcal{S}$ as

$$X_{\mathrm{dR}}(R) = \varinjlim_{I} X(\pi_0(R)/I),$$

where the colimit ranges over the poset of nilpotent ideals I in $\pi_0(R)$. One notable difference compared to X_{crys} is that the colimit in the definition of X_{dR} is filtered. Thus when X is locally of finite presentation, we recover the formula $X_{\mathrm{dR}}(R) \simeq X(\pi_0(R)^{\mathrm{red}})$. So far as we can tell, that in general has no analogue for X_{crys}. That has some serious drawbacks, as much of the theory of the de Rham space, as developed in [16], [9], and GR, relies essentially on its connection to reduction. See Remark 7.7 for an example of this difficulty in action.

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[1]Though we have elected to phrase things in terms of simplicial commutative rings, the contents of the note would be unchanged if we decided to work with connective $\mathbb{E}_\infty$-rings instead. This is due to the fact that the crystalline space depends only on the underlying classical scheme of a derived or spectral scheme.

Remark 1.3 (Functoriality)**.** The formation of the crystalline space $X \to X_{\text{crys}}$ extends to a functor, as becomes transparent if we rephrase its definition in terms of a Kan extension. Indeed, denote by $\text{CAlg}^{\heartsuit}_{\text{pd}}$ the category of pd-rings, whose objects are triples (R, I, γ) of a commutative ring R, a nilpotent ideal $I \subseteq R$, and a divided power structure γ on I, and whose morphisms are ring homomorphisms which preserve both the relevant ideals and pd-structures. It possesses two functors $p_0, p_1 : \text{CAlg}^{\heartsuit}_{\text{pd}} \to \text{CAlg}^{\heartsuit}$ given by $p_1(R, I, \gamma) = R$ and $p_0(R, I, \gamma) = R/I$. The construction $X \to X_{\text{crys}}$ is given as the composition

$$\text{Fun}(\text{sCAlg}, \mathcal{S}) \xrightarrow{\pi_0} \text{Fun}(\text{CAlg}^{\heartsuit}, \mathcal{S}) \xrightarrow{p_0^*} \text{Fun}(\text{CAlg}^{\heartsuit}_{\text{pd}}, \mathcal{S}) \xrightarrow{\text{LKan}_{p_1}} \text{Fun}(\text{CAlg}^{\heartsuit}, \mathcal{S}) \hookrightarrow \text{Fun}(\text{sCAlg}, \mathcal{S})$$

exhibiting its functoriality. That is to say, any natural transformation $f : X \to Y$ induces canonically another natural transformation $f_{\text{crys}} : X_{\text{crys}} \to Y_{\text{crys}}$.

Definition 1.4. Given a map $f : X \to Y$ in $\text{Fun}(\text{sCAlg}, \mathcal{S})$, let us define the *pd-completion of* Y *along* X to be $Y^{\text{pd}}_X = Y \times_{Y_{\text{crys}}} X_{\text{crys}}$. More explicitly, this functor associates to a simplicial commutative ring R the space

$$Y^{\text{pd}}_X(R) \simeq \varinjlim_{(I,\gamma)} X(\pi_0(R)/I) \times_{Y(\pi_0(R)/I)} Y(R),$$

where the colimit ranges over the poset of nilpotent ideals I in $\pi_0(R)$ and pd-structures γ on I.

Remark 1.5 (Underlying classical functors)**.** Given a functor $X : \text{sCAlg} \to \mathcal{S}$, the *undering classical functor of* X is the functor $\pi_0(X) : \text{CAlg}^{\heartsuit} \to \mathcal{S}$ is obtained by restriction along the inclusion $\text{CAlg}^{\heartsuit} \to \text{sCAlg}$ of ordinary commutative rings into simplicial commutative rings as discrete simplicial rings. If a natural transformation $X \to X'$ induces an equivalence $\pi_0(X) \simeq \pi_0(X')$, then clearly the induced map of crystalline spaces $X_{\text{crys}} \to X'_{\text{crys}}$, or more generally any pd-completion $Y^{\text{pd}}_X \to Y^{\text{pd}}_{X'}$, is an equivalence.

Analogously to how ordinary schemes may be viewed as discrete derived schemes, left Kan extension admits a fully faithful embedding $\text{Fun}(\text{CAlg}^{\heartsuit}, \mathcal{S}) \to \text{Fun}(\text{sCAlg}, \mathcal{S})$. Interpreting the underlying classical functor $\pi_0(X)$ with its image under this embedding gives a canonical morphism $\pi_0(X) \to X$, in fact exhibiting a right adjoint to the embedding in question. It follows from the above discussion that for any natural transformation $X \to Y$, the map of pd-completions $Y^{\text{pd}}_{\pi_0(X)} \to Y^{\text{pd}}_X$ is an equivalence. In particular $\pi_0(X)_{\text{crys}} \simeq X_{\text{crys}}$.

Warning 1.6. The previous remark implies that, for the purposes of studying the crystalline stack, we may as well work with ordinary schemes and stacks. In that sense, crystals are agnostic to any derived structure. Nevertheless, the correctness of Definition 1.1 *relies crucial on working in the setting of derived algebraic geometry.* That is because the colimit in said definition must be interpreted in the ∞-categorical sense. For instance, even when X is a smooth classical scheme, the crystalline space may very well be a stack, which amounts to the colimit having to be computed in the $(2,1)$-category of groupoids, or equivalentnly, the ∞-category $\mathcal{S}^{\leq 1} \subseteq \mathcal{S}$ of 1-truncated spaces.

Remark 1.7 (A universal property)**.** Imitating the first part of the Subsection SAG.18.2.1, it may be proved that the formation of pd-completion $(X \to Y) \mapsto Y^{\text{pd}}_X$ exhibits the right adjoint to the inclusion into the ∞-category $\text{Fun}(\text{sCAlg}, \mathcal{S})_{X/}$ of the full subcategory spanned by those maps $X \to Y$ which are *pd-complete* in the sense that the induced map $X_{\text{crys}} \to Y_{\text{crys}}$ is an equivalence.

Remark 1.8 (Relative crystalline space)**.** Under the standard equivalence of ∞-categories $\text{Fun}(\text{sCAlg}, \mathcal{S})_{/\operatorname{Spec} A} \simeq \text{Fun}(\text{sCAlg}_A, \mathcal{S})$, a natural transformation $X \to \operatorname{Spec} A$ may be identified with a functor $X : \text{sCAlg}_A \to \mathcal{S}$. In light of this the pd-completion $(\operatorname{Spec} A)^{\text{pd}}_X$ has the effect of modifying Definition 1.1 by replacing sCAlg with sCAlg_A. For this reason, we will denote[2] $(\operatorname{Spec} A)^{\mathrm{pd}}_X = (X/A)_{\mathrm{crys}}$ and refer to it as the *relative crystalline space of* X *over* A.

Remark 1.9 (Crystalline vs de Rham space, II)**.** Forgetting the pd-structure on the ideal in the colimit definiting the crystalline space, which is to say sending $(I,\gamma) \to I$, gives rise to a map $X_{\mathrm{crys}} \to X_{\mathrm{dR}}$, natural with respect to X. Analogously there is a natural map $Y^{\mathrm{pd}}_X \to Y^{\wedge}_X$ from the pd-completion to the formal completion of any map $X \to Y$ in $\operatorname{Fun}(\mathrm{sCAlg}, \mathcal{S})$. Here the formal completion $Y^{\wedge}_X$ of a map $X \to Y$ is to be understood as the pullback $Y \times_{Y_{\mathrm{dR}}} X_{\mathrm{dR}}$.

Proposition 1.10. *Let* $X \to \operatorname{Spec} \mathbf{Q}$ *be a map in* $\operatorname{Fun}(\mathrm{sCAlg}, \mathcal{S})$. *Then the natural transformations* $(X/\mathbf{Q})_{\mathrm{crys}} \to X_{\mathrm{crys}} \to X_{\mathrm{dR}}$ *are all equivalences.*

Proof. If X admits a map to $\operatorname{Spec} \mathbf{Q}$, then $X(R) \simeq X(R \otimes \mathbf{Q})$ for any simplicial commutative ring R. The claim now follows from the fact that any ideal in a $\mathbf{Q}$-algebra admits a unique divided power structure given by $\gamma_n(x) = \frac{x^n}{n!}$. □

In characteristic 0 the crystalline space thus coincides with the de Rham space. Since study of the latter already boasts a large presence in the literature, we shall focus mostly on the positive characteristic situation, where the two notions disagree.

2. Divided power envelopes

We wish to compare the notion of pd-completion, as defined in the previous section, to the more traditional meaning of that term in algebraic geometry. Since the latter is defined in terms of pd-envelopes, we need to extend those to our derived setting.

Definition 2.1. A *pd-immersion of derived schemes* consists of a closed immersion of derived schemes $f : X \to Y$ together with a pd-structure on the sheaf of ideals $\mathscr{I}_X$ in $\pi_0(\mathscr{O}_Y)$ defining the underlying closed immersion of classical schemes. If the sheaf of ideals $\mathscr{I}_X$ is also nilpotent, then f is a *pd-thickening*.

Let $f : X \to Y$ be a closed immersion of derived schemes. We wish to construct an ∞-category $\mathrm{DSch}^{\mathrm{pd-im}}_{X//Y}$ of factorizations $f : X \to Y' \to X$ in which the first arrow is a pd-immersions. Let us denote by $\mathrm{DSch}^{\mathrm{cl-im}}_{X/}$ the full subcategory of $\mathrm{DSch}_{X/}$ spanned by closed immersions. Viewing f as exhibiting Y as an object of this ∞-category, let $\mathrm{DSch}^{\mathrm{cl-im}}_{X//Y}$ to denote the overcategory $(\mathrm{DSch}^{\mathrm{cl-im}}_{X/})_{/Y}$, which is to say the ∞-category of factorizations $f : X \to Y' \to Y$ in which the first arrow is a closed immersion. In light of Definition 2.1, the extra data required to make a closed immersion into pd-immersion exists purely on the level of underlying ordinary schemes. Thus the desired ∞-category of factorizations of f through pd-immersions may be defined as the pullback

$$\mathrm{DSch}^{\mathrm{pd-im}}_{X//Y} \simeq \mathrm{DSch}^{\mathrm{cl-im}}_{X//Y} \times_{\mathrm{Sch}^{\mathrm{cl-im}}_{X/}} \mathrm{Sch}^{\mathrm{pd-im}}_{X/}$$

inside the ∞-category of ∞-categories. The projection onto the first factor gives the functor of forgetting the pd-structure.

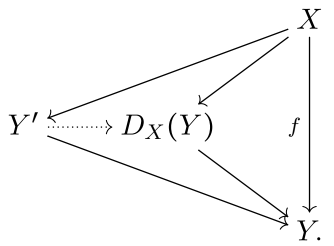
Lemma 2.2. *The functor* $\mathrm{DSch}^{\mathrm{pd-im}}_{X//Y} \to \mathrm{DSch}^{\mathrm{cl-im}}_{X//Y}$ *commutes with all small colimits.*

Proof. It suffices to show that the functor $\mathrm{Sch}^{\mathrm{pd-im}}_{X/} \to \mathrm{Sch}^{\mathrm{cl-im}}_{X/}$, given by discarding the pd-structure, commutes with all small colimits. This boils down to the standard observation that limits in the category of pd-rings are computes on the underlying rings, see for instance Lemma SP.23.3.2. □

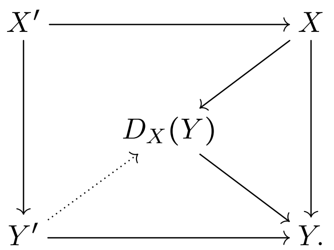
[2] There is a small potention for confusion, since we have co-opted the notation $(X/A)_{\mathrm{crys}}$, often used to denote the small crystalline site of an A-scheme X in the literature. Nonetheless, as we shall see, this is not entirely unappropriate either, since the relative crystalline space encodes essentially the same data as the crystalline site.

Definition 2.3. The *pd-envelope* of a closed immersion od derived schemes $f : X \to Y$ consists of the pd-immersion $X \to D_X(Y)$ and a factorization $f : X \to D_X(Y) \to Y$, obtained by evaluating the right adjoint of the functor $\mathrm{DSch}^{\mathrm{pd-im}}_{X//Y} \to \mathrm{DSch}^{\mathrm{cl-im}}_{X//Y}$, which exists by Lemma 2.2 and the Adjoint Functor Theorem, with the evident forgetful functor, of the initial object $X \to Y$ in $\mathrm{DSch}^{\mathrm{cl-im}}_{X//Y}$.

Remark 2.4 (Universal property of pd-envelopes)**.** Unwinding this, the pd-envelope of a closed immersion $f : X \to Y$ is a pd-immersion $X \to D_X(Y)$ together with a map $D_X(Y) \to Y$ factoring f, and such that for any other factorization $f : X \to Y' \to Y$ in which $X \to Y'$ is a pd-immersion, there exists an essentially unique dotted map of pd-immersions making the following diagram commute



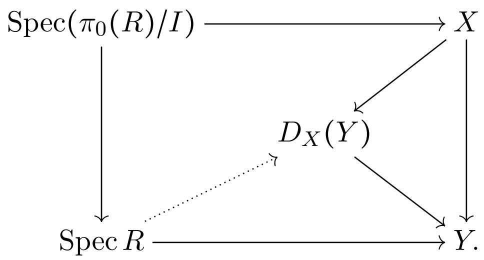
If fact, as familiar from the classical theory of pd-envelopes, the universal property can be extended slightly as follows. Let $X \to X'$ be a morphism of derived schemes and let $X' \to Y'$ denote a pd-thickening. Suppose further that we have a map of derived schemes $Y' \to Y$ making the outer square in the diagram below commute. Then there eixsts a unique map of derived schemes $Y' \to D_X(Y)$ making the entire diagram below commute, such that the induced map of relevant ideal sheaves preserves the pd-structures



Heuristically, the pd-envelope $D_X(Y)$ is the biggest pd-neighborhood of a closed subscheme X which still fits inside the ambient scheme Y. As such, the following observation should come at no surprise.

Proposition 2.5. *Let $X \to Y$ denote a closed immersion of derived schemes. Then the morphism $D_X(Y) \to Y$ exhibits an equivalence $D_X(Y)^{\mathrm{pd}}_X \to Y^{\mathrm{pd}}_X$.*

Proof. This follows from the universal property of the pd-envelope. Indeed, for any comutative simplicial ring R with a pd-structure on a nilpotent ideal I in $\pi_0(R)$, the space $X(\pi_0(R)/I) \times_{Y(\pi_0(R)/I)} X(R)$ parametrizes the space of commuting outer squares in the following diagram



The universal property of the pd-envelope implies the essentially unique existence of the dotted arrow, making the entire diagram commute. Thus the commuting outer square is equivalent to the commuting (deformed) inner square. The space in question is therefore equivalent to $X(\pi_0(R)/I)\times_{D_X(Y)(\pi_0(R)/I)} D_X(Y)(R)$, and passing to the colimit over ideals I in $\pi_0(R)$ and pd-structures on them yields the result. □

Proposition 2.6. *Let $X\to Y$ be a closed immersion of derived schemes for which the pd-immersion into the pd-envelope $X\to D_X(Y)$ is a pd-thickening, i.e. such that the corresponding sheaf of ideals on $\pi_0(D_X(Y))$ is nilpotent. Then there is an equivalence of functors $Y_X^{\mathrm{pd}}\simeq D_{\pi_0(X)}(Y)$.*

Proof. The nilpotence assumption shows that the pd-envelope in question is universal among pd-thickenings factoring through $\pi_0(X)\to Y$ (as opposed to only among pd-immersions, which it is without the assumption). Thus the functor $D_{\pi_0(X)}(Y)$ associates to any simplicial commutative ring R the space of homotopy commutative diagrams

$$\begin{array}{ccc} \mathrm{Spec}(\pi_0(R)/I) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, R & \longrightarrow & Y \end{array}$$

ranging over ideals $I\subseteq\pi_0(R)$ and pd-structures on the ideal I. But that is nothing other than $Y_X^{\mathrm{pd}}(R)$, and the claim is proved. □

Remark 2.7 (Importance of nilpotence hypothesis)**.** Let us explain how the proof of Proposition 2.6 fails without the nilpotnce assumption. In that case, the value of the pd-envelope $D_{\pi_0(X)}(Y)$ on a simplicial commutative ring R is the space parametrizing all homotopy commutative diagrams of the form

$$\begin{array}{ccc} \mathrm{Spec}(\pi_0(R)/I) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, R & \longrightarrow & Y \end{array}$$

ranging over ideals $I\subseteq\pi_0(R)$ and divided power structures on such I. The pd-completion $Y_X^{\mathrm{pd}}(R)$, on the other hand, only parametrizes all such diagrams for which the ideal I is nilpotent. The canonical map

$$Y_X^{\mathrm{pd}}(R)\to(D_{\pi_0(X)}(Y))(R),$$

arising from the inclusion of nilpotent ideals in $\pi_0(R)$ into all ideals in $\pi_0(R)$, is therefore not necessarily an equivalence.

Corollary 2.8. *Let $X\to Y$ be a map of derived schemes over $\mathbf{Z}/p^n$. Then there is an equivalence $Y_X^{\mathrm{pd}}\simeq D_X(Y)$ between its pd-completion and its pd-envelope.*

Proof. Observe that any ideal I in a $\mathbf{Z}/p^n$-algebra that admits a pd-structure γ, must satisfy $x^{p^n}=(p^n)!\gamma_{p^n}(x)=0$ for any element $x\in I$. Thus $I^{p^n}=0$, showing that any ideal supporting a pd-structure has to be nilpotent. The same holds for quasi-coherent ideal sheaves, hence any pd-immersion in characteristic p is automatically a pd-thickening. The conclusion now follows from Proposition 2.6. □

Variant 2.9 (Classical approach to pd-completion along closed immersion)**.** Even when the ideal $\overline{\mathscr{I}}\subseteq\pi_0(\mathscr{O}_{D_X(Y)})$ is not nilpotent, we can still say something. Let us assume that $f:X\to Y$ is a closed immersion of ordinary schemes. Since any closed immersion is affine, we have $X=\mathrm{Spec}_Y(f_*(\mathscr{O}_X))$ and $D_X(Y)=\mathrm{Spec}_Y(\mathscr{D}_X(Y))$ for a sheaf of $\mathscr{O}_Y$-algebras $\mathscr{D}_X(Y)$ with a pd-ideal $\overline{\mathscr{I}}$ and the quotient $\mathscr{D}_X(Y)/\overline{\mathscr{I}}\simeq f_*(\mathscr{O}_X)$. Let us define the *$n$-th order pd-neighborhood of X in Y* to be

$$D^n_{X/Y}=\mathrm{Spec}_Y\left(\mathscr{D}_X(Y)/\overline{\mathscr{I}}^{[n+1]}\right),$$

where a divided power $I^{[n]}$ of a pd-ideal (I,γ) is defined as the smallest pd-subideal of I containing I^n. More explicitly, $I^{[n]}$ is the ideal spanned by the products $\gamma_{i_1}(x_1)\cdots\gamma_{i_k}(x_k)$ for $x_i \in I$ and $i_1+\cdots+i_k \geq n$. In particular, the ideal $I/I^{[n]}$ inherits a pd-structure from I, showing that $X \to D^n_{X/Y}$ is a pd-thickening[3]. In fact, the universal property of the pd-envelope shows $X \to D^n_{X/Y}$ to be universal among pd-thickenings that factor $f: X \to Y$, such that their defining ideal sheaves are nilpotent of order $n+1$. Since the ideal of any pd-thickening is nilpotent to some order, we conclude that passing to the colimit along the tower of pd-neighborhoods $X = D^0_{X/Y} \to D^1_{X/Y} \to D^2_{X/Y} \to \cdots$ gives rise to pd-completion. That is to say, the pd-completion of a closed immersion of ordinary schemes $f: X \to Y$ can always be computes by the following formula

$$Y^{\mathrm{pd}}_X \simeq \varinjlim_n D^n_{X/Y} \simeq \varinjlim_n \mathrm{Spec}_Y\left(\mathscr{D}_X(Y)/\overline{\mathscr{I}}^{[n+1]}\right).$$

Remark 2.10 (pd-completon of rings)**.** On the level of affines the analogy with adic completion becomes even more transparent. As the formal completion of the closed immersion $\mathrm{Spec}(R/I) \to \mathrm{Spec}\, R$ is given by $\mathrm{Spf}(R^\wedge_I)$ for the I-adic completion[4] $R^\wedge_I = \varprojlim_n R/I^n$, so is its pd-completion given by $\mathrm{Spf}(R^{\mathrm{pd}}_I)$ for the pd-completion ring $R^{\mathrm{pd}}_I = \varprojlim_n \mathscr{D}_R(I)/\overline{I}^{[n]}$. In both cases the formal spectrum Spf is to be interpreted in terms of pro-representatiblity.

Remark 2.11 (pd-completions are ind-schemes)**.** For any pair of non-negative integers $n \leq m$, the canonical map $f_{nm}: D^n_{X/Y} \to D^m_{X/Y}$ corresponds to the map of quasi-coherent commutative $\mathscr{O}_Y$-algebras $\mathscr{D}_X(Y)/\overline{\mathscr{I}}^{[m]} \to \mathscr{D}_X(Y)/\overline{\mathscr{I}}^{[n]}$, induced by the inclusion of pd-ideals $\overline{\mathscr{I}}^{[m]} \subseteq \overline{\mathscr{I}}^{[n]}$. The algebra map is evidently surjective, thus f_{nm} is a closed immersion, and Variant 2.9 exhibits the pd-completion Y^{pd}_X as a sequential (so in particular, filtered) colimit along closed immersions of schemes. In the language of [10], we thus see that the pd-completion of a closed immersion of ordinary schemes, while not quite necessarily being a scheme, is always a classical ind-scheme. By Proposition 1.3.2 of *loc cit*[5], the functor Y^{pd}_X satisfies flat descent.

Proposition 2.12. *Let $f: X \to Y$ be a closed immersion of schemes, and let us denote by $\hat{f}: Y^{\mathrm{pd}}_X \to D_X(Y)$ the canonical map. The direct image functor induces a fully faithful embedding of ∞-categories $\hat{f}_*: \mathrm{QCoh}(Y^{\mathrm{pd}}_X) \to \mathrm{Mod}_{\mathscr{D}_X(Y)}(\mathrm{QCoh}(Y))$. Its essential image consists of the $\mathscr{D}_X(Y)$-modules $\mathscr{F}$ satisfying $\overline{\mathscr{I}}^{[n]}\pi_k(\mathscr{F})$ for $n \gg 0$ for all $k \in \mathbf{Z}$, where $\overline{\mathscr{I}}^{[n]}$ is the n-th devided power of the universal sheaf of pd-ideals $\overline{\mathscr{I}} \subseteq \mathscr{D}_X(Y)$.*

Proof. The identification of pd-completion as $Y^{\mathrm{pd}}_X \simeq \varinjlim_n D^n_{X/Y}$ in Variant 2.9 induces an equivalence of ∞-categories of quasi-coherent sheaves

$$\mathrm{QCoh}(Y^{\mathrm{pd}}_X) \simeq \varprojlim_n \mathrm{QCoh}(D^n_{X/Y}).$$

On the right-hand side we are computing the limit of the diagram of quasi-coherent sheaf with pullbacks along the maps $D^n_{X/Y} \to D^m_{X/Y}$ induced by the inclusion $\overline{\mathscr{I}}^{[m]} \subseteq \overline{\mathscr{I}}^{[n]}$ for all $n \leq m$. This limit is occuring in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of presentable ∞-categories and left ajdoint functors, so let us use the equivalence of ∞-categories $\mathcal{P}\mathrm{r}^{\mathrm{L}} \simeq (\mathcal{P}\mathrm{r}^{\mathrm{R}})^{\mathrm{op}}$ obtained

[3]Since we obviously have $\overline{I}^n \subseteq \overline{I}^{[n]}$, the ideal $\overline{I}/\overline{I}^{[n]}$ in $D^{n-1}_A(I) = D_A(I)/\overline{I}^{[n]}$ is nilpotent of order n.

[4]Because all our limits are ∞-categorical, this is more precisely *derived I-adic completion.* The same remark goes for the ring-level pd-completion discussed in the same sentence.

[5]The proof of which is entirely agnostic to whether we are doing derived algebraic geometry with simplicial rings, commutative differential graded algebras, or even connective $\mathbb{E}_\infty$-rings.

by switching to right adjoints. Then we are taking the colimit

$$\mathrm{QCoh}(Y_X^{\mathrm{pd}}) \simeq \varinjlim_n \mathrm{QCoh}(D^n_{X/Y}) \tag{1}$$

of pushforwards along the same maps $D^n_{X/Y} \to D^m_{X/Y}$.

Recall that both the pd-envelope map $\overline{f} : D_X(Y) \to Y$ and also the pd-neighborhoods $f_n : D^n_{X/Y} \to Y$ are all affine, corresponding to the sheafes of $\mathscr{O}_Y$-algebras $\mathscr{D}_X(Y)$ and $\mathscr{D}^n_{X/Y} = \mathscr{D}_X(Y)/\overline{\mathscr{I}}^{[n+1]}$ respectively. The functor $\overline{f}_*$ exhibits an equivalence of ∞-categories

$$\mathrm{QCoh}(D_X(Y)) \simeq \mathrm{Mod}_{\mathscr{D}_X(Y)}(\mathrm{QCoh}(Y))$$

by Proposition SAG.2.5.6.1, and the pushforward functors $(f_n)_*$ similarly exhibit

$$\mathrm{QCoh}(D^n_{X/Y}) \simeq \mathrm{Mod}_{\mathscr{D}^n_X(Y)}(\mathrm{QCoh}(Y)).$$

Furthermore, because all the maps f_n factor compatibly through $\overline{f}$, the pushforwards allows us to compatibly identify the ∞-categories $\mathrm{QCoh}(D^n_{X/Y})$ as full subcategories of $\mathrm{Mod}_{\mathscr{D}_X(Y)}(\mathrm{QCoh}(Y))$ spanned by the sheaves of modules $\mathscr{F}$ which have homotopy sheaves annihilated by $\overline{\mathscr{I}}^{[n+1]}$. The colimit in (1), which is taken over pushforwards and is as such compatible with the identifications of this paragraph, now takes place entirely inside the ∞-category $\mathrm{Mod}_{\mathscr{D}_X(Y)}(\mathrm{QCoh}(Y))$. To compute it we must take take the union of the previously-identified subcategories, which recovers the description of the essential image in the statement of the Proposition. □

Remark 2.13 (Analogy with sheaves on formal completion)**.** The result of Porposition 2.12 is an analogue of the classical identification between the quasi-coherent sheaves on the formal completion $Y^\wedge_X$ of a closed immersion $X \to Y$ with those quasi-coherent sheaves $\mathscr{F}$ on Y for which $\mathrm{Supp}(\mathscr{F}) \subseteq X$. It asserts that the difference in the pd-completion case is two-fold. Firstly, sheaves on $\mathscr{F}$ on Y must be considered togehter with a $\mathscr{D}_X(Y)$-module structure. Secondly, we do not require that a high-enough power of the ideal $\overline{\mathscr{I}}$ annihilates $\pi_*(\mathscr{F})$, but instead that a high enough divided power does. Since the inclusion $\overline{\mathscr{I}}^n \subseteq \overline{\mathscr{I}}^{[n]}$ holds for all $n \geq 0$, this is a stronger condition. Indeed, said inclusion of ideals induces the map $D_X(Y)^\wedge_X \to D_X(Y)^{\mathrm{pd}}_X \simeq Y^{\mathrm{pd}}_X$, which is the map of Remark 1.9 composed with the equivalence of Proposition 2.5.

3. Groupoid of pd-neighborhoods of the diagonal

Let us recall a little bit of the theory of the de Rham space from [9]. We say that a functor $X : \mathrm{sCAlg} \to \mathcal{S}$ is *classically fromally smooth* if the map $\pi_0(X(R)) \to \pi_0(X_{\mathrm{dR}}(R))$ is surjective for all simplicial commutative rings R. This condition is beneficial as its gives rise to a simplicial presentation of the de Rham space as

$$X_{\mathrm{dR}} \simeq \varinjlim \Big(\cdots \substack{\textstyle\longrightarrow\\[-4pt]\textstyle\longrightarrow\\[-4pt]\textstyle\longrightarrow\\[-4pt]\textstyle\longrightarrow} (X \times X \times X)^\wedge_X \substack{\textstyle\longrightarrow\\[-4pt]\textstyle\longrightarrow\\[-4pt]\textstyle\longrightarrow} (X \times X)^\wedge_X \rightrightarrows X \Big).$$

The right-hand side is the geometric realization of the so-called *infinitesimal groupoid*, is an incarnation of Grothendieck's stratifying site of X, while the left-hand side is an incarnation of the infinitesimal site of X. We wish to find an analogue of this story for X_{crys}, which will similarly relate incarnations of the pd-stratifying and crystalline sites.

Definition 3.1. A functor $X : \mathrm{sCAlg} \to \mathcal{S}$ is *classically pd-formally smooth* if the map $\pi_0(X(R)) \to \pi_0(X_{\mathrm{crys}}(R))$ is surjective for all simplicial commutative rings R.

Lemma 3.2. *If a functor $X : \mathrm{sCAlg} \to \mathcal{S}$ is classically formally smooth, then it is also classically pd-formally smooth.*

Proof. In more words, X is classically (pd-)formally smooth if, for every simplicial commutative ring R and every nilpotent ideal $I \subseteq \pi_0(R)$ (and a pd-structure on I), given solid arrows in the following diagram

$$\begin{array}{ccc} \operatorname{Spec}(\pi_0(R)/I) & \longrightarrow & X \\ \big\downarrow & \nearrow & \\ \operatorname{Spec} R, & & \end{array}$$

there exists some dotted arrow which makes it commute. Since commutativity of the diagram itself has nothing to do with a divided power structure that may or may not be present on the ideal I, the implication is obvious. □

Remark 3.3 (The role of π_0)**.** The proof of the above Lemma rests crucially on the fact that upon passing to connected components. In general though, the colimits appearing in the definition of the crystalline space and the de Rham space are quite different, owing to the fact that the same ideal may admit several inequivalent pd-structures.

This is already observed in the classical approach to crystalline cohomology: the comparison maps $c_{(U,T,\gamma)} : (\mathscr{F}_T)|_U \tilde{\to} \mathscr{F}_U$ of a crystal in quasi-coherent sheaves $\mathscr{F}$, corresponding to a pd-thickening $U \to T$, may very well give rise to *different automorphisms* of $\mathscr{F}_U$ for different choices of pd-structure γ on the defining ideal of U in $\mathscr{O}_T$, so they do not depend only on the underlying infinitesimal thickening $U \to T$. But as observed, this is statement about inequivalent paths, i.e. about π_1, and such distinctions are invisible on the level of connected components, i.e. on π_0.

Remark 3.4 (Classicaly pd-formally smooth over $\mathbf{F}_p$)**.** For a functor $X : \mathrm{sCAlg}_{\mathbf{F}_p} \to \mathcal{S}$, we could modify Definition 3.1 in the obvious way to obtain a notion of being classically pd-formally smooth over $\mathbf{F}_p$. Because an ideal I in an $\mathbf{F}_p$-algebra may only support a pd-structure if $I^p = 0$ (see the proof of Lemma 4.1), the condition of being relatively classically pd-formally smooth over $\mathbf{F}_p$ is strictly weaker than its non-pd analogue.

Proposition 3.5. *Let $X : \mathrm{sCAlg} \to \mathcal{S}$ be a classically pd-formally smooth functor. Then the map $X \to X_{\mathrm{crys}}$ induces an equivalence of functors*

$$X_{\mathrm{crys}} \simeq \varinjlim \Big(\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} (X \times X \times X)^{\mathrm{pd}}_X \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} (X \times X)^{\mathrm{pd}}_X \substack{\longrightarrow\\\longrightarrow} X \Big).$$

Proof. The assumption on X implies that the canonical map $|\check{\mathrm{C}}_\bullet(X \to X_{\mathrm{crys}})| \to X_{\mathrm{crys}}$, from the geometric realization of the Čech nerve of $X \to X_{\mathrm{dR}}$ to the crystalline space of X, is an equivalence of functors. It remains to identify the terms in the Čech nerve, which are by definition pullbacks $\check{\mathrm{C}}_\bullet(X \to X_{\mathrm{crys}}) \simeq X \times_{X_{\mathrm{crys}}} \cdots \times_{X_{\mathrm{crys}}} X$, with pd-completions of the diagonal maps $\Delta : X \to X \times \cdots \times X$. The latter are by definition of pd-completion equivalent to $(X \times \cdots \times X) \times_{(X \times \ldots X)_{\mathrm{crys}}} X_{\mathrm{crys}} \simeq (X \times \cdots \times X) \times_{X_{\mathrm{crys}} \times \ldots X_{\mathrm{crys}}} X_{\mathrm{crys}}$, where we have used that the construction of crystalline spaces commutes with finite products[6]. The desired conclusion is now an instance of the standard categorical observation that, in any ∞-category which admits finite limits, the square

$$\begin{array}{ccc} X \times_Y \cdots \times_Y X & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow \Delta \\ X \times \cdots \times X & \longrightarrow & Y \times \cdots \times Y \end{array}$$

is Cartesian for any morphism $X \to Y$. □

[6]This can for instance be seen as a consequence of the universality of colimits in any ∞-topos, one of the Giraud axioms characterizing ∞-topoi, applied to the presheaf ∞-topos $\mathrm{Fun}(\mathrm{sCAlg}, \mathcal{S})$.

Remark 3.6 (pd-stratification)**.** The formula for the crystalline space in the statement of the previous Proposition is closely related to the theory of pd-stratifications, seeing how the latter consist roughly of descent data for $(X \times \cdots \times X)^{\mathrm{pd}}_X \to X$.

This is also one possible venue to the connection with pd-differential operators. Indeed, given two quasi-coherent sheaves $\mathscr{F}$ and $\mathscr{G}$ on X, the *sheaf of pd-differential operators* (also called *crystalline differential operators*) *from* $\mathscr{F}$ *to* $\mathscr{G}$ is usually defined through the two projection maps $p_1, p_2 : (X \times X)^{\mathrm{pd}}_X \to X$ as

$$\begin{aligned} \mathscr{D}\!\mathit{iff}^{\mathrm{pd}}_X(\mathscr{F},\mathscr{G}) &= \underline{\mathrm{Map}}_{\mathscr{O}_X}((p_2)_* p_1^*(\mathscr{F}),\mathscr{G}) \\ &\simeq \underline{\mathrm{Map}}_{\mathscr{O}_X}(i^* i_*(\mathscr{F}),\mathscr{G}), \\ &\simeq i^*\underline{\mathrm{Map}}_{\mathscr{O}_{X_{\mathrm{crys}}}}(i_*(\mathscr{F}), i_*(\mathscr{G})), \end{aligned}$$

where $i : X \to X_{\mathrm{crys}}$ is the canonical map. The filtration of differential operators by order is obtained from the presentation $(X \times X)^{\mathrm{pd}}_X \simeq \varinjlim_n D^n_{X/X\times X}$ that we discussed in Variant 2.9: pd-differential operators of order $\leq n$ are obtained by restricting in the above formula along $D^n_{X/X\times X}$, the n-th order pd-neighborhood of the diagonal.

Everything in this section so far goes through just as well, and with unchanged proofs, in the relative setting too.

Definition 3.7. A natural transformation $X \to Y$ in $\mathrm{Fun}(\mathrm{sCAlg}, \mathcal{S})$ to be *classically pd-formally smooth* (resp. *classically formally smooth*) if the induced map $\pi_0(X(R)) \to \pi_0(Y^{\mathrm{pd}}_X(R))$ (resp. $\pi_0(X(R)) \to \pi_0(Y^{\wedge}_X(R))$) is surjective for all simplicial commutative rings R.

Lemma 3.8. *Any map* $X \to Y$ *in* $\mathrm{Fun}(\mathrm{sCAlg}, \mathcal{S})$ *which is classically formally smooth, is also classically pd-formally smooth.*

Proposition 3.9. *Let* $X \to Y$ *be a classically pd-formally smooth map in* $\mathrm{Fun}(\mathrm{sCAlg}, \mathcal{S})$*. Then the map* $X \to Y^{\mathrm{pd}}_X$ *induces an equivalence of functors*

$$Y^{\mathrm{pd}}_X \simeq \varinjlim \Big(\cdots \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} (X \times_Y X \times_Y X)^{\mathrm{pd}}_X \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} (X \times_Y X)^{\mathrm{pd}}_X \substack{\longrightarrow\\[-1ex]\longrightarrow} X \Big).$$

Remark 3.10 (Explicit formula for pd-completion)**.** When $X \to Y$ is a smooth and separated morphism of classical schemes, we may combine the previous Proposition with Variant 2.9 to obtain a rather explicit grip on the pd-completion of Y along X. Any smooth morphism is classically formally smooth, and as such classically pd-formally smooth by Lemma 3.8. This shows that the formula of Proposition 3.9 is applicable. Furthermore the separatedness assumption implies that the diagonal maps $X \to X \times_Y \cdots \times_Y X$ are all closed immersions, so their pd-completions may be computed by the procedure outlined in Variant 2.9. More precisely, let $D^n_{X/Y}(k)$ denote the n-th order pd-neighborhood of the diagonal inside the $(k+1)$-fold fibered product[7] $X \times_Y \cdots \times_Y X$. Then we may obtain the pd-completion as

$$Y^{\mathrm{pd}}_X \simeq \varinjlim_{([k],n)\in \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{Z}_{\geq 0}} D^n_{X/Y}(k),$$

where the terms in the colimit are explicity given by

$$D^n_{X/Y}(k) = \mathrm{Spec}_{X^{\times_Y (k+1)}} \Big(\mathscr{D}_X\big(X^{\times_Y (k+1)}\big) / \overline{\mathscr{I}(k)}^{[n+1]} \Big)$$

[7]By analogy with the situation for formal completion, see [9], we expect that the conclusion of Proposition 3.9 holds under much weaker assumptions than stated. However, even if the need for smoothness in the case of a morphism of classical schemes was circumvented, the explicit construction of pd-completion described in Remark 3.10 would only hold for morphisms which are separated and flat. The latter being necessary to ensure the fibered products $X \times_Y \cdots \times_Y X$, which are in our setting inherently derived, will in fact be ordinary schemes, making Variant 2.9 applicable to it.

in which $\overline{\mathscr{I}(k)}$ is the sheaf of ideals defining the closed immersion $X \to \mathscr{D}_X(X^{\times_Y(k+1)})$, the pd-thickening of the relative diagonal.

Remark 3.11 (Čech-Alexander resolution, I)**.** One useful approach to crystalline cohomology, similar to the one outlined in the preceding Remark, goes by the name of the Čech-Alexander complex. It translates in our setting to a method for computing the relative crystalline space of a smooth S-scheme X in terms of of an closed embedding $X \to Y$ into a smooth S-scheme Y. This allows us to view the Čech nerve $\check{\mathrm{C}}_\bullet(Y \to \operatorname{Spec} R)$ as a simplicial object in the category $(\mathrm{Sch}_\mathrm{R})_{X/}$, where the structure morphism of the term $\check{\mathrm{C}}_n(Y \to \operatorname{Spec} R) = Y \times_R \cdots \times_R Y$ is the composite of the immersion $X \to Y$ and the relative diagonal $\Delta_{Y/R} : Y \to Y \times_R \times \cdots \times_R Y$. It may be shown, using the smoothness hypothesis, that applying pd-completion along X preserves the colimit $|\check{\mathrm{C}}_\bullet(Y \to \operatorname{Spec} R)| \simeq \operatorname{Spec} R$. This gives rise to an equivalence $\left|(Y \times_R \cdots \times_R Y)^{\mathrm{pd}}_X\right| \simeq (\operatorname{Spec} R)^{\mathrm{pd}}_X \simeq (X/R)_{\mathrm{crys}}$, presented more diagramatically as

$$(X/R)_{\mathrm{crys}} \simeq \varinjlim \Big(\cdots \xrightarrow{\quad} (Y \times_R Y \times_R Y)^{\mathrm{pd}}_X \rightrightarrows (Y \times_R Y)^{\mathrm{pd}}_X \longrightarrow Y^{\mathrm{pd}}_X \Big),$$

and this is the Čech-Alexander resolution of the crystalline space.

4. Crystalline space in positive characteristic

Let us restrict to the positive characteristic case. That is to say, we will discuss the crystalline theory of functors $X : \mathrm{sCAlg} \to \mathcal{S}$ admitting a natural transformation $X \to \operatorname{Spec} \mathbf{F}_p$, or equivalentnly functors $X : \mathrm{sCAlg}_{\mathbf{F}_p} \to \mathcal{S}$. The key result will be the following innocuous Lemma, in which we denote by $\varphi_R : R \to R$ for any commutative $\mathbf{F}_p$-algebra R the Frobenius homomorphism $\varphi_R(x) = x^p$.

Lemma 4.1. *Let R be a commutative $\mathbf{F}_p$-algebra and I and ideal in R which supports a divided power structure. Then there exists a unique ring homomorphism $\tilde{\varphi} : R/I \to R$ making the following diagram commute*

$$\begin{array}{ccc} R & \xrightarrow{\varphi_R} & R \\ \downarrow & \overset{\tilde{\varphi}}{\nearrow} & \downarrow \\ R/I & \xrightarrow[\varphi_{R/I}]{} & R/I, \end{array}$$

where the unmarked arrows are the quotient map $R \to R/I$.

Proof. Since the ideal I admits a divided power structure γ, we have $x^p = p!\gamma_p(x) = 0$ for every element $x \in I$. Consequently we have $(a + x)^p = a^p$ for all $a \in R$ and $x \in I$, which implies that the Frobenius morphism φ_R factors uniquely through the quotient map $R \to R/I$. This gives rise to the unique homomorphism $\tilde{\varphi} : R/I \to R$ making the upper triangle of the diagram in the statement of the Lemma commute. For the lower triangle, note that $\tilde{\varphi}$ it is given explicitly by sending a coset $a + I \in R/I$ to the element $a^p \in R$ for any representative a. Hence it is clear that it produces the Frobenius of the quotient ring upon composition with $R \to R/I$. $\square$

Let us define the *absolute Frobenius morphism* of a functor $X : \mathrm{sCAlg}_{\mathbf{F}_p} \to \mathcal{S}$ to be the natural transformation $F : X \to X$ given for any $R \in \mathrm{sCAlg}_{\mathbf{F}_p}$ by $X(\varphi_R) : X(R) \to X(R)$. We will denote the limit of the tower $\cdots \xrightarrow{F} X \xrightarrow{F} X \xrightarrow{F} X$ by X^{1/p^∞} (it is denoted by X_{perf} in [5]) and called the *perfection of* X. It is given explicity as $X^{1/p^\infty}(R) \simeq X(R^\flat)$, where $R^\flat = \varprojlim(\ldots \xrightarrow{\varphi_R} R \xrightarrow{\varphi_R} R \xrightarrow{\varphi_R} R)$ is the *tilting* (sometimes also called *Fontainization*) of a commutative $\mathbf{F}_p$-algebra R.

Proposition 4.2. *For any functor $X : \mathrm{sCAlg}_{\mathbf{F}_p} \to \mathcal{S}$ there is an equivalence of functors $(X/\mathbf{F}_p)_{\mathrm{crys}} \simeq \pi_0(X)^{\mathrm{pd}}_F$ between the relative crystalline space of X over $\mathbf{F}_p$ and the pd-completion of the absolute Frobenius map $F : \pi_0(X) \to \pi_0(X)$.*

Proof. The comparison map $X^{\mathrm{pd}}_F \to X_{\mathrm{crys}}$ between the pd-completion along the Frobenius map and the crystalline space of X is obtained by passing to the colimit from the projection onto the first factor

$$X(R/I) \times_{F, X(R/I)} X(R) \to X(R/I) \tag{2}$$

where R is an arbitrary commutative $\mathbf{F}_p$-algebra with a pd-structure on an ideal I. We may identify the left-hand side as the space of commuting diagrams

$$\begin{array}{ccc} \mathrm{Spec}(R/I) & \longrightarrow & X \\ {\scriptstyle \mathrm{Spec}(\varphi_{R/I})}\downarrow & & \uparrow \\ \mathrm{Spec}(R/I) & \longrightarrow & \mathrm{Spec}\, R. \end{array}$$

in which the left vertical and lower horizontal maps are prescribed. The map (2) is obtained by restricting to the upper horizontal arrow in the diagram. But Lemma 4.1 guarantees the existence of an anti-diagonal map making the lower triangle commute, exhibiting a retraction of this diagram onto its upper horizontal morphism. Thus the map (2) is a homotopy equivalence, and passing to the colimit over (I, γ), the result follows. □

Variant 4.3 (Other fields of positive characteristic, I)**.** Proposition 4.2 generlizes easily to any field κ of positive characteristic. The one change we need is to use the relative Frobenius map $F_{X/\kappa} : X \to X^{(p)}$, which is a morphism in $\mathrm{Fun}(\mathrm{sCAlg}_\kappa, \mathcal{S})$, as opposed to the absolute Frobenius map, which is ony a morphism in $\mathrm{Fun}(\mathrm{sCAlg}_{\mathbf{F}_p}, \mathcal{S})$. For any functor $X : \mathrm{sCAlg}_\kappa \to \mathcal{S}$ we have a natural transformation $(X/\kappa)_{\mathrm{crys}} \simeq \pi_0(X^{(p)})^{\mathrm{pd}}_{\pi_0(X)}$, identifying the crystalline space with the pd-completion of the relative Frobenius map.

The conclusion of Proposition 4.2 leads us to consider X^{pd}_F, the pd-completion of the Frobenius map $F : X \to X$. Under the assumption that the absolute Frobenius map is a closed immersion, usually called that X is *semi-perfect*, we already know that the pd-completion will coincide with the pd-envelope.

Corollary 4.4. *Let X be a semi-perfect derived $\mathbf{F}_p$-scheme. Then $(X/\mathbf{F}_p)_{\mathrm{crys}} \simeq D_F(\pi_0(X))$, i.e. the relative crystalline space of X is equivalent to the pd-envelope of the absolute Frobenius map $F : \pi_0(X) \to \pi_0(X)$.*

Proof. Combine Proposition 4.2 and Corollary 2.8. □

Still fixing a functor $X : \mathrm{sCAlg}_{\mathbf{F}_p} \to \mathcal{S}$, we would like to obtain a description of the relative crystalline space $(X/(\mathbf{Z}/p^n))_{\mathrm{crys}}$ analogous to the one for $(X/\mathbf{F}_p)_{\mathrm{crys}}$ provided by Proposition 4.2. We need an enchancement of Lemma 4.1, and following Drinfeld we employ the following in that role.

Lemma 4.5 (Proposition 2.2.1 in [6])**.** *Let A be a commutative $\mathbf{F}_p$-algebra, and let $f : R \to A$ be a homomorphism of commutative $\mathbf{Z}/p^n$ algebras, whose kernel supports a divided power structure. Then there exists a unique homorphism $\tilde{f} : W_n(A^\flat) \to R$ making the diagram*

$$\begin{array}{ccc} W_n(A^\flat) & \xrightarrow{\tilde{f}} & R \\ \downarrow & & \downarrow{\scriptstyle f} \\ A^\flat & \longrightarrow & A, \end{array}$$

commute.

Proof. Recall that the ring of truncated Witt vectors $W_n(R)$ is defined so that its underlying additive group is R^{n+1}, while the multiplication is characterized by making the map $w_n : R^{n+1} \to R$, defined by $w_n(a_0, a_1, \dots, a_n) = a_0^{p^n} + pa_1^{p^{n-1}} + \cdots + p^n a_n$, into a ring homomorphism.

Let I denote the kernel of $R \to A$. For any $x \in I$ we have $x^p = p!\gamma_p(x) \in pI$, and hence by induction and the binomial formula $(x+a)^{p^i} - a^{p^i} \in p^i I$ for all $i \geq 0$ and all $a \in R$. This computation reveals that the value $w_n(a_0, \dots, a_n) \in R$ depends only on the coset $(a_0, \dots, a_n) + I \in R/I \simeq A$. It follows that the homomorphism $w_n : W_n(R) \to R$ factors as $W_n(R) \xrightarrow{W_n(f)} W_n(A) \xrightarrow{\tilde{w}_n} R$. In fact, the same type of argument as in the proof of Lemma 4.1 shows that the entire diagram

$$\begin{array}{ccc} W_n(R) & \xrightarrow{w_n} & R \\ {\scriptstyle W_n(f)}\downarrow & \nearrow^{\tilde{w}_n} & \downarrow{\scriptstyle f} \\ W_n(A) & \xrightarrow[w_n]{} & A \end{array}$$

commutes.

With the lift $\tilde{w}_n : W_n(A) \to R$ in hand, we may construct the sought homomorphism $\tilde{f}$ as the compositum

$$W_n(A^\flat) \xrightarrow{W_n(\varphi_{A^\flat}^{-n})} W_n(A^\flat) \xrightarrow{W_n(\sharp)} W_n(A) \xrightarrow{\tilde{w}_n} R,$$

where we have used the fact that the Frobenius of $A^\flat$ is invertible, and where $\sharp : A^\flat \to A$ is the canonical map from the tilt. To see the homomorphism $\tilde{f} : W_n(A^\flat) \to R$ thus constructed makes the desired diagram commute, we must identify the composition $f \circ \tilde{f}$ with $W_n(A^\flat) \to A^\flat \to A$, in which the left arrow is what we call w_0, and right arrow is $\sharp$. The commutative diagram from the previous paragraph shows (because $W_n(f)$ is surjective) that $f \circ \tilde{w}_n = w_n$, so it remains to study the composition

$$W_n(A^\flat) \xrightarrow{W_n(\varphi_{A^\flat}^{-n})} W_n(A^\flat) \xrightarrow{W_n(\sharp)} W_n(A) \xrightarrow{w_n} A.$$

By commutativity of the diagram

$$\begin{array}{ccccc} W_n(A^\flat) & \xrightarrow{W_n(\varphi_{A^\flat}^{-n})} & W_n(A^\flat) & \xrightarrow{W_n(\sharp)} & W_n(A) \\ {\scriptstyle w_n}\downarrow & & {\scriptstyle w_n}\downarrow & & {\scriptstyle w_n}\downarrow \\ A^\flat & \xrightarrow{\varphi_{A^\flat}^{-n}} & A^\flat & \xrightarrow{\sharp} & A, \end{array}$$

it remains only to note that $w_n(a) = w_0(a)^{p^n} = \varphi_{A^\flat}^n(w_0(a))$ for all $a \in W_n(A^\flat)$, because A and hence $A^\flat$ are $\mathbf{F}_p$-algebras, and all the remaining terms of the Witt polynomial w_n include powers of p. Thus $\tilde{f}$ satisfies the required property.

Conversly, suppose $\tilde{f} : W_n(A^\flat) \to R$ is any ring homomorphism making the diagram in the statement of the Lemma commute. We must show it is unique.

By the observations of the second paragraph of the proof, the element $\tilde{f}(a)^{p^n} \in R$ for any $a \in W_n(A^\flat)$ depends only on the coset of $\tilde{f}(a)$ in $R/I \simeq A$, i.e. on $f(\tilde{f}(a)) = w_0(a)^\sharp$. That is to say, for any element $x \in R$ such that $x + I = f(x) = w_0(a)^\sharp$ holds in $R/I \simeq A$, we will have $x^{p^n} = \tilde{f}(a)^{p^n}$ in R.

The key is now to show that the values $\tilde{f}(a)^{p^n}$, which have seen in the previous paragraph to be unique, will in fact entirely determine the map $\tilde{f}$. This is a consequence of the ring $W_n(A^\flat)$ being generated by the image of the Teichmuller lift map $\omega : A^\flat \to W_n(A^\flat)$, since we have $\tilde{f}(\omega(a)) = \tilde{f}(\omega(\varphi_{A^\flat}^{-n}(a))^{p^n}$ for any $a \in A^\flat$. $\square$

Let us define the *tilting of a functor* $X : \mathrm{sCAlg}_{\mathbf{F}_p} \to \mathcal{S}$ by left Kan extension from affines. That is to say, the tilting of X is functor $X^\flat : \mathrm{sCAlg}_{\mathbf{F}_p} \to \mathcal{S}$ such that $(\operatorname{Spec} A)^\flat \simeq \operatorname{Spec}(A^\flat)$, and we requiring that the constrction $X \mapsto X^\flat$ commutes with all small colimits. The truncated Witt vectors W_n are extended to functors in the same way.

Proposition 4.6. *For any functor $X : \mathrm{sCAlg}_{\mathbf{F}_p} \to \mathcal{S}$ there is an equivalence of functors $(X/(\mathbf{Z}/p^n))_{\mathrm{crys}} \simeq W_n(\pi_0(X)^\flat)^{\mathrm{pd}}_{\pi_0(X)}$ between the relative crystalline space and the pd-completion of the map $\pi_0(X) \to W_n(\pi_0(X)^\flat)$.*

Proof. Since formation of the crystalline space commutes with colimits, it suffices to prove the calim for $X \simeq \operatorname{Spec}(A)$ for some simplicial commutative $\mathbf{Z}/p^n$-algebra A. Furthermore since the crystalline space can not tell A and $\pi_0(A)$ apart, we may also assume that A is an ordinary commutative $\mathbf{Z}/p^n$-algebra.

Now given any commutative $\mathbf{Z}/p^n$-algebra R and a pd-structure γ on an ideal $I \subseteq R$, Lemma 4.5 may be used to show that

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{\mathbf{Z}/p^n}}(A, R/I) & \longrightarrow & \mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{\mathbf{Z}/p^n}}(W_n(A^\flat), R) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{\mathbf{Z}/p^n}}(A, R/I) & \longrightarrow & \mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{\mathbf{Z}/p^n}}(W_n(A^\flat), R/I) \end{array}$$

is a pushout square. Passing to the colimit in (I, γ) produces the series of homotopy equivalences

$$\begin{aligned} (\operatorname{Spec}(A)/(\mathbf{Z}/p^n))_{\mathrm{crys}}(R) &\simeq \varinjlim_{(I,\gamma)} (\operatorname{Spec} A)(R/I) \\ &\simeq \varinjlim_{(I,\gamma)} \operatorname{Spec}(W_n(A^\flat))(R) \times_{\operatorname{Spec}(W_n(A^\flat))(R/I)} (\operatorname{Spec} A)(R/I) \\ &\simeq \operatorname{Spec}(W_n(A^\flat))^{\mathrm{pd}}_{\operatorname{Spec}(A)}(R), \end{aligned}$$

in which the second equivalence follows from the above observation. □

Corollary 4.7. *Let X be a semi-perfect derived $\mathbf{F}_p$-scheme. Then there is an equivalence $(X/(\mathbf{Z}/p^n))_{\mathrm{crys}} \simeq D_{\pi_0(X)}(W_n(\pi_0(X)^\flat))$, i.e. the relative crystalline space of X is equivalent to the pd-envelope of the closed immersion $\pi_0(X) \to W_n(\pi_0(X)^\flat)$.*

Proof. We may assume with no loss that X is a semi-perfect ordinary $\mathbf{F}_p$-scheme. The semi-perfectness assumption guarantees the tilting map $X \to X^\flat$ to be a closed immersion. Since $X^\flat \to W_n(X^\flat)$ is also a closed immersion, it follows that we may apply Corollary 2.8 to Proposition 4.6 and deduce the desired conclusion. □

Remark 4.8 (Yoga)**.** The yoga of the above two results, in comparison to Proposition 4.2 and Corollary 4.4 is roughly as follows. While it suffices, when working over $\mathbf{F}_p$, to only think about the Frobenius $F : X \to X$, when working over $\mathbf{Z}/p^n$ we must also think of higher iterates of the Frobenius, manifesting itself in $X^\flat$. On the other hand, deformations of $\mathbf{F}_p$-algebras to $\mathbf{Z}/p^n$-algebras are as usual controlled by the truncated Witt vectors W_n, accounting for their appearence as well. This also motivates the following section, where we will see that working over $\mathbf{Z}_p$ requires the use of the full ring of (p-typical) Witt vectors.

5. Crystalline space over the p-adics

In the usual treatments of crystalline cohomology, e.g. [2] and Chapter SP.55, attention is restricted only to such pd-thickenings in which $p^n = 0$ for $n \gg 0$. This amounts in our language to working over the p-adic integers $\mathbf{Z}_p$. More precisely however, we must work over the formal scheme $\operatorname{Spf}\mathbf{Z}_p$, instead of the affine scheme $\operatorname{Spec}\mathbf{Z}_p$. The former is to be interpreted in the pro-representability sense, which is to say that

$$(\operatorname{Spf}\mathbf{Z}_p)(R) \simeq \varinjlim_n \operatorname{Spec}(\mathbf{Z}/p^n)(R) \simeq \varinjlim_n \operatorname{Map}_{\mathrm{sCAlg}}(\mathbf{Z}/p^n, R),$$

sending a discrete ring R to the set of its p-torsion elements. This leads to the following definition.

Definition 5.1. The *p-adic crystalline space* of a functor $X : \mathrm{sCAlg}_{\mathbf{F}_p} \to \mathcal{S}$ is a functor $(X/\operatorname{Spf}\mathbf{Z}_p)_{\mathrm{crys}} : \mathrm{sCAlg} \to \mathcal{S}$ defined as the colimit of the tower

$$(X/\mathbf{F}_p)_{\mathrm{crys}} \to (X/(\mathbf{Z}/p^2))_{\mathrm{crys}} \to (X/(\mathbf{Z}/p^3))_{\mathrm{crys}} \to \cdots,$$

obtained from the corresponding ring homomorphisms $\mathbf{Z}/p^n \to \mathbf{Z}/p^{n-1}$. Equivalently, for any simplicial commutative ring R for which p is nilpotent in $\pi_0(R)$, the p-adic crystalline space is defined as

$$(X/\operatorname{Spf}\mathbf{Z}_p)_{\mathrm{crys}}(R) \simeq \varinjlim_{(I,\gamma)} X(\pi_0(R)/I),$$

where the colimit ranges over the poset of nilpotent ideals I in $\pi_0(R)$ and pd-structures γ on I. For a general simplicial commutative ring R the space $(X/\operatorname{Spf}\mathbf{Z}_p)_{\mathrm{crys}}(R)$ is determined by Kan extension.

Remark 5.2 (p-adic completion)**.** Given any functor $X : \mathrm{sCAlg} \to \mathcal{S}$, we define its *$p$-adic completion* $X^\wedge_p$ to be the left Kan extension back to sCAlg of the restriction of X to the full subcategory[8] $\mathrm{sCAlg}^{\mathrm{cont}}_{\mathbf{Z}_p} \subseteq \mathrm{sCAlg}$, spanned by all simplicial commutative rings R for which p is nilpotent in $\pi_0(R)$. Since $\mathrm{sCAlg}^{\mathrm{cont}}_{\mathbf{Z}_p}$ coincides with the colimit of the tower of subcategories[9] $\mathrm{sCAlg}_{\mathbf{F}_p} \subseteq \mathrm{sCAlg}_{\mathbf{Z}/p^2} \subseteq \mathrm{sCAlg}_{\mathbf{Z}/p^2} \subseteq \cdots \subseteq \mathrm{sCAlg}$, and the left Kan extension along the inclusion $\mathrm{sCAlg}_{\mathbf{Z}/p^n} \subseteq \mathrm{sCAlg}$ of the restriction $X|_{\mathrm{sCAlg}_{\mathbf{Z}/p^n}}$ may readily be identified with the product $X \times \operatorname{Spec}(\mathbf{Z}/p^n)$, it follows that

$$\begin{aligned} X^\wedge_p &\simeq \varinjlim_n (X \times \operatorname{Spec}(\mathbf{Z}/p^n)) \\ &\simeq X \times \big(\varinjlim_n \operatorname{Spec}(\mathbf{Z}/p^n)\big) \\ &\simeq X \times \operatorname{Spf}\mathbf{Z}_p. \end{aligned}$$

Working over $\operatorname{Spf}\mathbf{Z}_p$ is therefore equivalent to restricting ourselves to the p-complete setting. Since the p-adic crystalline space is by construction an object over $\operatorname{Spf}\mathbf{Z}_p$, this hopefully clarifies why its value is only determined by the expected formula on $\mathrm{CAlg}^{\mathrm{cont}}_{\mathbf{Z}_p}$.

The Witt vectors of a functor $X : \mathrm{sCAlg}_{\mathbf{F}_p} \to \mathcal{S}$ is defined to be the functor $X : \mathrm{sCAlg} \to \mathcal{S}$ obtained by left Kan extension from affines, for which we set $W(\operatorname{Spec} A) \simeq \operatorname{Spf}(W(A))$. Recall that truncated Witt vectors were extended to functors in Section 4 in the same way, showing that the usual formula $W(A) \simeq \varprojlim W_n(A)$ relating truncated and p-typical Witt vectors, extends to give $W(X) \simeq \varinjlim W_n(X)$ for all $X \in \operatorname{Fun}(\mathrm{sCAlg}_{\mathbf{F}_p}, \mathcal{S})$.

[8]There is reason for our choice of notation for the subcategory in question. Recall that the ∞-category of simplicial commutative rings is equivalent to the the ∞-category of topological commutative rings $\mathrm{CAlg}^{\mathrm{cont}}$. One construction of the ring $\mathbf{Z}_p$ proceeds by completing the p-adic topology of the integers, so it can in particular be viewed as a topological commutative ring. The ∞-category that we have chosen to call $\mathrm{CAlg}^{\mathrm{cont}}_{\mathbf{Z}_p}$ in this section coincides under the equivalence $\mathrm{sCAlg} \simeq \mathrm{CAlg}^{\mathrm{cont}}$ with $\mathrm{CAlg}^{\mathrm{cont}}_{\mathbf{Z}_p/}$, i.e. the ∞-category of topological $\mathbf{Z}_p$-algebras.

[9]We are implicitly using that being a $\mathbf{Z}/p^n$-algebra is a *propery* of a simplicial commutative ring, and not *extra structure*. This seems trivial, but is one point of profund difference to the theory of $\mathbb{E}_\infty$-rings, where that is no longer the case. Roughly speaking, we are using the fact that simplicial commutative rings have no power operations.

Proposition 5.3. *For any functor $X : \mathrm{sCAlg}_{\mathbf{F}_p} \to \mathcal{S}$ there is an equivalence of functors $(X/\operatorname{Spf}\mathbf{Z}_p)_{\mathrm{crys}} \simeq \big(W(\pi_0(X)^\flat)^{\mathrm{pd}}_{\pi_0(X)}\big)^\wedge_p$ between the relative crystalline space and the p-adically completed pd-completion of the map $\pi_0(X) \to W(\pi_0(X)^\flat)$.*

Proof. Pass to the colimit as $n \to \infty$ from Proposition 4.6, observing that the appearence of the p-adic completion is clear from the discussion in Remark 5.2. □

Variant 5.4 (Other fields of positive characteristic, II)**.** Continuing with Variant 4.3, the analogue of Proposition 4.6 for a perfect field κ of characteristic p requires us to re-define the tilting of an $\mathbf{F}_p$-algebra A to $A^\flat \simeq \varprojlim\big(\cdots \xrightarrow{F_{A/\kappa}} A^{(p^2)} \xrightarrow{F_{A/\kappa}} A^{(p)} \xrightarrow{F_{A/\kappa}} A\big)$. That is to say, just as mentioned in Variant 4.3, the absolute Frobenius maps must be replaed with the relative Frobenius maps. The analoues of Proposition 4.6 and 5.3 identify relative crystalline spaes of a functor $X : \mathrm{sCAlg}_\kappa \to \mathcal{S}$ as $(X/W_n(\kappa))_{\mathrm{crys}} \simeq W_n(\pi_0(X^\flat))^{\mathrm{pd}}_{\pi_0(X)}$ for any non-negative integer n, and $(X/\operatorname{Spf}W(\kappa))_{\mathrm{crys}} \simeq W(\pi_0(X^\flat))^{\mathrm{pd}}_{\pi_0(X)}$ respectively.

At this point it is convenient, as well as well-motivated by Proposition 5.3 to introduce *Fontaine's crystalline period ring* $\mathbf{A}_{\mathrm{crys}}(A)$ for any semi-perfect $\mathbf{F}_p$-algebra A. The ring $\mathbf{A}_{\mathrm{crys}}(A)$ is defined to be the p-adically completed pd-envelope of $W(A^\flat)$ along the kernel of the canonical surjection $W(A^\flat) \to A^\flat \to A$ (note that is for this map to be a surjection that we need the semi-perfectness assumption). We extend the functor $\mathbf{A}_{\mathrm{crys}}$ to $\mathrm{Fun}(\mathrm{sCAlg}_{\mathbf{F}_p}, \mathcal{S})$ by setting $\mathbf{A}_{\mathrm{crys}}(\operatorname{Spec}A) \simeq \operatorname{Spf}(\mathbf{A}_{\mathrm{crys}}(A))$, and requiring the construction $X \to \mathbf{A}_{\mathrm{crys}}(X)$ to commute with all small colimits. The following observation, at least in the affine setting, goes back to [7].

Proposition 5.5 (Fontaine)**.** *Let X be a semi-perfect derived $\mathbf{F}_p$-scheme. Then there is an equivalence $(X/\operatorname{Spf}\mathbf{Z}_p)_{\mathrm{crys}} \simeq \mathbf{A}_{\mathrm{crys}}(\pi_0(X))$, i.e. the relative crystalline space of X is equivalent to the p-adically complete pd-envelope of the closed immersion $\pi_0(X) \to W(\pi_0(X)^\flat)$.*

Proof. As usual, it suffices to prove the resut for $X \simeq \operatorname{Spec}A$ where A is an ordinary commutative $\mathbf{F}_p$-algebra. Then we may pass to the colimit as $n \to \infty$ in Corollary 4.7. □

Remark 5.6 (Drinfeld's crystalline stack)**.** Motivated by ideas of Bhatt-Morrow-Scholze, Drinfeld proposed a stacky approach to crystals in [5]. For this purpose, he introduces the notion of *Frobenius-smoothness* for an $\mathbf{F}_p$-scheme X, one way of phrasing which is to require that the absolute Frobenius map $F : X \to X$ is syntomic. The assumption of Frobenius smoothness suffices for the map $X^{1/p^\infty} \to X$ to induce a surjection $\pi_0\big(X^{1/p^\infty}_{\mathrm{crys}}(R)\big) \to \pi_0\big((X/\operatorname{Spf}\mathbf{Z}_p)_{\mathrm{crys}}(R)\big)$ for any simplicial commutative ring R. This allows us to use descent, in the form of the canonical map from the realization of the Čech nerve $|\check{C}_\bullet(X^{1/p^\infty} \to X)| \to X$ inducing an equivalence on crystalline spaces. The terms of the Čech nerve $\check{C}_\bullet(X^{1/p^\infty} \to X) \simeq X^{1/p^\infty} \times_X \cdots \times_X X^{1/p^\infty}$ are all semi-perfect $\mathbf{F}_p$-schemes, allowing us to apply Proposition 5.5. This gives rise to the groupoid presentation of the p-adic crystalline space of X as

$$(X/\operatorname{Spf}\mathbf{Z}_p)_{\mathrm{crys}} \simeq \varinjlim\Big(\cdots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \mathbf{A}_{\mathrm{crys}}\big(X^{1/p^\infty} \times_X X^{1/p^\infty}\big) \rightrightarrows \mathbf{A}_{\mathrm{crys}}\big(X^{1/p^\infty}\big)\Big).$$

Since Drinfeld takes the right-hand side as the definition, this shows that our p-adic crystalline space and his stack coincide whenever the latter is defined.

Remark 5.7 (Čech-Alexander resolution, II)**.** In Remark 3.11 we discussed a version of the Čech-Alexander presentation for the relative crystalline space. Though $(X/\operatorname{Spf}\mathbf{Z}_p)_{\mathrm{crys}}$ is not technically an example of the latter, it is very close to being so, allowing us to extend the Čech-Alexander method in the following setup. Let X be a smooth $\mathbf{F}_p$-scheme and Y a smooth p-adic formal scheme over $\operatorname{Spf}\mathbf{Z}_p$, together with a closed immersion $f : X \to Y$, such that f exhibits an isomorphism $X \simeq Y \times_{\operatorname{Spf} \mathbf{Z}_p} \operatorname{Spec} \mathbf{F}_p$. We say that Y is a *smooth lift of X over $\mathbf{Z}_p$*, and X is its *mod p-reduction* or *special fiber.* Using Lemma 2.8 to simplify the Čech-Alexander resolution for the relative crystalline space over $\mathbf{Z}/p^n$, passing to the colimit $n \to \infty$, and using that the ideal defining the closed immersion $X \to Y^n$ is p^n-torsion, produces the formula

$$(X/\operatorname{Spf} \mathbf{Z}_p)_{\mathrm{crys}} \simeq \varinjlim \Big(\cdots \mathrel{\substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow}} D_X(Y \times Y \times Y)^\wedge_p \mathrel{\substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow}} D_X(Y \times Y)^\wedge_p \rightrightarrows D_X(Y)^\wedge_p \Big).$$

In fact, the comparison between the sheaves on $(X/\operatorname{Spf} \mathbf{F}_p)_{\mathrm{crys}}$ with crystals in quasi-coherent sheaves on X given in [5] is proved by expressing the former via a Čech-Alexander resolution of this form.

6. Crystals and the crystalline site

In this section we define crystals on a functor, and discuss the connection with the original crystalline site approach to crystals of Grothendieck. To make the analogy cleaner, we first make a digression about the value of pd-completion on derived schemes.

Given a functor $X : \mathrm{sCAlg} \to \mathcal{S}$, we may evaluate $X(S)$ for any derived scheme S in the usual way, by Kan extension along $\operatorname{Spec} : \mathrm{sCAlg} \to \mathrm{DSch}^{\mathrm{op}}$. Explicitly this is given by $X(S) \simeq \varinjlim_{U \subseteq X} X(\mathscr{O}(U))$ with the colimit ranging over the poset of affine opens in X. The next Lemma will express pd-completion in terms of evaluating on derived schemes.

Lemma 6.1. *The value of pd-completion of a map $X \to Y$ in* $\operatorname{Fun}(\mathrm{sCAlg}, \mathcal{S})$ *on a derived scheme T is given by*

$$Y^{\mathrm{pd}}_X(T) \simeq \varinjlim_{S \to \pi_0(T)} X(S) \times_{Y(S)} Y(T),$$

with the colimit taken over pd-thickenings (in the category of ordinary schemes) $S \to \pi_0(T)$.

Proof. Since the formula on the right hand side commutes with colimits, it is determined as a functor $\mathrm{DSch}^{\mathrm{op}} \to \mathcal{S}$ by its restriction to affine schemes. As such, it suffices to show the equivalence when $T \simeq \operatorname{Spec} R$ is an affine derived scheme. Since S is a closed subscheme of the affine scheme $\operatorname{Spec} \pi_0(R)$, it must be affine itself. The right-hand side in the statement of the Lemma thus reproduces the formula for pd-completion in Definition 1.4. □

Definition 6.2. Given a functor $X : \mathrm{sCAlg} \to \mathcal{S}$, the *$\infty$-category of crystals (in quasi-coherent sheaves) on X* is defined as $\mathrm{Crys}(X) = \mathrm{QCoh}(X_{\mathrm{crys}})$. The functor $\mathrm{Crys}(X) \to \mathrm{QCoh}(X)$, induced by the canonical map $X \to X_{\mathrm{crys}}$, is viewed as sending a crystal to its underlying quasi-coherent sheaf.

Remark 6.3 (Informal intepretation of crystals)**.** In view of the definition of quasi-coherent sheaves on a functor, see e.g. Definition SAG.6.2.2.1, and the above Lemma, we find that we may write the ∞-category of crystals on X as

$$\mathrm{Crys}(X) \simeq \varprojlim_{S \in \mathrm{DSch}_{/X}} \varprojlim_{T \in \mathrm{Sch}^{\mathrm{pd-th}}_{S/}} \mathrm{QCoh}(T), \tag{3}$$

in which the first limit ranges over maps of derived schemes $S \to X$ and the second limit ranges over all pd-thickenings $S \to T$ by ordinary schemes T, or hence equivalently, all pd-thickenings by ordinary schemes of $\pi_0(S)$. A crystal $\mathscr{F} \in \mathrm{Crys}(X)$ therefore informally consists of associating to every map of a scheme S into X, and every pd-thickening $S \to T$ by an ordinary scheme T, a quasi-coherent sheaf[10] $\mathscr{F}_{(S,T)} \in \mathrm{QCoh}(T)$. Furthermore, for any map of pd-thickenings $f : T \to T'$, there is an equivalence $f^*(\mathscr{F}_{(S,T')}) \simeq \mathscr{F}_{(S,T)}$ of quasi-coherent sheaves on T.

[10]Note that we are speaking about quasi-coherent sheaves in the sense of derived algebraic geometry. Therefore even though T is an ordinary scheme, we are denoting by $\mathrm{QCoh}(T)$ the derived ∞-category of $\mathrm{QCoh}(T)^\heartsuit$, the ordinary category of classical quasi-coherent sheaves on T. That is to say, $\mathscr{F}_{(S,T)}$ may be viewed as a chain complex of classical quasi-coherent sheaves on T, defined only up to quasi-isomorphism.

Remark 6.4 (Importance of ∞-categories)**.** Since the inclusion of the underlying classical functor $\pi_0(X) \to X$ does not effect the crystalline space, it induces an equivalence of ∞-categories $\mathrm{Crys}(X) \simeq \mathrm{Crys}(\pi_0(X))$. But even though the crystals do not see derived structure, their definition must still be given in the world of derived algebraic geometry. As explained in Warning 1.6, this is because in order to obtain the correct notion, the limit in (3) in the definition must be interpreted ∞-categorically.

The definition of crystals extends to the relative setting in the expected way.

Variant 6.5. Let R be a simplicial commutative ring and $X : \mathrm{sCAlg}_R \to \mathcal{S}$ a functor. The ∞*-category of relative crystals on X over R* is defined as $\mathrm{Crys}(X/R) = \mathrm{QCoh}((X/R)_{\mathrm{crys}})$. Similarly, the ∞*-category of p-adic crystals* on a functor $X : \mathrm{sCAlg}_{\mathbf{F}_p} \to \mathcal{S}$ is defined as $\mathrm{Crys}(X/\operatorname{Spf}\mathbf{Z}_p) \simeq \mathrm{QCoh}((X/\operatorname{Spf}\mathbf{Z}_p)_{\mathrm{crys}})$.

Remark 6.6 (Recovering the crystalline site)**.** Let R be a commutative ring and X an scheme. The presentation of crystals by a formula analogous to (3) still holds for $\mathrm{Crys}(X)$, so long as we replace schemes with schemes. Let us examine the category on which the total limit in the (analogue of) formula (3) is indexed. Its objects consist of pairs $(S \to X, S \to T)$, or (S,T) for short, of any map from a derived R-scheme S into X, and a pd-thickening by an ordinary scheme T of S. The morphisms $(S,T) \to (S',T')$ in this category consist of commutative diagrams

$$\begin{array}{ccccc} & & S & \longrightarrow & T \\ & \swarrow & \downarrow & & \downarrow \\ X & \longleftarrow & S' & \longrightarrow & T' \end{array}$$

in DSch, in which the map $T \to T'$ commutes with the pd-structure on the defining ideals of S and S' respectively. Note that, since X is assumed to be classical, any map $S \to X$ from a derived R-scheme S into X will factor uniquely through the map $\pi_0(S) \to X$. This means that the full subcategory of the category in question, spanned by those (S,T) for which S is an ordinary scheme, is cofinal, allowing us to index the colimit on it instead. We may therefore restrict to this subcategory with no loss of generality. The category we have thus obtained is precisely the big crystalline site of X, as defined in [2] and Definition SP.55.8.1. Let us denote it $\mathrm{CRIS}(X)$.

Remark 6.7 (Grothendieck topology on the crystalline site)**.** As the name suggests, the crystalline site is not merely a category, but also inherits Grothendieck topologies from X. To see how this arises in our setup, let us consider replacing an arbitrary functor $X : \mathrm{sCAlg} \to \mathcal{S}$ by its sheafification with respect to one of the standard Grothendieck topologies on (affine) schemes: Zariski, étale, smooth, flat; pick your poison. Let $L : \mathrm{Fun}(\mathrm{sCAlg}, \mathcal{S}) \to \mathrm{Shv}(\mathrm{sCAlg}^{\mathrm{op}})$ denote sheafification with respect to the chosen topology. By Proposition SAG.6.2.3.1 the map $X_{\mathrm{crys}} \to L(X_{\mathrm{crys}})$ induces an equivalence of ∞-categories on quasi-coherent sheaves. Recall that the sheafification of X may be obtained by applying (transfinitely many times) the construction $X^+(S) = \varinjlim_{U \to S} X(U)$ with the colimit ranging over coverings $U \to S$. Thus we see that the topology only effects the scheme S in an object $(S,T) \in \mathrm{CRIS}(X)$, suggesting the topology on the crystalline space to be defined entirely on the level of S. That is to say, $\{(S_i,T_i) \to (S,T)\}_i$ constitutes a covering in $\mathrm{CRIS}(X)$ if and only if $\{S_i \to S\}_i$ is a covering of schemes in the chosen topology. Thus we have recovered the traditional way to topologize the crystalline site.

Remark 6.8 (The infinitesimal site)**.** Applying to X_{dR} similar analysis as we undertook for X_{crys} in the previous two Remarks, reproduces the big infinitesimal site. The only difference with the crystalline site is the absence of pd-structures, instead considering all infinitesimal (that is to say, nilpotent) immersions in place of pd-immersions in the discussion above. As follows from Proposition 1.10, the two sites coincide in characteristic zero.

Remark 6.9 (Crystals in other categories)**.** In the famous letter to Tate, Grothendieck emphasizes the ubiquity of crystals in various substances, not exclusively in quasi-coherent sheaves. Crystals in categories manifest as quasi-coherent sheaves with values in categories, as discussed in Chapter SAG.10, on the crystalline stack.

7. Right crystals and crystalline D-modules

Following the yoga of Gaitsgory-Rozenblyum, the version of the ∞-category of crystals on X discussed in the previous section should be considered as *left crystals*, while *right crystals on X* should be defined as $\mathrm{Crys}^{\mathrm{R}}(X) = \mathrm{IndCoh}(X_{\mathrm{crys}})$. For our purposes we may define ind-coherent sheaves on a functor naively[11] by taking $\mathrm{IndCoh}(X) \simeq \mathrm{Ind}(\mathrm{Coh}(X))$ for a derived scheme X, and left Kan extending along the inclusion $\mathrm{DSch} \to \mathrm{Fun}(\mathrm{sCAlg}, \mathcal{S})$.

Remark 7.1 (Utility of right crystals in characteristic 0)**.** In the characteristic 0 story exposited in [9], one major utility of right crystals in the sense of $\mathrm{IndCoh}(X_{\mathrm{dR}})$, observed already in the highly influential preprint of [1], is that the functoriality of ind-coherent sheaves makes for a simple identification of crystals with D-modules. That is to say, when X is a smooth scheme, the !-pullback along the canonical map $X \to X_{\mathrm{dR}}$ induces an equivalence of ∞-categories $\mathrm{IndCoh}(X_{\mathrm{dR}}) \simeq \mathrm{Mod}_{\mathscr{D}_X}(\mathrm{QCoh}(X))$, where $\mathscr{D}_X = \mathscr{D}\!\mathit{iff}(\mathscr{O}_X, \mathscr{O}_X)$ is the sheaf of differential operators on X.

In this section, we will develop the analogue of the preceding Remark in our setting. Recall from Remark 3.6 the sheaf of pd-differential operators $\mathscr{D}\!\mathit{iff}^{\mathrm{pd}}_X(\mathscr{F}, \mathscr{G})$ between $\mathscr{F}, \mathscr{G} \in \mathrm{QCoh}(X)$. We call $\mathscr{D}^{\mathrm{pd}}_X = \mathscr{D}\!\mathit{iff}^{\mathrm{pd}}_X(\mathscr{O}_X, \mathscr{O}_X)$ the *sheaf of crystalline differential operators on X*.

Remark 7.2 (Comparison with ordinary differential operators)**.** Let us denote by $p_1, p_2 : (X \times X)^{\mathrm{pd}}_X \rightrightarrows X$ and $q_1, q_2 : (X \times X)^{\wedge}_X \rightrightarrows X$ two projections from the pd-completion and formal completion respectively of the diagonal inside $X \times X$. The sheaves of crystalline and ordinary differential operators, the latter in the sense of [11], are given by inner mapping objects in $\mathrm{QCoh}(X)$

$$\mathscr{D}^{\mathrm{pd}}_X \simeq \underline{\mathrm{Map}}_{\mathscr{O}_X}((p_1)_* p_2^*(\mathscr{O}_X), \mathscr{O}_X), \qquad \mathscr{D}_X \simeq \underline{\mathrm{Map}}_{\mathscr{O}_X}((q_1)_* q_2^*(\mathscr{O}_X), \mathscr{O}_X).$$

Recall from Remark 3.10 (and its formal analogue) that the completions in question admit exhaustive filtrations

$$\begin{aligned} (X \times X)^{\mathrm{pd}}_X &\simeq \varinjlim (X \simeq D^0_X(1) \to D^1_X(1) \to D^2_X(1) \to \cdots), \\ (X \times X)^{\wedge}_X &\simeq \varinjlim (X \simeq P^0_X(1) \to P^1_X(1) \to P^2_X(1) \to \cdots) \end{aligned}$$

by increasing-order pd-neighborhoods $D^n_X(1)$ resp. infinitesimal neighborhoods $P^n_X(1)$ of the diagonal. These correspond to the order filtration on differential operators. Given the explicit definitions of infinitesimal neighborhoods and pd-neighborhoods of the diagonal in terms of the defining ideals $\mathscr{I} \subseteq \mathscr{O}_{X \times X}$ and $\overline{\mathscr{I}} \subseteq \mathscr{D}_{X \times X}(X)$ of the respective diagonally immersed copies of X, we obtain

$$\begin{aligned} \mathscr{D}^{\mathrm{pd}}_X &\simeq \varinjlim_{n \geq 0} \underline{\mathrm{Map}}_{\mathscr{O}_X}\big(\mathscr{D}_X(X \times X)/\overline{\mathscr{I}}^{[n+1]}, \mathscr{O}_X\big), \\ \mathscr{D}_X &\simeq \varinjlim_{n \geq 0} \underline{\mathrm{Map}}_{\mathscr{O}_X}(\mathscr{O}_{X \times X}/\mathscr{I}^{n+1}, \mathscr{O}_X). \end{aligned}$$

More concretely, $\mathscr{D}_X$ coincides with differential operators on X defined inductively in terms of commutators with functions, while $\mathscr{D}^{\mathrm{pd}}_X$ is the enveloping sheaf of algebras of the standard tangent bundle Lie algebroid on X.

[11] Following GR, we might wish to impose some finite presentation assumptions throughout.

Proposition 7.3. *Let X be a proper separated smooth scheme. The !-pullback along the canonical map $i : X \to X_{\mathrm{crys}}$ is monadic and exhibits an equivalence of ∞-categories*

$$\mathrm{Crys}^{\mathrm{R}}(X) \simeq \mathrm{Mod}_{\mathscr{D}_X^{\mathrm{pd}}}(\mathrm{QCoh}(X))$$

between right crystals and crystalline D-modules on X.

Proof. First we must establish that the "forgetful functor" $i^! : \mathrm{IndCoh}(X_{\mathrm{crys}}) \to \mathrm{IndCoh}(X)$ is monadic. Because X is a smooth separated scheme, we may use the presentation for X_{crys} of Proposition 3.5 and pass to ind-coherent sheaves to obtain an equivalence of ∞-categories

$$\mathrm{IndCoh}(X_{\mathrm{crys}}) \simeq \mathrm{Tot}\Big(\mathrm{IndCoh}\big((X \times \cdots \times X)^{\mathrm{pd}}_X\big)\Big).$$

Using Theorem HA.4.7.5.2, the desired monadicity will follow if we prove that the cosimplicial ∞-category on the right-hand side of the above equivalence satisfies the Beck-Chevalley condition. That is to say, we must show that for every map $[k] \to [l]$ in Δ, the diagram

$$\begin{array}{ccc} \mathrm{IndCoh}\big((X^{k+1})^{\mathrm{pd}}_X\big) & \longrightarrow & \mathrm{IndCoh}\big((X^{k+2})^{\mathrm{pd}}_X\big) \\ \big\downarrow & & \big\downarrow \\ \mathrm{IndCoh}\big((X^{l+1})^{\mathrm{pd}}_X\big) & \longrightarrow & \mathrm{IndCoh}\big((X^{l+2})^{\mathrm{pd}}_X\big), \end{array}$$

obtained by taking !-pullbacks of the relevant projections, is left adjointable.

Using the n-th pd-neighborhoods of the diagonal $D^n_X(k)$ as in Remark 3.10, we have $(X^{k+1})^{\mathrm{pd}}_X \simeq \varinjlim_n D^n_X(k)$, thus it suffices to show left adjointability of the !-pullback diagram

$$\begin{array}{ccc} \mathrm{IndCoh}(D^n_X(k)) & \longrightarrow & \mathrm{IndCoh}(D^n_X(k+1)) \\ \big\downarrow & & \big\downarrow \\ \mathrm{IndCoh}(D^n_X(l)) & \longrightarrow & \mathrm{IndCoh}(D^n_X(l+1)) \end{array}$$

for all $n \geq 0$. We need to show that the !-pullback of the map $D^n_X(k+1) \to D^n_X(k)$, coming from the projection $X^{k+1} \to X^k$, admits a left adjoint, and that said left adjoint satisfies the relevant projection formula. Since X is a proper scheme, the projection $X^{k+1} \to X$ is also proper, thus everthing follows from the standard pushforward-functoriality of ind-coherent sheaves (as developed e.g. in GR). This concludes the proof of monadicity for $i^!$.

Now we know that $i^!$ exhibits an equivalence between the ∞-category of right crystals on X and the ∞-category of modules in $\mathrm{IndCoh}(X)$ over the monad $i^! i_*$, induced by the adjunction between i_* and $i^!$. Using base-change for the pullback square[12]

$$\begin{array}{ccc} (X \times X)^{\mathrm{pd}}_X & \xrightarrow{p_1} & X \\ {\scriptstyle p_2}\big\downarrow & & \big\downarrow{\scriptstyle i} \\ X & \xrightarrow{i} & X_{\mathrm{crys}} \end{array}$$

gives an identification of the monad $i^! i_* \simeq (p_2)_* p_1^!$. Because X is a smooth scheme, we may employ the canonical equivalence $\mathrm{IndCoh}(X) \simeq \mathrm{QCoh}(X)$. Using the compatibility of the !-pullback with the $\mathrm{QCoh}(X)$-action on $\mathrm{IndCoh}(X)$, and the usual projection formula for quasi-coherent sheaves, we find that

$$(p_2)_* p_1^!(\mathscr{F}) \simeq (p_2)_*(p_1^!(\mathscr{O}_X) \otimes p_1^*(\mathscr{F})) \simeq (p_2)_* p_1^!(\mathscr{O}_X) \otimes_{\mathscr{O}_X} \mathscr{F}$$

[12]That we are allowed to do this is another consequence of having verified the Beck-Chevalley requirement for descent of Theorem HA.4.7.5.2.

for any quasi-coherent sheaf $\mathscr{F}$ on X. Thus it suffices to identify $(p_2)_* p_1^!(\mathscr{O}_X)$ with the crystalline differential operators D_X. That follows from the natural homotopy equivalences

$$\begin{aligned}
\mathrm{Map}_{\mathrm{QCoh}(X)}(\mathscr{F}, (p_2)_* p_1^!(\mathscr{O}_X)) &\simeq \mathrm{Map}_{\mathrm{QCoh}(X)}\big((p_1)_* p_2^*(\mathscr{F}), \mathscr{O}_X\big) \\
&\simeq \mathrm{Map}_{\mathrm{QCoh}(X)}\big((p_1)_* p_2^*(\mathscr{O}_X) \otimes_{\mathscr{O}_X} \mathscr{F}, \mathscr{O}_X\big) \\
&\simeq \mathrm{Map}_{\mathrm{QCoh}(X)}\big(\mathscr{F}, \underline{\mathrm{Map}}_{\mathscr{O}_X}((p_1)_* p_2^*(\mathscr{O}_X), \mathscr{O}_X)\big) \\
&\simeq \mathrm{Map}_{\mathrm{QCoh}(X)}(\mathscr{F}, \mathscr{D}_X^{\mathrm{pd}})
\end{aligned}$$

for any quasi-coherent sheaf $\mathscr{F}$ on X. □

Remark 7.4. The first part of the above proof, in which we established monadicity, is essentially a spelled-out proof of Proposition 3.3.3, Chapter III.3 in GR. In light of Remark 2.11 and an ind-properness observation, we could have also invoked that result directly. We have chosen to spell it out instead in hopes of clarifying the relationship with the proof of Proposition 7.5.

Using the explicit description of quasi-coherent sheaves on pd-completions of Proposition 2.12, together with Lurie's quasi-coherent Grothendieck duality from Section SAG.6.4, we can find an analogous description of left crystals as crystalline D-modules.

Proposition 7.5. *Let X be a proper separated smooth scheme. The pushforward along the canonical map $i : X \to X_{\mathrm{crys}}$ exhibits an equivalence of ∞-categories*

$$\mathrm{Crys}(X) \simeq \mathrm{Mod}_{\mathscr{D}_X^{\mathrm{pd}}}(\mathrm{QCoh}(X))$$

between left crystals and crystalline D-modules on X.

Proof. We argue similarly to the proof of Proposition 7.3. First we invoke Proposition 3.5 to obtain an equivalence of ∞-categories

$$\mathrm{QCoh}(X_{\mathrm{crys}}) \simeq \mathrm{Tot}\big(\mathrm{QCoh}\big((X \times \cdots \times X)_X^{\mathrm{pd}}\big)\big),$$

with the simplicial structure on the right-hand side coming from pullbacks along the relevant maps. Using the anti-equivalence $\mathcal{P}\mathrm{r}^{\mathrm{L}} \simeq (\mathcal{P}\mathrm{r}^{\mathrm{R}})^{\mathrm{op}}$ of exchanging left and right adjoints, which in our case amounts to passing from pullbacks to pushforwards along the same maps, we get an ∞-categorical geometric realization

$$\mathrm{QCoh}(X_{\mathrm{crys}}) \simeq \Big|\mathrm{QCoh}\big((X \times \cdots \times X)_X^{\mathrm{pd}}\big)\Big|,$$

formula for left crystals. As in the proof of Proposition 7.3, we now strive to apply Theorem HA.4.7.5.2 (more precisely, its opposite-variance analogue) to obtain the desired monadicity conclusion. The relevant Beck-Chevalley property to check now is that the diagram

$$\begin{array}{ccc}
\mathrm{QCoh}\big((X^{k+1})_X^{\mathrm{pd}}\big) & \xrightarrow{(p_X^{\mathrm{pd}})_*} & \mathrm{QCoh}\big((X^{k})_X^{\mathrm{pd}}\big) \\
\Big\downarrow{\scriptstyle ((q')_X^{\mathrm{pd}})_*} & & \Big\downarrow{\scriptstyle (q_X^{\mathrm{pd}})_*} \\
\mathrm{QCoh}\big((X^{l+1})_X^{\mathrm{pd}}\big) & \xrightarrow{((p')_X^{\mathrm{pd}})_*} & \mathrm{QCoh}\big((X^{l})_X^{\mathrm{pd}}\big),
\end{array} \tag{4}$$

obtained by pushforwards along appropriate projection maps p, p' and q, q', is right adjointable. Let us study the horizontal morphism more carefully.

Any of the coordinate projections $p : X^{k+1} \to X^k$ induces by compatibility with diagonals and functoriality of pd-completion a pushforward functor

$$(p_X^{\mathrm{pd}})_* : \mathrm{QCoh}\big((X^{k+1})_X^{\mathrm{pd}}\big) \to \mathrm{QCoh}\big((X^{k})_X^{\mathrm{pd}}\big). \tag{5}$$

To obtain a better understanding of this functor, we will use the explicit description of quasi-coherent sheaves on pd-completions garnered by Proposition 2.12.

Indeed, the quasi-coherent pushforward along p is a lax symmetric-monoidal functor $p_* : \mathrm{QCoh}(X^{k+1}) \to \mathrm{QCoh}(X^k)$. There is a canonical map $\varphi : \mathscr{D}_X(X^k) \to p_*(\mathscr{D}_X(X^{k+1}))$ of sheaves of $\mathscr{O}_{X^k}$-algebras, arising from the undersal property of pd-completion. The pushforward p_* thus extends to a functor between module ∞-categories

$$p_*^{\mathrm{Mod}} : \mathrm{Mod}_{\mathscr{D}_X(X^{k+1})}(\mathrm{QCoh}(X^{k+1})) \to \mathrm{Mod}_{\mathscr{D}_X(X^k)}(\mathrm{QCoh}(X^k)),$$

still given by the same underlying functor p_*, with the added superscript introduced only for the sake of clarity. The map φ of sheaves of algebras is furthermore compatible with pd-structures, implying that $\varphi\big(\overline{\mathscr{I}}(k)^{[n]}\big) \subseteq p_*\big(\overline{\mathscr{I}}(k+1)^{[n]}\big)$ holds for every $n \geq 0$. This shows that p_*^{Mod} carries the full subcategories, corresponding to quasi-coherent sheaves on the respective pd-completions according to Proposition 2.12, to each other, inducing a functor of the same form as (5). Since the identification of Poroposition 2.12 is also given by pushforwards, we find that the obtained functor is the pushforward $(p^{\mathrm{pd}}_X)_*$, as promised. The preceding discussion may be summarized in the commutative diagram

$$\begin{array}{ccc} \mathrm{QCoh}\big((X^{k+1})^{\mathrm{pd}}_X\big) & \xrightarrow{(p^{\mathrm{pd}}_X)_*} & \mathrm{QCoh}\big((X^k)^{\mathrm{pd}}_X\big) \\ \downarrow & & \downarrow \\ \mathrm{Mod}_{\mathscr{D}_X(X^{k+1})}(\mathrm{QCoh}(X^{k+1})) & \xrightarrow{p_*^{\mathrm{Mod}}} & \mathrm{Mod}_{\mathscr{D}_X(X^k)}(\mathrm{QCoh}(X^k)), \end{array} \tag{6}$$

in which the unlabled vertical arrows are fully faithful embeddings.

In order to show that the diagram (4) is right adjointable, we must first show that the functor $(p^{\mathrm{pd}}_X)_*$ admits a right adjoint. Since X is a proper smooth scheme, the projection $p : X^{k+1} \to X^k$ is also proper and smooth. By Definition SAG.6.4.0.1, the pushforward functor p_* thus admits a right adjoint $p^!$. Since X, being smooth, is in particular also flat, it has Tor-aplitude 0 in the language of Section SAG.6.1. Therefore Corollary SAG.6.4.2.7 implies that $p^!$ is compatible with the monoidal structure on quasi-coherent sheaves, in the sense that there is an equivalence $p^!(\mathscr{F}) \simeq p^*(\mathscr{F}) \otimes p^!(\mathscr{O}_{X^k})$ for any $\mathscr{F} \in \mathrm{QCoh}(X^k)$. It follows that the right adjoint $p^!$ is compatible with module structures, and as such extends to a right adjoint to the functor p_*^{Mod}. The monoidal compatibility furthermore implies that this adjoints preserves the full subcategories annihilated by high-enough powers of defining pd-ideals $\overline{\mathscr{I}}$, so we may read off from (6) that it gives rise to a right adjoint to $(p^{\mathrm{pd}}_X)_*$. The desired base-change formula, exhibiting right adjointability of the diagram (4), will similarly follow from the base-change formula $(p')^! q_* \simeq (q')_* p^!$ guaranteed by Proposition SAG.6.4.2.1.

In light of Theorem HA.4.7.5.2, we have now verified that the adjunction

$$i_* : \mathrm{QCoh}(X_{\mathrm{crys}}) \rightleftarrows \mathrm{QCoh}(X) : i^!$$

(with the existence of the exceptional pullback $i^!$ is part of the Theorem) is monadic. The same calculation as in the second half of the proof of Proposition 7.3 identifies the monad of this adjunction with the monad of tensoring with the sheaf $\mathscr{D}^{\mathrm{pd}}_X$ of crystalline differential operators on X. □

Corollary 7.6. *Let X be a proper separated smooth scheme. The canonical functor $\mathrm{QCoh}(X_{\mathrm{crys}}) \to \mathrm{IndCoh}(X_{\mathrm{crys}})$ is an equivalence of ∞-categories, exhibiting an equivalence $\mathrm{Crys}(X) \simeq \mathrm{Crys}^{\mathrm{R}}(X)$ between left and right crystals on X.*

Remark 7.7 (Properness assumption)**.** We believe the properness assumption in the statement of all the above results could be dropped, but do not know how to prove it. Our proofs of Propositions 7.3 and 7.5 both use properness in an essential way to verify the Beck-Chevalley propery. The approach taken in Subsection 4.2 of Chapter III.4 in GR (in the characteristic 0 setting) avoids this by showing that the !-pullback along the canonical map $i : X \to X_{\mathrm{dR}}$ admits a left adjoint directly. The key result there is Proposition

3.1.2 of *loc cit*, asserting that i is inf-schematic. The proof of that uses essentially that the formation of the de Rham space $X \mapsto X_{\mathrm{dR}}$ is right adjoint to the reduction functor $X \mapsto X_{\mathrm{red}}$. Since we are not aware of an analogue in the crystalline setting, we are unable to follow the same approach.

8. Crystalline space over a pd-scheme

In the entirety of the above discussion, we have been slightly lax regarding the type of pd-structures we were allowing. Namely, it is convenient to introduce a slightly more general version of the crystalline space, where we restrict the pd-structures we are considering to only those compatible with a pre-chosen pd-scheme.

Definition 8.1. A *derived pd-scheme* is a triple $(S, \mathscr{I}, \gamma)$ of a derived scheme S, a sheaf of ideals $\mathscr{I} \subseteq \pi_0(\mathscr{O}_S)$, and a divided power structure γ on $\mathscr{I}$. When no confusion is likely to arise, we will abuse notation and refer to the derived pd-scheme $(S, \mathscr{I}, \gamma)$ as S.

Remark 8.2 (Affine derived pd-schemes)**.** Let $\mathrm{sCAlg}_{\mathrm{pd}}$ denot the ∞-category of *derived pd-rings*, which means triples (R, I, γ) of a simplicial commutative ring $R \in \mathrm{sCAlg}$, an ideal $I \subseteq \pi_0(R)$, and a pd-structure γ on I. Morphisms are required to preserve both the ideals and the pd-structures thereon. Derived pd-schemes analogously form an ∞-category $\mathrm{DSch}^{\mathrm{pd}}$. Consider the functor $\mathrm{Spd} : \mathrm{sCAlg}_{\mathrm{pd}}^{\mathrm{op}} \to \mathrm{DSch}^{\mathrm{pd}}$ given by $(A, I, \gamma) \mapsto (\operatorname{Spec} A, \widetilde{I}, \widetilde{\gamma})$. Here $\widetilde{I}$ is the quasi-coherent sheaf on $\operatorname{Spec} A$ corresponding to the A-module I under the equivalence $\mathrm{Mod}_A \simeq \mathrm{QCoh}(\operatorname{Spec} A)$, and $\widetilde{\gamma}$ is the essentially unique pd-structure on $\widetilde{I}$ corresponding to γ on I. The functor Spd gives a fully faithful contravariant embedding of derived pd-rings into derived pd-schemes as affine derived pd-schemes. It is furthermore clear that any derived pd-scheme is locally isomorphic to an affine one.

Remark 8.3 (pd-schemes and pd-rings)**.** Note that a derived pd-scheme $(S, \mathscr{I}, \gamma)$ is specified by a pd-immersion $Z \to S$ of an ordinary scheme $Z = \operatorname{Spec}_S(\pi_0(\mathscr{O}_S)/\mathscr{I})$ into S. If we denote by $\mathrm{DSch}^{\mathrm{pd}}$ the evidently defined ∞-category of pd-schemes, then we obtain a pair of "forgetful" functors $p_0, p_1 : \mathrm{DSch}^{\mathrm{pd}} \rightrightarrows \mathrm{DSch}$ given by $p_1(S, \mathscr{I}, \gamma) = S$ and $p_0(S, \mathscr{I}, \gamma) = Z$ respectively. This is compatible through the functor Spd with (a slight upgrade to non-discrete pd-rings) of the functors p_0, p_1 discussed in Remark 1.3. More precisely, there is a commutative diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{sCAlg}_{\mathrm{pd}}^{\mathrm{op}} & \xrightarrow{\mathrm{Spd}} & \mathrm{DSch}^{\mathrm{pd}} \\ \downarrow{\scriptstyle p_i} & & \downarrow{\scriptstyle p_i} \\ \mathrm{sCAlg}^{\mathrm{op}} & \xrightarrow{\mathrm{Spec}} & \mathrm{DSch} \end{array}$$

where $p_0, p_1 : \mathrm{sCAlg}_{\mathrm{pd}} \to \mathrm{sCAlg}$ are given by $p_1(R, I, \gamma) = R$ and $p_0(R, I, \gamma) = \pi_0(R)/I$.

The functor of points of a derived pd-scheme S is naturally a functor of the form $\mathrm{sCAlg}_{\mathrm{pd}} \to \mathcal{S}$, giving a fully faithfyl embedding $\mathrm{DSch}^{\mathrm{pd}} \to \mathrm{Fun}(\mathrm{sCAlg}_{\mathrm{pd}}, \mathcal{S})$. Though we are ultimately interested in the relative crystalline space over a derived pd-scheme, we can phrase everything with no more effort in the context of a functor $\mathrm{sCAlg}_{\mathrm{pd}} \to \mathcal{S}$.

Definition 8.4. The *crystalline space* of a functor $S : \mathrm{sCAlg}_{\mathrm{pd}} \to \mathcal{S}$ is the functor $S_{\mathrm{crys}} : \mathrm{sCAlg} \to \mathcal{S}$ given by

$$S_{\mathrm{crys}}(R) \simeq \varinjlim_{(I,\gamma)} S(R, I, \gamma),$$

where the colimit ranges of the poset of all nilpotent ideals $I \subseteq \pi_0(R)$ and pd-structures γ on I.

Remark 8.5 (Functoriality)**.** The functoriality of the crystalline space, as discussed in Remark 1.3, extends to the context of Definition 8.4. Recall from said Remark that we use $\mathrm{CAlg}^{\heartsuit}_{\mathrm{pd}}$ to denote[13] the category of pd-rings with nilpotent ideals. There is a clear fully faithful embedding $\iota : \mathrm{CAlg}^{\heartsuit}_{\mathrm{pd}} \to \mathrm{sCAlg}_{\mathrm{pd}}$, compatible with the projection functors $p_0, p_1 : \mathrm{CAlg}^{\heartsuit}_{\mathrm{pd}} \to \mathrm{CAlg}^{\heartsuit}$ discussed in Remark 1.3 and Remark 8.3. The construction $S \to S_{\mathrm{crys}}$ may be identified with

$$\mathrm{Fun}\left(\mathrm{sCAlg}_{\mathrm{pd}}, \mathcal{S}\right) \xrightarrow{\iota^*} \mathrm{Fun}\left(\mathrm{CAlg}^{\heartsuit}_{\mathrm{pd}}, \mathcal{S}\right) \xrightarrow{\mathrm{LKan}_{p_1}} \mathrm{Fun}\left(\mathrm{CAlg}^{\heartsuit}, \mathcal{S}\right) \hookrightarrow \mathrm{Fun}\left(\mathrm{sCAlg}, \mathcal{S}\right).$$

This functioriality asserts more explicitly that any natural transformation $f : S \to S'$ in $\mathrm{Fun}\left(\mathrm{sCAlg}_{\mathrm{pd}}, \mathcal{S}\right)$ induces canonically a natural transformation $f_{\mathrm{crys}} : S_{\mathrm{crys}} \to S'_{\mathrm{crys}}$, which is to say a morphism in the ∞-category $\mathrm{Fun}\left(\mathrm{sCAlg}, \mathcal{S}\right)$.

Remark 8.6 (pd-completion as crystalline space)**.** The crystalline space of a functor $X : \mathrm{sCAlg}_{\mathrm{pd}} \to \mathcal{S}$ of Definition 1.1 is recovered by pre-composing X with the functor $p_0 : \mathrm{sCAlg} \to \mathrm{CAlg}$. Similarly, given a natural transformation $X \to Y$ in $\mathrm{Fun}\left(\mathrm{sCAlg}, \mathcal{S}\right)$, we can define a functor $S : \mathrm{sCAlg}_{\mathrm{pd}} \to \mathcal{S}$ by $S = (X \circ p_0) \times_{(Y \circ p_0)} (Y \circ p_1)$. Then $S_{\mathrm{crys}} \simeq Y^{\mathrm{pd}}_X$, recovering Definition 1.4 of pd-completion.

The functor of points of a derived pd-scheme S is a functor $S : \mathrm{sCAlg}_{\mathrm{pd}} \to \mathcal{S}$, while its "underlying" functor $\mathrm{sCAlg} \to \mathcal{S}$ is the functor of points of the underlying scheme $p_1(S)$. Explicitly, we have $(p_1(S))(R) \simeq S(R, 0, 0)$ for any simplicial commutative ring R, where $(R, 0, 0)$ denotes the derived pd-ring of the trivial pd-structure on the zero ideal (an inclusion $\mathrm{sCAlg} \subseteq \mathrm{sCAlg}_{\mathrm{pd}}$ splitting both p_0 and p_1). This makes just as much sense for an arbitrary functor $S : \mathrm{sCAlg}^{\mathrm{pd}} \to \mathcal{S}$, as it does for the functor of points of a pd-scheme, and we use the same notation $p_1(S)$.

Definition 8.7. Consider functors $X : \mathrm{sCAlg} \to \mathcal{S}$ and $X : \mathrm{sCAlg}_{\mathrm{pd}} \to \mathcal{S}$, and a natural transformation $X \to p_1(S)$. The *relative crystalline space of X over S* is given by $(X/S)_{\mathrm{crys}} \simeq X_{\mathrm{crys}} \times_{p_1(S)_{\mathrm{crys}}} S_{\mathrm{crys}}$. More concretely, for any simplicial commutative ring R we have

$$(X/S)_{\mathrm{crys}}(R) \simeq \varinjlim_{(I,\gamma)} X(\pi_0(R)/I) \times_{S(\pi_0(R)/I,0,0)} S(R, I, \gamma),$$

where the colimit ranges of the poset of all nilpotent ideals $I \subseteq \pi_0(R)$ and pd-structures γ on I.

Remark 8.8 (As crystalline space)**.** The relative crystalline space is a special case of Definition 8.4. More precisely, $(X/S)_{\mathrm{crys}}$ is equivalent to the crystalline space of the functor $(X \circ p_0) \times_{p_1(S) \circ p_0} S$, which might or might not seem needlessly complicated.

Remark 8.9 (Informal interpretation)**.** The relative crystalline space over a pd-scheme S has the effect of restricting the pd-ideals considered in the colimit to only those which are appropriately compatible with the pd-scheme S. Informally, the points of $(X/S)_{\mathrm{crys}}$ correspond to commutative diagrams in derived schemes

$$\begin{array}{ccc} \mathrm{Spec}(\pi_0(R)/I) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, R & \longrightarrow & S, \end{array}$$

in which the lower horizontal arrow is a projection under p_1 of a morphism of derived pd-schemes $\mathrm{Spd}\, R \to S$. With this in mind, reasoning like in Section 6 shows that the quasi-coherent sheaves over $(X/S)_{\mathrm{crys}}$ for an ordinary scheme X and a pd-scheme S recovers the theory of crystals as discussed in [3] and [15].

[13]The notation we have chosen is slightly unfortunate, as we are calling $\mathrm{sCAlg}_{\mathrm{pd}}$ the ∞-category of derived pd-rings, be their pd-ideals nilpotent or not, while $\mathrm{CAlg}^{\heartsuit}_{\mathrm{pd}}$ consits only of the pd-rings with nilpotent pd-ideals. Since we never require the non-nilpotent discrete analogue of the category of pd-rings, we have chosen to stick with this notation, as we find $\mathrm{CAlg}^{\heartsuit}_{\mathrm{pd-nil}}$ rather crowded and cumbersome.

For the most part, the base pd-scheme is taken to be affine, which is to say $S = \operatorname{Spd} A$ for a pd-ring (A, I, γ). The discussion of Section 2 works in the relative case just as well as in the absolute one, if we use pd-envelopes compatible with γ, i.e. envelopes of the form $D_{B,\gamma}(J)$ as discussed at length in the literature, e.g. Chapter 3 of [2], or Section SP.55.2. With this in mind, Sections 4 and 5 may be taken over $\operatorname{Spf} \mathbf{Z}_p$ (or $\operatorname{Spd} W(\kappa)$, in the setting of Variants 4.3 and 5.4) with respect to its usual divided power structure on the principal ideal (p). This rules out some exotic and "non-geometric" pd-structures, such as the non-standard ones on the p-adic integers.

Caveat 8.10. It is only in the setting outlined in the previous paragraph that comparisons with classical constructions, such as Proposition 5.5 actually recover the usual classical structures. But as remarked above, this is merely the matter of what kind of pd-envelopes are being used, so technically speaking, results such as Proposition 5.5 are true as stated, if the alleged classical objects (the crystalline period ring $\mathbf{A}_{\mathrm{crys}}$ in the case of Proposition 5.5) are not entirely the same as their usual counterparts. None of the proofs change, since literally the only things that changes are that the colimit in the definition of the crystalline space is only taken over such pd-ideals which are compatible with the standard pd-structure on $(p) \subseteq \mathbf{Z}_p$, and the version of pd-envelopes that must be used is the one compatible with it. Thus we have chosen to eschew these technicalities for the sake of clearer exposition in the previous Sections.

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