MUSINGS ON $\mathbb{E}_n$-CRYSTALS

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This is a collection of thoughts on $\mathbb{E}_n$-crystals, a certain type of higher analogue of D-modules in derived algebraic geometry. They appear (under a pseudonym) in the work of Beraldo, and will be featured prominently in upcoming work of Ben-Zvi and Safronov.

1. Ordinary crystals

In the immortal words of Grothendieck, crystals are characterized by two properties: growth and rigidity. In particular, a crystal extends along arbitrary infinitesimal extensions (growth), and remains invariant under this extension (rigidity).

One way of realizing this is to define the *infinitesimal groupoid* of, say, a derived scheme X, as completion along the diagonal $(X \times X)^\wedge_X \to X \times X$ (extended to a Segal groupoid in the standard way). Thinking of groupoids as generalized equivalence relations, this encodes the equivalence relation of two points being infinitesimally close together. The category of *crystals*[1] *on* X may be defined as

$$\mathrm{Crys}(X) := \mathrm{IndCoh}(X)^{(X\times X)^\wedge_X},$$

which is to say, sheaves equivariant for the infinitesimal groupoid.

Utilizing stacks, this may be phrased in terms of the quotient $X_{\mathrm{dR}} := X/(X \times X)^\wedge_X$ called the *de Rham space of* X. It may be defined directly via functor of points as $X_{\mathrm{dR}}(R) = X(\pi_0(R)_{\mathrm{red}})$. In terms of the de Rham space, crystals on X are

$$\mathrm{Crys}(X) \simeq \mathrm{IndCoh}(X_{\mathrm{dR}}).$$

When working in characteristic zero, there is a third approach. We may define the *sheaf of differential operators on* X to be the enveloping sheaf of algebras $\mathcal{D}_X = U(T_X)$ of the tangent Lie algebroid T_X on X. Then we have

$$\mathrm{Crys}(X) \simeq \mathrm{Mod}_{\mathcal{D}_X}(\mathrm{IndCoh}(X)),$$

identifying crystals with *D-modules.*

2. Higher infinitesimals

One heuristic perspective on derived algebraic geometry is that the homotopy theory built into algebraic geometry allows for additional "higher" directions in which deformation is possible. As summarized succinctly in [SAG, Section 0.1.3]:

Derived Algebraic Geometry = Algebraic Geometry + Deformation Theory.

For instance, while $\mathrm{Spec}\, k[t]/(t^2)$ is a classical square-zero thickening of $\mathrm{Spec}\, k$, we may consider $\mathrm{Spec}\, k[u]$ with $|u| = -1$ as an analogous derived square-zero thickening.

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[1] In this note, the word *crystal* always refers to crystals in ind-coherent (or equivalently, quasi-coherent) sheaves.

From this perspective, the idea of $\mathbb{E}_n$-crystals is quite reasonable: just as ordinary crystals exhibit growth and rigidity along ordinary infinitesimal extensions, one might ask for a notion of crystals that behaves similarly for these new derived infinitesimal extensions.

In order to make this precise, let us review a convenient setting for formal derived algebraic geometry.

3. Formal moduli problems

Fix k to be a field of characteristic 0 throught the rest of this note. Let $\mathrm{CAlg}_k^{\mathrm{cn}}$ denote the ∞-category of *derived k-algebras*, by which we mean either connective $\mathbb{E}_\infty$-algebras over k, or equivalently, coconnective commutative differential graded k-algebras. In regards to definitions and conventions from derived algebraic geometry, we mostly follow [GR].

Let the functor $X : \mathrm{CAlg}_k^{\mathrm{cn}} \to \mathcal{S}$ be locally almost of finite type and admit deformation theory[2]. We think of X as being the functor of points of a scheme or algebraic stack.

Definition 1 ([GR, Section 5.1.3])**.** A *formal moduli problem under* X is a map $X \to Y$ in $\mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{cn}}, \mathcal{S})$ such that

(i) The functor Y is locally almost of finite type and admits deformation theory.
(ii) The induced map $X_{\mathrm{dR}} \to Y_{\mathrm{dR}}$ is an equivalence in $\mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{cn}}, \mathcal{S})$.

The formal moduli problems under X span the full subcategory $\mathrm{FMP}_{X/} \subseteq \mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{cn}}, \mathcal{S})_{X/}$.

Remark 2. Informally, a fomal moduli problem Y under X is an algebro-geometric object, which is in some way (e.g. by filtered colimits) built out of infinitesimal extensions of X.

The following examples will be important for us:

- The identity map $X \to X$ exhibits X as the initial object in $\mathrm{FMP}_{X/}$.
- The canonical map $X \to X_{\mathrm{dR}}$ exhibits X_{dR} as the final object in $\mathrm{FMP}_{X/}$.
- Let $X \to Y$ be a map in $\mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{cn}}, \mathcal{S})$, where Y is locally almost of finite type and admits deformation theory (e.g. a relative Artin stack). Then the formal completion
$$Y_X^\wedge := Y_{\mathrm{dR}} \times_{X_{\mathrm{dR}}} X$$
induces a map $X \to Y_X^\wedge$, which exhibits $Y_X^\wedge$ as a formal moduli problem over X.

Remark 3. The last example gives rise to the adjunction
$$\mathrm{oblv} : \mathrm{FMP}_{X/} \rightleftarrows \mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{cn}}, \mathcal{S})_{X/}^{\mathrm{laft-def}} : (-)_X^\wedge, \tag{1}$$
showing that formal completions are "cofree" formal moduli problems.

4. $\mathbb{E}_n$-Formal groupoids

The ∞-category of *formal groupoids over* X is defined [GR, Section 5.2.2] to be the full subcategory
$$\mathrm{FGrpd}(X) \subseteq \mathrm{Grpd}(\mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{cn}}, \mathcal{S})_{X/}^{\mathrm{laft-def}}) \times_{\mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{cn}}, \mathcal{S})_{X/}^{\mathrm{laft-def}}} \{X\}$$
of groupoid objects $\mathcal{G}^\bullet$, together with equivalences on their 0-simplices $\mathcal{G}^0 \simeq X$, spanned by those such groupoids for which all the composed face maps $\mathcal{G}^\bullet \to X$ are inf-schemetic

[2]Recall from [GR, Definition 7.1.2] that admiting a deformation theory is equivalent to the functor being convergent (nilcomplete, in the language of [SAG]), admitting a pro-cotangent complex, and being infinitesimally cohesive. This is very close to satisfying the criteria of the Derived Artin Representability Theorem [SAG, Theorem 18.3.0.1], essentially only missing the integrability hypothesis. This is sensible, seeing how integrability, in the sense of [SAG, Definition 17.3.4.1], has to do precisely with passing from the formal to the global.

and induce equivalences $\mathcal{G}^\bullet_{\mathrm{dR}} \simeq X_{\mathrm{dR}}$. That is to say, at every simplicial level, $\mathcal{G}^\bullet$ is a formal moduli problem *over* X, in the sense of [GR, Subsection 5.1.1].

According to [GR, Theorem 5.2.3.2], there is a canonical equivalence of ∞-categories

$$\Omega_X : \mathrm{FMP}_{X/} \simeq \mathrm{FGrpd}(X) : \mathrm{B}_X, \tag{2}$$

where the *formal loops* functor is given by $\Omega_X Y \simeq X \times_Y X$, and the *formal delooping* functor $\mathrm{B}_X\mathcal{G} \simeq X/\mathcal{G}$ gives the quotient of X by the formal groupoid $\mathcal{G}$.

Definition 4. Let $\mathrm{FGrpd}_{\mathbb{E}_1}(X) \coloneqq \mathrm{FGrpd}(X)$. For all integers $n \geq 1$, we inductively define $\mathbb{E}_{n+1}$*-formal groupoids over* X to be groupoid objects inside the ∞-category $\mathrm{FGrpd}_{\mathbb{E}_n}(X)$ of $\mathbb{E}_n$-formal groupoids over X, togehter with an identification of their 0-simplices with the trivial $\mathbb{E}_n$-formal groupoid X

Remark 5. The definition of $\mathbb{E}_n$-formal groupoids is motivated by the Dunn Additivity Theorem [HA, Theorem 5.1.2.2], which says roughly that $\mathbb{E}_n \simeq \mathbb{E}_1 \otimes \cdots \otimes \mathbb{E}_1$. That is to say, an $\mathbb{E}_n$-structure is equivalent to n compatible $\mathbb{E}_1$-structures.

Level-wise application of the equivalence (2) proves the following:

Proposition 6. *There is a canonical equivalence of* ∞*-categories*

$$\Omega^n_X : \mathrm{FMP}_{X/} \simeq \mathrm{FGrpd}_{\mathbb{E}_n}(X) : \mathrm{B}^n_X.$$

Remark 7. Proposition 6 may be viewed as a kind of formal algebraic geometry analogue of the Boardman-Vogt-May Recognition Principle, identifying n-connected spaces and $\mathbb{E}_n$-groups in homotopy theory.

Remark 8. There is a qualitative difference between ordinary, i.e. $\mathbb{E}_1$-formal groupoids, and their $\mathbb{E}_n$ counterparts for $n \geq 2$. This is because, through the projection onto the first factor, there always exists a map $\Omega_X Y \simeq X \times_Y X \to X$. Thus while for an $\mathbb{E}_1$-formal groupoid $\mathcal{G}$, the quotient projection $X \to \mathrm{B}_X\mathcal{G}$ exhibits $\mathrm{B}_X\mathcal{G}$ as only a formal moduli problem *over* X, the classifying stack $\mathrm{B}_X\mathcal{G}$ of any $\mathbb{E}_n$-formal groupoid for $n \geq 2$ is also a formal moduli problem *under* X. Note that this procedure is not canonical, as it requires choosing a factor of X to project onto. Alas, it will have profound effects for the special behavior of $n = 1$ as opposed to $n \geq 2$ situations. The special properties in the case $n \geq 2$ in Propositions 30 and 44, and Section 13, may all be traced back to this fact.

5. Lower formal loops

Given any formal moduli problem Y over X, Proposition 6 says that we can extract the $\mathbb{E}_n$-formal groupoid $\Omega^n_X Y$ over X. This situation is familiar from homotopy theory: $\mathbb{E}_n$-structures appear through n-fold loop spaces. To gain a better grasp on them, let us examine how these formal loops look.

Lemma 9. *For any* $Y \in \mathrm{FMP}_{X/}$*, there is an equivalence of underlying formal moduli problems over* X

$$\Omega^n_X Y \simeq (Y^{S^n})^\wedge_X,$$

where Y^{S^n} *denotes the cotensoring with the space* S^n *in the* ∞*-category* $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_k, \mathcal{S})$.

Proof. This follows from all the functors in sight preserving (finite) limits. The formal completion functor $(-)^\wedge_X : \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_k, \mathcal{S})^{\mathrm{laft-def}}_{X/} \to \mathrm{FMP}_{X/}$ is a right adjoint by (1), and as such preserves limits. Thus $\Omega^n_X Y$, which is by definition the cotensoring of Y with S^n in $\mathrm{FMP}_{X/}$, may be obtained by formal completion from the cotensoring of Y with S^n in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_k, \mathcal{S})^{\mathrm{laft-def}}_{X/}$. Since the forgetful functor $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_k, \mathcal{S})^{\mathrm{laft-def}}_{X/} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_k, \mathcal{S})^{\mathrm{laft-def}}$ clearly preseves all limits, it preserves cotensoring. It remains to observe that, as noted in [GR, Remark 1.7.1.3], the subcategory $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_k, \mathcal{S})^{\mathrm{laft-def}} \subseteq$

$\mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{cn}}, \mathcal{S})$ is closed under all finite limits, and so is closed under cotensoring with spaces. $\square$

We make the following definition in analogy with Lemma 9, and in line with the standard definitions $S^0 \simeq \mathrm{pt} \coprod \mathrm{pt}$ and $S^{-1} = \varnothing$.

Definition 10. The 0*-th formal loops* on a formal moduli problem Y over X is

$$\Omega_X^0 Y := (Y^{S^0})_X^\wedge \simeq (Y \times Y)_X^\wedge$$

and its (-1)*-st formal loops* is defined as

$$\Omega_X^{-1} Y = (Y^{\varnothing})_X^\wedge \simeq \mathrm{pt}_X^\wedge \simeq X_{\mathrm{dR}}.$$

With this extended definition of formal loops, we find the infinitesimal groupoid from Section 1 in the guise of

$$(X \times X)_X^\wedge \simeq \Omega_X^0 X \simeq \Omega_X X_{\mathrm{dR}}. \tag{3}$$

The latter of these is especially meaningful - it exhibits the infinitesimal groupoid as an object in $\mathrm{FGrpd}_{\mathbb{E}_1}(X) \simeq \mathrm{FGrpd}(X)$, thus exhibiting its groupoid structure as coming from formal loops.

6. Formal loops on X_{dR}

Since X_{dR} is final among formal moduli problems over X, the equivalence of ∞-categories of Proposition 6 implies that $\Omega_X^n X_{\mathrm{dR}}$ is the final $\mathbb{E}_n$-formal groupoid over X.

Remark 11. In light of (3), the infinitesimal groupoid $(X \times X)_X^\wedge$ is the final formal groupoid over X. That offers another justification for the first definition of crystals in Section 1: they are Ind-coherent sheaves on X with the maximal amount of equivariance that may be imposed by a formal groupoid.

Lemma 12. *For every $n \geq 0$, there are canonical equivalences*

$$\Omega_X^n X_{\mathrm{dR}} \simeq \Omega_X^{n-1} X \simeq (X^{S^{n-1}})_X^\wedge,$$

in the ∞-category $\mathrm{FMP}_{/X}$.

Proof. The desired equivalences are obtained by applying the functor Ω_X^n to the equivalence (3). $\square$

Remark 13. The preceding Lemma identifies (the underlying formal moduli problem of) the final object in $\mathrm{FGrpd}_{\mathbb{E}_n}(X)$ with the formal completion of the "higher derived loop space" $X^{S^{n-1}}$. In [Töe], this is denoted $B_{\mathbb{E}_n} X$ and called the space of $\mathbb{E}_n$-branes in X.

Example 14. The $\mathbb{E}_n$-formal groupoids $\Omega_X^n X_{\mathrm{dR}}$ are familiar objects for small values of n.

- For $n = 0$ we get $\Omega_X^0 X_{\mathrm{dR}} \simeq X_{\mathrm{dR}}$, the de Rham space of X.
- For $n = 1$, we get $\Omega_X^1 X_{\mathrm{dR}} \simeq (X \times X)_X^\wedge$, the infinitesimal groupoid over X.
- For $n = 2$, we get $\Omega_X^2 X_{\mathrm{dR}} \simeq \widehat{\mathscr{L}}X$, the completed derived free loop space on X.

As is usually the case with $\mathbb{E}_n$-objects of any sort, the higher n versions are less familiar.

Proposition 15. *Suppose that the functor $X_{\mathrm{CAlg}_k^\heartsuit}$ takes values in the subcategory $\mathcal{S}_{\leq i} \subseteq \mathcal{S}$ of i-truncated spaces for some integer $i \geq 0$. Then there is a canonical equivalence*

$$\Omega_X^n X_{\mathrm{dR}} \simeq X^{S^{n-1}}$$

of functors over X for all $n \geq i + 2$.

Proof. We must show that the map $X^{S^n} \to (X^{S^n})^\wedge_X$ is an equivalence for all $n \geq i+1$. The truncatedness hypothesis implies that the diagonal map $X \to X^{S^n}$ induces an equivalence on de Rham spaces. Thus we get $(X^{S^n})^\wedge_X \simeq X_{\mathrm{dR}} \times_{(X^{S^n})_{\mathrm{dR}}} X^{S^n} \simeq X^{S^n}$ by the definition of formal completion in terms of de Rham spaces. □

Remark 16. That is to say, when n sufficiently transcends the "degree of derivedness" of X, then the formal $\mathbb{E}_n$-groupoid $\Omega^n_X X_{\mathrm{dR}}$ loses all its formal character, and recovers the higher derived loop space $X^{S^{n-1}}$.

Remark 17. From the sequence of equatorial inclusions of spheres

$$\varnothing = S^{-1} \to S^0 \to S^1 \to S^2 \to \cdots \to S^n \to \cdots \to S^\infty \simeq \mathrm{pt},$$

we obtain by cotensoring with X in $\mathrm{FMP}_{X/}$ the tower

$$\Omega^\infty_X X \to \cdots \to \Omega^n_X X \to \cdots \to \Omega^2_X X \to \Omega^1_X X \to \Omega^0_X X \to \Omega^{-1}_X X.$$

This may be rewritten using Lemma 9 as

$$X \to \cdots \to \Omega^{n+1}_X X_{\mathrm{dR}} \to \cdots \to \Omega^3_X X_{\mathrm{dR}} \to \widehat{\mathscr{L}}X \to (X \times X)^\wedge_X \to X_{\mathrm{dR}}.$$

One interpretation of the formal loop spaces $\Omega^n_X X_{\mathrm{dR}}$ is thus that they start off with $n = 0$ at the terminal object X_{dR} of the ∞-category $\mathrm{FMP}_{X/}$ and, as $n \to \infty$, they approach its initial object X.

7. $\mathbb{E}_n$-CRYSTALS

Recall from Section 1 that usual crystals on X may be defined as Ind-coherent sheaves on X, equivariant for the infinitesimal groupoid $(X \times X)^\wedge_X$. In (3) we recognized the infinitesimal groupoid as $\Omega_X X_{\mathrm{dR}}$, the final object in $\mathrm{FGrpd}(X) \simeq \mathrm{FGrpd}_{\mathbb{E}_1}(X)$. To obtain an $\mathbb{E}_n$-analogue of crystals, it thus seems reasonable to consider equivariance with repsect to the final $\mathbb{E}_n$-formal groupoid $\Omega^n_X X_{\mathrm{dR}}$.

Definition 18. The *∞-category of $\mathbb{E}_n$-crystals on X* is

$$\mathrm{Crys}_{\mathbb{E}_n}(X) := \mathrm{IndCoh}(X)^{\Omega^n_X X_{\mathrm{dR}}}.$$

That is to say, an $\mathbb{E}_n$-crystal on X is an Ind-coherent sheaf on X, which is equivariant with respect to the $\mathbb{E}_n$-formal groupoid $\Omega^n_X X_{\mathrm{dR}}$.

Remark 19. The $\mathbb{E}_n$-structure on $\Omega^n_X X_{\mathrm{dR}}$ equips $\mathrm{Crys}_{\mathbb{E}_n}(X)$ with the structure of an $\mathbb{E}_{n-1}$-monoidal ∞-category.

Definition 18 is an $\mathbb{E}_n$-analogue of defining crystals through the infinitesimal groupoid. As discussed in Section 1, the de Rham space description is obtained by passing to the quotient of X by the infinitesimal groupoid. The following Proposition is the $\mathbb{E}_n$-analogue:

Proposition 20. *For every integer $n \geq 1$, there is a canonical equivalence of ∞-categories*

$$\mathrm{Crys}_{\mathbb{E}_n}(X) \simeq \mathrm{IndCoh}(\Omega^{n-1}_X X_{\mathrm{dR}}) \simeq \mathrm{IndCoh}\big((X^{S^{n-2}})^\wedge_X\big).$$

Proof. Since the inf-schematic hypothesis is automatic in our setting, see [GR, Subsection 5.1.3.1], it follows from [GR, Proposition 5.2.2.6] that

$$\mathrm{IndCoh}(X)^{\Omega^n_X X_{\mathrm{dR}}} \simeq \mathrm{IndCoh}(\mathrm{B}_X \Omega^n_X X_{\mathrm{dR}}),$$

and the conclusion follows since $\mathrm{B}_X \Omega^n_X X_{\mathrm{dR}} \simeq \Omega^{n-1}_X X_{\mathrm{dR}} \simeq (X^{S^{n-2}})^\wedge_X$ by Lemma 12. □

Example 21. For small values of n, and with the help of Example 14, Proposition 20 reduces to fairly familiar statements:

- For $n = 1$, we recover $\mathrm{Crys}(X) \simeq \mathrm{IndCoh}(X_{\mathrm{dR}})$, the usual de Rham description of crystals.

- For $n = 2$, we obtain $\mathrm{Crys}_{\mathbb{E}_2}(X) \simeq \mathrm{IndCoh}\left((X \times X)^{\wedge}_{X}\right)$. That may be rewritten as

$$\mathrm{IndCoh}(X)^{\widehat{\mathscr{L}}X} \simeq \mathrm{IndCoh}_X(X \times X)$$

and identifies the completed-loop-space-equivariant sheaves on X with the sheaves on $X \times X$ supported along the diagonal.
- For $n = 3$, we obtain $\mathrm{Crys}_{\mathbb{E}_3}(X) \simeq \mathrm{IndCoh}(\widehat{\mathscr{L}X})$. Thus $\mathbb{E}_3$-differential operators are sheaves on the (completed) derived free loop space - another reasonably well-studied object.

Remark 22. From the above description of $\mathbb{E}_2$-crystals, we may recognize $\mathrm{Crys}_{\mathbb{E}_2}(X)$ as Beraldo's category $\mathbb{H}(X)$ (at least ignoring the distinction between the usual IndCoh and Berlado's modified IndCoh_0) of what he calls *categorified D-modules* in [Ber20]. In [Ber19], Beraldo also considers $\mathbb{E}_n$-crystals for arbitrary n, using the notation $\mathrm{Sph}(X, n)$ for what we call $\mathrm{Crys}_{\mathbb{E}_{n+2}}(X)$ (again, up to the distinction between IndCoh and IndCoh_0).

Example 23. Repeating two of Beraldo's selling points in this and the next exercise, we first compute the $\mathbb{E}_2$-crystals on the classifying stack $\mathrm{B}G$ of a group scheme G. We obtain

$$\begin{aligned}\mathrm{Crys}_{\mathbb{E}_2}(\mathrm{B}G) &\simeq \mathrm{IndCoh}\left(\mathrm{B}G \times_{\mathrm{B}G_{\mathrm{dR}}} BG\right)\\ &\simeq \mathrm{IndCoh}(G\backslash G_{\mathrm{dR}}/G)\\ &\simeq \mathrm{IndCoh}(G_{\mathrm{dR}}/G)^G\\ &\simeq \mathrm{IndCoh}(\mathrm{B}\widehat{G})^G\\ &\simeq \mathrm{Mod}_{\mathfrak{g}}^G,\end{aligned}$$

the ∞-category of Harish-Chandra bimodules for G. In the above computation we used the short exact sequence of groups schemes $\widehat{G} \to G \to G_{\mathrm{dR}}$ for the fourth equivalence, and the identification between the representations of the formal group $\widehat{G}$ and representations of the Lie algebra $\mathfrak{g}$.

Example 24. A similar computation for $\mathbb{E}_4$-crystals on $\mathrm{B}G$ gives

$$\begin{aligned}\mathrm{Crys}_{\mathbb{E}_4}(\mathrm{B}G) &\simeq \mathrm{IndCoh}\left(\mathrm{B}G \times_{\Omega^2_{\mathrm{B}G}\mathrm{B}G_{\mathrm{dR}}} BG\right)\\ &\simeq \mathrm{IndCoh}\left(\mathrm{B}G \times_{\Omega_{\mathrm{B}G}\mathrm{B}G} BG\right)\\ &\simeq \mathrm{IndCoh}\left(\mathrm{B}G \times_{\widehat{G}/G} BG\right)\\ &\simeq \mathrm{IndCoh}(\mathrm{B}G \times_{\mathfrak{g}/G} \mathrm{B}G),\end{aligned}$$

the Statake category (the renorormalized spherical category of [AG]). This is another ∞-category that shows up in geometric representation theory, appearing on one side of the *Derived Geometeric Satake Theorem* [BF, Theorem 5]. Following more precisely the equivalent statement of this Theorem from [AG, Theorem 12.3.3.], it may be stated as assering the equivalence of ∞-categories

$$\mathrm{Crys}^{\mathrm{ren}}_{\mathbb{E}_1}\left(G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O})\right) \simeq \mathrm{Crys}_{\mathbb{E}_4}(B\check{G}).$$

Here the supercript ren indicates that these are *renormalized* D-modules, i.e. the full subcategory compactly generated by those $\mathbb{E}_1$-crystals whose pullback to the affine Grassmanian along the quotient map $G(\mathcal{K})/G(\mathcal{O}) \to G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O})$ is compact.

8. Descent along neighborhoods of the diagonal

One way of getting hold of ordinary crystals, is to require the existence of compatible lifts of sheaves along i-th infinitesimal neighborhoods of the diagonal for all $i \geq 2$. From the perspective of the infinitesimal groupoid, discussed in Section 1, this comes from considering the descent data along the projection $X \to X_{\mathrm{dR}} \simeq X/(X \times X)^{\wedge}_{X}$ onto the groupoid quotient. Indeed, the Čech nerve of this map is given by

$$\check{\mathrm{C}}^{\bullet}(X \to X_{\mathrm{dR}}) \simeq (\underbrace{X \times \cdots \times X}_{\bullet+1})^{\wedge}_{X}.$$

and we get by descent the identification of crystals as

$$\mathrm{Crys}(X) \simeq \mathrm{Tot}(\mathrm{IndCoh}_X(X \times \cdots \times X)).$$

An entirely analogous proceedure works for $\mathbb{E}_n$-crystals as well. According to Propositon 20, the quotient map in question is $X \to \mathrm{B}_X\Omega^n_X X_{\mathrm{dR}} \simeq (X^{S^{n-2}})^{\wedge}_{X}$. We compute its Čech nerve to be

$$\begin{aligned} \check{\mathrm{C}}^{\bullet}\big(X \to (X^{S^{n-2}})^{\wedge}_{X}\big) &\simeq \check{\mathrm{C}}^{\bullet}\big(X \to X^{S^{n-2}}\big)^{\wedge}_{X} \\ &\simeq \big(\underbrace{X \times_{X^{S^{n-2}}} \cdots \times_{X^{S^{n-2}}} X}_{\bullet+1}\big)^{\wedge}_{X} \\ &\simeq \big(X^{S^{n-1}\vee\cdots\vee S^{n-1}}\big)^{\wedge}_{X}. \end{aligned}$$

Thus $\big(X^{(S^{n-1})^{\vee i}}\big)^{\wedge}_{X}$ is the analogue in the theory of $\mathbb{E}_n$-crystals of the i-th infinitesimal neighborhood of the diagonal. In particular, it gives rise to the description of $\mathbb{E}_n$-crystals as

$$\mathrm{Crys}_{\mathbb{E}_n}(X) \simeq \mathrm{Tot}\big(\mathrm{IndCoh}_X\big(X^{S^{n-1}\vee\cdots\vee S^{n-1}}\big)\big),$$

which follows from how the ∞-category of Ind-coherent sheaves equivariant for a formal groupoid is defined in [GR, Subsection 5.2.2.5]. When $n = 1$, we get $S^0 \simeq \mathrm{pt} \coprod \mathrm{pt}$, and so recover the usual story of infinitesimal neighborhoods of the diagonal.

9. Lie Algebroids

The theory of Lie algebroids in derived algebraic geometry is laid out in [GR, Chapter 8]. There Lie algebroids over X are technically defined to be formal groupoids over X, taking the fundamental equivalence of ∞-categories

$$\mathrm{Lie}_X : \mathrm{FGrpd}(X) \simeq \mathrm{LieAlgd}(X) : \exp_X \tag{4}$$

as a definition, and mearly tweaking the meaning of the free and forgetful functors.

Remark 25. It is also shown in [GR, Section 8.8] that Lie algebroids are equivalent to modules for a monad on the ∞-category $\mathrm{LieAlg}(\mathrm{IndCoh}(X))$ given roughly by

$$\mathfrak{g} \mapsto T_X[-1] \ltimes \mathfrak{g}$$

for a canonically defined action of the Lie algebra $T_X[-1]$ (the Lie algebra of the $\mathbb{E}_1$-group stack ΩX) on $\mathfrak{g}$. While not precisely the same, this feels close enough to the usual definition of ordinary Lie algebroids to make this derived version seem "morally" acceptable.

Recall from [GR, Section 8.2.1] that the forgetful functor $\mathrm{LieAlgd}(X) \to \mathrm{IndCoh}(X)_{X/}$, discarding the Lie algebroid structure, is given in terms of the equivalence of ∞-categories (2) as the functor $\mathrm{FMP}_{/X}$ sending $Y \mapsto T_{X/Y}$. Thus in terms of the Lie algebroids, it sends $\mathcal{G} \mapsto T_{X/\mathrm{B}_X\mathcal{G}} \simeq T_{\mathcal{G}/X}|_X$. Thus the underlying functor into $\mathrm{IndCoh}(X)$ of the functor Lie_X from (4) is $\mathrm{Lie}_X(\mathcal{G}) \simeq T_{\mathcal{G}/X}|_X$.

Example 26. Consider the case $X = \mathrm{pt}$. Then formal groupoids are merely formal groups. For any formal group G, we get $\mathrm{Lie}_{\mathrm{pt}}(G) \simeq T_{G,\mathbf{1}} = \mathfrak{g}$, which is certainly what we expect the Lie algebra of G to be.

Lemma 27. *The equivalence* (4) *induces an equivalence of* ∞*-categories*

$$\mathrm{Lie}_X : \mathrm{FGrpd}_{\mathbb{E}_n}(X) \simeq \mathrm{Mon}_{\mathbb{E}_{n-1}}(\mathrm{LieAlgd}(X))$$

for all integers $n \geq 1$.

Proof. We may view (4) as an $n = 0$ variant of the statement, since the ∞-category of Lie algebroids is pointed at the zero object 0, and thus agrees with its $\mathbb{E}_0$-objects.

In light of Definition 4 of $\mathbb{E}_n$-formal groupoids, it suffices to proceed inductively on n. Thus assume the conclusion of the Proposition is true for some n.

It is then clear that (4) induces an equivalence between $\mathrm{FGrpd}_{\mathbb{E}_{n+1}}(X)$ and the ∞-category of those groupoid objects in $\mathrm{Mon}_{\mathbb{E}_{n-1}}(\mathrm{LieAlgd}(X))$, whose 0-simplices are given by $\mathrm{Lie}_X(X)$. Of course, $\mathrm{Lie}_X(X) \simeq T_{X/X} \simeq 0$, and so such groupoid objects are precisely the group objects. It follows that Lie_X induces the equivalence of ∞-categories.

$$\mathrm{FGrpd}_{\mathbb{E}_{n+1}}(X) \simeq \mathrm{Grp}(\mathrm{Mon}_{\mathbb{E}_{n-1}}(\mathrm{LieAlgd}(X))).$$

We claim that the inclusion $\mathrm{Grp}_{\mathbb{E}_1}(\mathrm{LieAlgd}(X)) \to \mathrm{Mon}_{\mathbb{E}_1}(\mathrm{LieAlgd}(X))$ is an equivalence of ∞-categories. Passing from Lie algebroids to formal groupoids via (4), this follows from the observation [GR, Subsection 5.2.2.1] that any Segal object on X is automatically a groupoid object. Consequently we get

$$\mathrm{FGrpd}_{\mathbb{E}_{n+1}}(X) \simeq \mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Mon}_{\mathbb{E}_{n-1}}(\mathrm{LieAlgd}(X))),$$

and an application of the Dunn Additivity Theorem [HA, Theorem 5.1.2.2] concludes the proof. □

10. Shifted tangent bundles

Let us apply Lemma 27 to the $\mathbb{E}_n$-formal groupoid $\Omega^n_X X_{\mathrm{dR}}$ that we have been studying so far. We compute its Lie algebroid to be

$$\begin{aligned} \mathrm{Lie}_X(\Omega^n_X X_{\mathrm{dR}}) &\simeq T_{X/\mathrm{B}_X \Omega^n_X X_{\mathrm{dR}}} \\ &\simeq T_{X/\Omega^{n-1}_X X_{\mathrm{dR}}} \\ &\simeq T_{X/X_{\mathrm{dR}}}[1-n] \\ &\simeq T_X[1-n], \end{aligned}$$

where we have used for the third equivalence that the functor $T_{X/}$ commutes with limits (which it, in its capactity as a forgetful functor, certainly does), and for the last one the well-known that $T_{X_{\mathrm{dR}}} \simeq 0$. We find that

Proposition 28. *The shifted tangent complex* $T_X[1-n]$ *admits a canonical structure of an* $\mathbb{E}_{n-1}$*-monoid in the* ∞*-category of Lie algebroids over* X*, exhibiting it as a final object of* $\mathrm{Mon}_{\mathbb{E}_{n-1}}(\mathrm{LieAlgd}(X))$.

This may be turned around, to leverage a somewhat more concrete understanding of the $\mathbb{E}_n$-formal groupod $\Omega^n_X X_{\mathrm{dR}}$. In what follows, we may take the *n-shifted tangent bundle on X* to be the mapping stack $T[n]X := \underline{\mathrm{Map}}(\mathrm{Spec}\, k[\varepsilon]/(\varepsilon^2), X)$, where the generator ε is in degree n. The augmentation of derived rings $k \to k[\varepsilon]/(\varepsilon^2) \to k$ exhibits $T[n]X$ as a stack both over and under X, giving the zero section and the bundle projection. Define the *formal n-shifted tangent bundle on X* to be the completion $\widehat{T}[n]X := (T[n]X)^\wedge_X$ along the zero section. The following is a "higher" variant of the Hochschild-Kostant-Rosenberg Theorem, and is quite well-known in a variety of forms, e.g. [Pre, Proposition 2.3.2] and [Töe, Corollary 5.4].

Remark 29. Recall from Remark 8 that for $n \geq 2$, the underlying formal moduli problems of the $\mathbb{E}_n$-groupoids $\Omega^n_X X_{\mathrm{dR}}$ are *pointed over* X. Since the equivalence of ∞-categories $\mathrm{FMP}_{X/} \simeq \mathrm{LieAlgd}(X)$, obtained by combining (2) and (4), restricts by [GR, Theorem 7.3.1.4] to an equivalence $\mathrm{FMP}_{X//X} \simeq \mathrm{LieAlg}(\mathrm{IndCoh}(X))$, it follows that $\mathrm{Lie}_X(\Omega^n_X X_{\mathrm{dR}}) \simeq T_X[1-n]$ is not only a Lie algebroid over X, but furthermore a Lie algebra. That it so say, its anchor map $T_X[1-n] \to T_X$ is the zero map. This difference to the $n = 1$ case, where the Lie algebroid is $\mathrm{id} : T_X \to T_X$, is at the heart of what follows.

Proposition 30 (Formal HKR)**.** *For every $n \geq 1$ there is an equivalence*

$$\exp : \widehat{T}[1-n]X \simeq \Omega^n_X X_{\mathrm{dR}}$$

of the underlying formal moduli problems over and under X.

Proof. Let us recall [GR, Corollary 7.3.2.2], which says that for any group object $\mathcal{G}$ in formal moduli problems over X, there is an equivalence of underlying formal moduli probelms

$$\mathcal{G} \simeq \mathrm{Vect}_X(\mathrm{Lie}_X(\mathcal{G})). \tag{5}$$

Here the vector prestack is defined in [GR, Section 7.1.4]. Applying this to $\mathscr{G} = \Omega^n_X X_{\mathrm{dR}}$, in light of the computation of $\mathrm{Lie}_X(\Omega^n_X X_{\mathrm{dR}})$ at the start of this section, we get an equivalence

$$\Omega^n_X X_{\mathrm{dR}} \simeq \mathrm{Vect}_X(T_X[1-n]).$$

The shifted tangent bundle $T[1-n]X$ being affine imples that the formal shifted tangent bundle $\widehat{T}[1-n]X$ is also inf-affine in the sense of [GR, Section 7.2], and so equals the right-hand side in the equivalence above. ☐

Example 31. For low values of n, Formal HKR recover the familiar statements:

- For $n = 1$, we obtain $\widehat{T}X \simeq (X \times X)^{\wedge}_X$.
- For $n = 2$, we obtain $\widehat{T}[-1]X \simeq \widehat{\mathscr{L}}X$, the (formal version of) the usual Hochschild-Kostant-Rosenberg Theorem.

Remark 32. The proof of Formal HKR was an application of the equivalence (5) from [GR], which is an incarnation of the exponential map. Indeed, let $X = \mathrm{pt}$, and G a group scheme. Then the equivalence (5) equates the formal completion at the unit in G, and the geometric vector space corresponding to its Lie algebra $\mathfrak{g}$. Of course, the map $\mathfrak{g} \to \widehat{G}$ in question is nothing but the exponential map. This justifies using the name exp for the map exhibiting Formal HKR.

Remark 33. An analogous non-formal version of HKR is proved in [Töe, Corollary 5.4] for derived Artin stacks. For X a derived Artin stack, they Töen shows that a choice of a formality equivalence of operads $\mathbb{E}_n \simeq \mathrm{H}^*(\mathbb{E}_n) = \mathbb{P}_n$ for any $n \geq 2$ induces an equivalence between higher Hochschild cohomology $\mathrm{HH}^{\mathbb{E}_n}(X) := \Gamma(X; \mathscr{L}^n X)$ and shifted polyvector fields $\mathcal{O}(T^*[n]X)$. In fact, the equivalence is shown to be compatible with the natural (n-shifted) Lie algebra structures on both sides.

Remark 34. The proof of Formal HKR depends on viewing $\Omega^n_X X_{\mathrm{dR}}$ as an object of $\mathrm{FMP}_{X//X}$. While it has a canonical structure of a formal moduli problem *under* X, its moduli structure *over* X must be chosen. There are such choices, corresponding to choosing a base-point for the n-sphere S^n. In this sense, the formal HKR Theorem is *not canonical.* The dependence on a choice of base-point for $\Omega^n_X X_{\mathrm{dR}}$ is related to the choice of a formality isomorphism $\mathbb{E}_n \simeq \mathbb{P}_n$ in the approach from [Töe] discussed in Remark 33

Remark 35. From the perspective of differential topology, Formal HKR Theorem may be viewed as an analogue of the Tubular Neighborhood Theorem. Indeed, consider the diagonal map $X \to \mathscr{L}^n X$. Its derived normal bundle is[3] $T_{X/\mathscr{L}^n X}[1] \simeq T_{X/X\times X}[1-n]$, and since $T_{X/X\times X} \simeq T_X[-1]$, it is actually $T_X[-n]$. In light of Lemma 12, Formal HKR now asserts the equivalence $\widehat{\mathscr{L}^n} X \simeq \widehat{T}_{X/X\times X}[-1]$ between the formal neighborhood of the image of the diagonal map $X \to \mathscr{L}^n X$, and the formal neighborhood of the zero section in its normal bundle.

Remark 36. Let G be an algebraic group. For $X = \mathrm{B}G$ we find the shifted tangent complex to be $T_{\mathrm{B}G}[n] \simeq \mathfrak{g}[-n]/G$, the quotient of the Lie algebra by the (shifted) adjoint action. Applying this to $\mathbb{E}_4$-crystals, we find in light of Proposition 20 that

$$\mathrm{Crys}_{\mathbb{E}_4}(\mathrm{B}G) \simeq \mathrm{IndCoh}(\Omega^2_{\mathrm{B}G}\mathrm{B}G) \simeq \mathrm{IndCoh}(\widehat{\mathfrak{g}}[-2]/G).$$

In fact, Proposition 15 and Remark 16 may be leveraged to remove the formal completion from $\mathfrak{g}$ in the formula. Recall from Example 24 that $\mathbb{E}_4$-crystals on $\mathrm{B}G$ may be identified with the Satake category. Thus the Formal HKR recovers the "Koszul dual description" of the Satake category from [AG, Proposition 12.4.2].

11. $\mathbb{E}_n$-differential operators

An easy modification of the argument leading to Proposition 28 gives:

Lemma 37. *For any Lie algebroid $\mathfrak{L}$ on X, there exists a canonical $\mathbb{E}_n$-monoid structure on the shift $\mathfrak{L}[-n]$.*

This procedure is clearly functorial. We may combine it with the enveloping algebra functor (which is appropriately oplax monoidal to induce maps on $\mathbb{E}_n$-modules) and form the composite

$$\mathrm{LieAlgd}(X) \xrightarrow{[1-n]} \mathrm{Mon}_{\mathbb{E}_{n-1}}(\mathrm{LieAlgd}(X)) \xrightarrow{U} \mathrm{Mon}_{\mathbb{E}_{n-1}}(\mathrm{Alg}(\mathrm{IndCoh}(X)).$$

By once more envoking Dunn Additivity, we may identify this with a functor

$$U_{\mathbb{E}_n} : \mathrm{LieAlgd}(X) \to \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{IndCoh}(X))$$

which we call the *(universal) enveloping $\mathbb{E}_n$-algera* over X.

Remark 38. Specializing to the case $X = \mathrm{pt}$, this agrees with the universal enveloping $\mathbb{E}_n$-algebra functors discussed elswhere in the literature, e.g. [GR, Remark 6.1.4] and [AF]. In particular, it is the left adjoint to "forgetful functor" $\mathrm{Alg}_{\mathbb{E}_n} \to \mathrm{LieAlg}$, given by $A \mapsto A[n-1]$, which discards the $\mathbb{E}_{n-1}$-structure, and converts the remaining $\mathbb{E}_1$-algebra structure into a Lie algebra one (forgetting the mutiplication, and remembering only the commutators).

Definition 39. The sheaf of *$\mathbb{E}_n$-differential operators on X* is the Ind-coherent sheaf of $\mathbb{E}_n$-algebras on X given by $\mathcal{D}_X^{\mathbb{E}_n} := U_{\mathbb{E}_n}(T_X)$.

Example 40. When $n = 1$, we get $\mathcal{D}_X^{\mathbb{E}_1} \simeq \mathcal{D}_X$, and so recover the ordinary sheaf of differential operators on X.

Remark 41. Ordinary differential operators on X may be informally described as generated inside k-linear endomorphisms on $\mathcal{O}_X$ by two types of generators:

- functions f on X, acting on $\mathcal{O}_X$ by multiplication $g \mapsto fg$,
- vector fields ξ on X, acting on $\mathcal{O}_X$ by derivations $g \mapsto \xi(g)$.

[3] Indeed, for any map $X \to Y$ the fiber sequence $T_{X/Y} \to T_X \to T_Y|_X$ shows that the normal bundle, which should be defined as the cofiber of the second map, is just $T_{X/Y}[1]$.

The compatibility between them is encoded in the commutation condition $[\xi, f] = \xi(f)$. The $\mathbb{E}_n$-differential operators admit an analogous informal description: they are spanned inside the *derived* k-linear endomorphisms on $\mathcal{O}_X$ (thus allowing also operations of the form $\mathcal{O}_X \to \mathcal{O}_X[r]$ for any $r \in \mathbf{Z}$) by

- functions f on X, acting on $\mathcal{O}_X$ by multiplication,
- $(1-n)$-shifted vector fields ξ on X, acting on $\mathcal{O}_X$ by degree $1-n$ derivations.

Thus the function generators f are in degree 0, but now unlike ordinary differential operators, the vector field generators ξ are placed into degree $1-n$.

Remark 42. Recall that $\mathcal{D}_X$ may be viewed as a quantization of the cotangent bundle T^*X. The symplectic space on the latter makes it the archetypcal example of a configuration space of classical mechanics. The shifted cotangent bundle $T^*[n-1]X$ is $(n-1)$-*shifted symplectic* in the sense of [PTVV]. As explained there[4], that means that its deformation quantization should come equipped with an $\mathbb{E}_n$-algebra structure. As they satisfy $\mathcal{D}_X^{\mathbb{E}_n} \simeq U(T_X[1-n])$ by the definition of the enveloping $\mathbb{E}_n$-algebra, this suggests that $\mathbb{E}_n$-*differential operators may be viewed as a quantization of the shifted cotangent bundle* $T^*[n-1]X$.

Finally we can give the $\mathbb{E}_n$-analogue of the last of the three alternative descriptions of crytals from Section 1: we show that $\mathbb{E}_n$-crystals correspond to $\mathbb{E}_n$-D-modules.

Proposition 43. *For every integer $n \geq 1$, there is a canonical equivalence of ∞-categories*

$$\mathrm{Crys}_{\mathbb{E}_n}(X) \simeq \mathrm{Mod}_{\mathcal{D}_X^{\mathbb{E}_n}}(\mathrm{IndCoh}(X)).$$

Proof. We know from [GR, Section 8.4.2] that, in line with the (definitional) equivalence (4), there is an equivalence of ∞-categories between the Ind-coherent sheaves on X which are equivariant for a formal algebroid $\mathcal{G}$, and the Ind-coherent sheaves on X which are modules for the corresponding Lie algebroid $\mathrm{Lie}_X(\mathcal{G})$. In symbols, this amounts to the assertion that the forgetful functor $\mathrm{IndCoh}(X)^{\mathcal{G}} \to \mathrm{IndCoh}(X)$ factors through $U(\mathrm{Lie}_X(\mathcal{G}))$-modules, and induces an equivalence of ∞-categories

$$\mathrm{IndCoh}(X)^{\mathcal{G}} \simeq \mathrm{Mod}_{U(\mathrm{Lie}_X(\mathcal{G}))}(\mathrm{IndCoh}(X)).$$

Let us apply this to the $\mathbb{E}_n$-formal groupoid $\mathcal{G} = \Omega^n_X X_{\mathrm{dR}}$. We know that the corresponding Lie algebroid is $\mathrm{Lie}_X(\Omega^n_X X_{\mathrm{dR}}) \simeq T_X[1-n]$ from Section 10. By the definition of the enveloping $\mathbb{E}_n$-algebra, we get $U(T_X[1-n]) \simeq U_{\mathbb{E}_n}(T_X)$, which is by definition the sheaf $\mathcal{D}_X^{\mathbb{E}_n}$ of $\mathbb{E}_n$-differential operators on X. $\square$

The informal description of $\mathbb{E}_n$-differential operators from Remark 41 may be combined with the Poincare-Birkhoff-Witt Theorem, to show the following rather surprising result.

Proposition 44. *For every integer $n \geq 2$, there is a canonical equivalence*

$$\mathcal{D}_X^{\mathbb{E}_n} \simeq \mathrm{Sym}^*(T_X[1-n])$$

in the ∞-category $\mathrm{Alg}(\mathrm{IndCoh}(X))$.

Proof. By Definition 39, $\mathbb{E}_n$-differential operators are the universal enveloping $\mathbb{E}_n$-algebra

$$\mathcal{D}_X^{\mathbb{E}_n} \simeq U_{\mathbb{E}_n}(T_X) \simeq U(T_X[1-n]).$$

Note that here the shift $[1-n]$ plays the role of the loops operator Ω^{Lie} in the ∞-category of Lie algebroids on X, e.g. [GR, Remark 6.6.1.4]. In fact, since $n \geq 2$, this is taking

[4] The phrasing in [CPTVV] is in terms of shifted Poisson structures. Alas, as observed in [CPTVV, Remark 3.4.2], as consequence of well-known formality results in characteristic zero, there exist equivalences of operads $\mathbb{P}_n \simeq \mathbb{E}_n$ for all $n \geq 2$.

place in the ∞-category of Lie algebras $\mathrm{LieAlg}(\mathrm{IndCoh}(X))$. Indeed, by [GR, Proposition 6.1.7.3], the Lie algebra loop operator satisfies $\Omega^{\mathrm{Lie}}(\mathfrak{g}) \simeq (\mathfrak{g}[-1])^{\mathrm{triv}}$, showing that $T_X[1-n]$ is equipped with the trivial Lie algebra structure[5]. Now we may apply the PBW Theorem in the form of [GR, Theorem 6.5.2.4], which says that $U(\mathfrak{g}^{\mathrm{triv}}) \simeq \mathrm{Sym}^*(\mathfrak{g})$, and obtain the desired equivalence. □

Remark 45. Proposition 44 may be interpreted as saying that it is not the underlying associative, i.e. $\mathbb{E}_1$-algebra on $\mathbb{E}_n$-differential operators $\mathcal{D}_X^{\mathbb{E}_n}$, but rather its full $\mathbb{E}_n$-algebra structure, that holds the interesting data. Similarly, the truly interesting part of the ∞-category of $\mathbb{E}_n$-crystals is not the ∞-category itself, but rather its $\mathbb{E}_{n-1}$-monoidal structure.

12. The center of categorified rings of differential operators

In this section we summarize a result of Beraldo. This requires a slightly modified version of the ∞-category of $\mathbb{E}_n$-crystals.

Definition 46. The *∞-category of derived $\mathbb{E}_n$-crystals on X* is

$$\mathrm{Crys}^{\mathrm{der}}_{\mathbb{E}_n}(X) \coloneqq \mathrm{Crys}_{\mathbb{E}_n}(X) \times_{\mathrm{IndCoh}(X)} \mathrm{QCoh}(X).$$

This ∞-category appears in Beraldo's work, under the name $\mathrm{Sph}(X, n-2)$ in [Ber19], and as $\mathfrak{D}^{\mathrm{der}}(X)$ for $n = 1$ in [Ber20]. In particular, using Beraldo's technology, the right-hand-side in the above definition may be expressed as IndCoh_0.

Remark 47. We follow [Ber20] in nomenclature, since we find it quite evocative: the key difference between $\mathrm{Crys}_{\mathbb{E}_n}(X)$ and $\mathrm{Crys}^{\mathrm{der}}_{\mathbb{E}_n}(X)$ is that the former only depends on the underlying classical stack of X (and even further only its reduction), while the latter genuinely depends on the derived structure of X. This is because $\mathrm{QCoh}(X)$ is susceptible to the derived structure, while X_{dR}, and hence its ∞-categories of sheaves, are not.

For the purpose of stating Beraldo's result, recall e.g. from [BZNF, Section 5.3] or [HA, Section 5.3.1], that for any $n \geq i \geq 1$, the *$\mathbb{E}_i$-center of an $\mathbb{E}_n$-category* $\mathcal{C}$ is defined to be $\mathcal{Z}_{\mathbb{E}_i}(\mathcal{C}) = \mathrm{End}_{\mathrm{Mod}^{\mathbb{E}_i}_{\mathcal{C}}}(\mathcal{C})$, the endomorphism ∞-category of $\mathcal{C}$ among $\mathbb{E}_i$-modules over itself. The $\mathbb{E}_i$-center also goes by the name *$\mathbb{E}_i$-Hochschild cochains*, e.g. in [Ber19].

Theorem 48 (Beraldo, [Ber19])**.** *Let X be a perfect*[6] *stack. For any integers $n \geq 1$ and $1 \leq i \leq n-1$, there is a canonical equivalence of ∞-categories*

$$\mathcal{Z}_{\mathbb{E}_i}\left(\mathrm{Crys}^{\mathrm{der}}_{\mathbb{E}_n}(X)\right) \simeq \mathrm{Crys}^{\mathrm{der}}_{\mathbb{E}_{n-i}}\left(X^{S^i}\right).$$

Remark 49. The above Theorem of Beraldo may be viewed as an anlogue of [BZNF, Corollary 5.12], which identifies the $\mathbb{E}_i$-center of quasi-coherent sheaves on a perfect stack X as

$$\mathcal{Z}_{\mathbb{E}_i}(\mathrm{QCoh}(X)) \simeq \mathrm{QCoh}\left(X^{S^i}\right)$$

for all $i \geq 1$. This is possible because $\mathrm{QCoh}(X)$ is a symmetric monoidal ∞-category. On the other hand, $\mathbb{E}_n$-crystals are only an $\mathbb{E}_n$-monoidal ∞-category, and as such admit only $\mathbb{E}_i$-centers for $i \leq n$.

Remark 50. The appearence of derived $\mathbb{E}_n$-crystals in the statements of the above Theorem, instead of the non-derived version (which is to say, Beraldo's IndCoh_0 instead of the usual IndCoh) is due to IndCoh, unlike QCoh, not behaving as nicely with respect to integral kernels, see [BZNP]. Derived $\mathbb{E}_n$-crystals mix $\mathbb{E}_n$-crystals with QCoh, allowing to utilize the results of [BZNF].

[5] This might seem contradictory, but what is going on is that the original Lie algebra structure on T_X is informing the way in which $T_X[1-n]$ is a $\mathbb{E}_n$-algebra in $\mathrm{LieAlg}(\mathrm{IndCoh}(X))$.

[6] As explained in [Ber19, Remark 2.2.1], this is taken here to mean algebraic quasi-compact derived stacks with an affine diagonal and perfect cotangent complex.

Other than accomodating the above computation of their center, derived $\mathbb{E}_n$-crystals are also useful for making precise the relationship between $\mathbb{E}_n$-crystals and $(n-1)$-shifted cotangent bundles.

Proposition 51. *For every $n \geq 2$ there is a canonical equivalence of ∞-categories*

$$\mathrm{Crys}^{\mathrm{der}}_{\mathbb{E}_n}(X) \simeq \mathrm{QCoh}(T^*[n-1]X).$$

Proof. Using Proposition 43, we may identify $\mathbb{E}_n$-crystals with $\mathbb{E}_n$-D-modules. Then Proposition 44 gives rise to the sequence of equivalences of ∞-categories

$$\begin{aligned} \mathrm{Crys}^{\mathrm{der}}_{\mathbb{E}_n}(X) &\simeq \mathrm{Mod}_{\mathcal{D}^{\mathbb{E}_n}_X}(\mathrm{IndCoh}(X)) \times_{\mathrm{QCoh}(X)} \mathrm{IndCoh}(X) \\ &\simeq \mathrm{Mod}_{\mathrm{Sym}^*(T_X[1-n])}(\mathrm{IndCoh}(X)) \times_{\mathrm{QCoh}(X)} \mathrm{IndCoh}(X) \\ &\simeq \mathrm{Mod}_{\mathrm{Sym}^*_{\mathcal{O}_X}(T_X[1-n])}(\mathrm{QCoh}(X)) \\ &\simeq \mathrm{QCoh}(T^*[n-1]X), \end{aligned}$$

the last of which made use of the fact that the bundle projection $T^*[n-1]X \to X$ is an affine morphism, corresponding to the quasi-coherent sheaf of algebras $\mathrm{Sym}^*_{\mathcal{O}_X}(T_X[1-n])$. □

Remark 52. The preceding Proposition may be viewed as partial justification for the quantization claim of Remark 42. On the other hand, as consequence of Formal HKR and Proposition 12, the ∞-category of $\mathbb{E}_n$-crystals for $n \geq 2$ may also be described as

$$\mathrm{Crys}_{\mathbb{E}_n}(X) \simeq \mathrm{IndCoh}(\Omega^{n-1}_X X_{\mathrm{dR}}) \simeq \mathrm{IndCoh}(\widehat{T}[2-n]X).$$

The reason for this off-by-one difference is that the shifted tangent and cotangent spaces $\widehat{T}[2-n]X$ and $T^*[n-1]X$ are related by Koszul duality.

13. The $\mathbb{E}_n$-Hodge filtration is split for $n \geq 2$

One incarnation of the quantization statement in Remark 42, in the classical $n = 1$ case, is Simpson's stacky interpretation of the Hodge filtration on de Rham cohomology from [Sim]. In light of the well-known correspondence between filtrations and $\mathbf{A}^1$-actions, Simpson constructs an $\mathbf{A}^1$-family of stacks X_{Hodge}, such that

- its generic fiber is the de Rham space, i.e. $(X_{\mathrm{Hodge}})_\lambda \simeq X_{\mathrm{dR}}$ for all $\lambda \in \mathbf{A}^1 - \{0\}$,
- its special fiber is the 1-shifted tangent space[7], i.e. $(X_{\mathrm{Hodge}})_0 \simeq \widehat{T}[1]X$.

This maye be summarized through the diagram

$$\begin{tikzcd} \widehat{T}[1]X \arrow[r] \arrow[d] & X_{\mathrm{Hodge}} \arrow[d] & X_{\mathrm{dR}} \times \mathbf{G}_m \arrow[l] \arrow[d] \\ \{0\} \arrow[r] & \mathbf{A}^1 & \mathbf{G}_m, \arrow[l] \end{tikzcd}$$

both of the squares of which are pullbacks, and which exhibits X_{Hodge} as a deformation of $\widehat{T}[1]X$ into X_{dR}. To be able to give an analogous description for higher n, we discuss the canonical source of such deformations.

Construction 53 (Deformation to the normal cone)**.** Given $Y \in \mathrm{FMP}_{X/}$, its *deformation to the normal cone*, as constructed in [GR, Section 9.2.4], is a canonically defined formal moduli problem $Y_{\mathrm{scale}} \in \mathrm{FMP}_{X//Y\times\mathbf{A}^1}$. We view it as an $\mathbf{A}^1$-family of formal moduli problems under X, and it satisfies

$$Y_{\mathrm{scale}} \times_{\mathbf{A}^1} \mathbf{G}_m \simeq Y \times \mathbf{G}_m, \qquad Y_{\mathrm{scale}} \times_{\mathbf{A}^1} \{0\} \simeq \widehat{T}[1](X/Y).$$

[7] In analogy with the classical Hodge filtration, this is called the *Dolbeault space of* X in [Sim].

That is to say, Y_{scale} has Y as its generic fiber, while its special fiber is the completed relative 1-shifted tangent space $\widehat{T}[1](X/Y) \simeq \mathrm{Vect}_X(T_{X/Y}[1])$, which may be viewed as a the derived normal bundle of X in Y.

Remark 54. We given an informal sketch of how Y_{scale} is constructed, that we learned from [EY]. We use the equivalence $\mathrm{FMP}_{X/} \simeq \mathrm{LieAlgd}(X)$ of (2), and describe Lie algebroids in terms of the forgetful functor $\mathrm{LieAlgd}(X) \to \mathrm{IndCoh}(X)_{/T_X}$. Thus the formal moduli problem Y over X corresponds to the Lie algebroid $T_{X/Y} \xrightarrow{f} T_X$. By scaling, which is to say, by considering $T_{X/Y} \xrightarrow{\lambda f} T_X$ for $\lambda \in \mathbf{A}^1$, we obtain an $\mathbf{A}^1$-family of Lie algebroids on X. Passing back through the equivalences of ∞-categories, this gives rise to the desired $\mathbf{A}^1$-family Y_{scale} in $\mathrm{FMP}_{X/}$. Its fiber $(Y_{\text{scale}})_\lambda$ corresponds to the Lie algebroid

- $T_{X/Y} \xrightarrow{\lambda f} T_X$, when $\lambda \neq 0$. Multiplication by λ^{-1} induces an equivalence of Lie algebroid with $T_{X/Y} \xrightarrow{f} T_X$. Thus the generic fiber of the $\mathbf{A}^1$-family is $(Y_{\text{scale}})_\lambda \simeq Y$.
- $T_{X/Y} \xrightarrow{0} T_X$, when $\lambda = 0$. Under the equivalence of ∞-categories (2), this corresponds to the formal moduli problem $\mathrm{Vect}_X(T_{X/Y}[1]) \simeq \widehat{T}[1](X/Y)$ under X. Thus the special fiber of the $\mathbf{A}^1$-family in question is $(Y_{\text{scale}})_0 \simeq \widehat{T}[1](X/Y)$.

As observed in [GR, Example 9.2.4.3], applying the deformation to the normal cone construction to the de Rham space $(X_{\mathrm{dR}})_{\text{scale}} \simeq X_{\mathrm{Hodge}}$, recovering (a version of) Simpson's Hodge stack. Since the $\mathbb{E}_n$-analogue of X_{dR} is $\mathrm{B}_X\Omega^n_X X_{\mathrm{dR}} \simeq \Omega^{n-1}_X X_{\mathrm{dR}}$, we obtain the appropriate analogue of the Hodge stack as $X^{\mathbb{E}_n}_{\mathrm{Hodge}} := (\Omega^{n-1}_X X_{\mathrm{dR}})_{\text{scale}}$. The computation at the beginning of section 10 shows its special fiber to be

$$(X^{\mathbb{E}_n}_{\mathrm{Hodge}})_0 \simeq \widehat{T}[1](X/\Omega^{n-1}_X X_{\mathrm{dR}}) \simeq \widehat{T}[2-n]X$$

by the computation at the beginning of section 10. When $n = 1$, this recovers the Dolbeault space $\widehat{T}[1]X$. But when $n \geq 2$, the Formal HKR Theorem identifies the right-hand side with $\Omega^{n-1}_X X_{\mathrm{dR}}$ - the generic fiber of the $\mathbf{A}^1$-family in question. This exhibits $X^{\mathbb{E}_n}_{\mathrm{Hodge}}$ as the trivial $\mathbf{A}^1$-family with fiber $\Omega^{n-1}_X X_{\mathrm{dR}}$ for all $n \geq 2$.

Remark 55. Though we exhibited a *trivialization* for the $\mathbf{A}^1$-family $X^{\mathbb{E}_n}_{\mathrm{Hodge}}$, it depended on the choice of a Formal HKR isomorphism, or equivalently, a formality isomorphism of operads $\mathbb{E}_n \simeq \mathbb{P}_n$. Thus while $X^{\mathbb{E}_n}_{\mathrm{Hodge}}$ is a *trivializable* family, it is not canonically *trivial.*

Remark 56. The $\mathbb{E}_n$-variant of the Hodge stack $X^{\mathbb{E}_n}_{\mathrm{Hodge}}$ admits an interpretation from the perspective of the familiar identifications between $\mathbf{A}^1/\mathbf{G}_m$ and filtrations, and $\mathrm{B}\,\mathbf{G}_m$ and gradings. The complex $\Gamma(X; \Omega^{n-1}_X X_{\mathrm{dR}})$, which is by Lemma 12 for all $n \geq 2$ a version of the $(n-2)$-th higher Hochschild homology of X, with a filtration. This is to be viewed as the analogue of the Hodge filtration, in accordance with which Its associated graded is $\Gamma(X; \widehat{T}[1-n]X) \simeq \mathcal{O}(\widehat{T}^*_X[n-1])$. But unlike the $n = 1$ case, the relevant Hodge-to-de Rham spectral sequence always splits for $n \geq 2$, and thus the filtration exhibits an equivalence between the $\mathbb{E}_n$-de Rham cohomology of X and the shifted cotangent complex $\widehat{T}^*_X[n-1]$.

Remark 57. The Hodge filtration on de Rham cohomology corresponds to the canonical degree filtration on differential operators. Viewing the latter as a universal enveloping algebra, this is just the PBW filtration. From this perspective, the splitting of the relevant $\mathbb{E}_n$-filtration is not at all surprising, and in fact encoded in Proposition 44.

14. Loop spaces and connections

Finally we discuss a relationship between $\mathbb{E}_n$-crystals for different n. We will do this as an application of Preygel's Localization Theorem from [Pre]. Upcoming work of Ben-Zvi *et. al.* will give a simpler perspective on this result, deriving it from equivariance properties of the $\mathbb{E}_n$-operads.

In what follows, we require a version of the Tate construction for Ind-coherent sheaves, called the *t-Tate construction* in [Pre].

Construction 58. The setting is: Y is a derived scheme (or derived algebraic space) with an action of a g-dimensional Lie group G. That equips the k-linear ∞-category $\mathrm{Coh}(Y)$ with a G-action, and its categorical G-invariants $\mathrm{Coh}(Y)^G$ are naturally a k^G-linear ∞-category. Here $k^G \simeq C^*(\mathrm{B}G;k)$ are the (derived) G-invariants of the trivial G-module k. Recall the Tate construction: there is a certain norm map $\mathrm{Nm} : \Sigma^g k_G \to k^G$, whose cofiber k^{tG} is a derived k-algebra. The *t-Tate construction* for $\mathrm{IndCoh}(Y)$ is defined as the base-change

$$\mathrm{IndCoh}(Y)^{\tau G} := \mathrm{Ind}(\mathrm{Coh}(Y)^G \otimes_{k^G} k^{tG}).$$

For general Y (derived stacks, or nice-enough functors) with a G-action, we extend the definition of the t-Tate construction by descent.

Example 59. When $G = \mathrm{SU}(2)$ is the circle group, a classical computation, see e.g. [NS, Section 1.4], shows that $k^{\mathrm{SU}(2)} \simeq k[[\beta]]$ for some generator β in degree 2, and $k^{t\mathrm{SU}(2)} \simeq k((\beta))$. Thus the t-Tate construction has the effect of inverting the action of the Bott element β. In particular, for Y equipped with a trivial $\mathrm{SU}(2)$-action, we find that

$$\mathrm{IndCoh}(Y)^{\tau \mathrm{SU}(2)} \simeq \mathrm{IndCoh}(Y) \otimes_k k((\beta)).$$

This works analogously for torus groups $G = \mathrm{SU}(2)^r$ for any integer $r \geq 1$.

Proposition 60. *Fix an equivalence of derived rings $C^*(\mathrm{BSO}(2)^r;k) \simeq k[[\beta_1,\dots,\beta_r]]$ with generators β_i in degree -2. The map $\Omega_X^{n+2r-1}X_{\mathrm{dR}} \to \Omega_X^{n-1}X_{\mathrm{dR}}$ is $\mathrm{SO}(2)^r$-equivariant and induces an equivalence of ∞-categories*

$$\mathrm{Crys}_{\mathbb{E}_{n+2r}}(X)^{\tau \mathrm{SO}(2)^r} \simeq \mathrm{Crys}_{\mathbb{E}_n}(X) \otimes_k k((\beta_1,\dots,\beta_r)).$$

Proof. Due to both sides satisfying descent (by our definition of the t-Tate construction), it suffices to assume that X is a derived algebraic space. Recalling Lemma 12, we are considering the adjunction induced on Ind-coherent sheaves from the map $(X^{S^{n+2r-2}})^\wedge_X \to (X^{S^{n-2}})^\wedge_X$. This map of formal moduli problems is $\mathrm{SO}(2)^r$-equivariant, with the equivariance coming from the $\mathrm{SO}(2)^r$-equivariant space map $S^{n-2} \to S^{n+2r-2}$. This is the equatorial inclusion, and coincides with the inclusion of the $\mathrm{SO}(2)^r$-fixed pointes into the $(n+2r-2)$-sphere. We are thus in the setting to which Preygel's Localization Theorem [Pre, Theorem 5.3] applies, and it concludes the proof. □

Remark 61. Informally, after Tate-periodization, we may freely convert any number of $\mathbb{E}_2$-degrees in higher crystals into the same number of Bott element actions. As the upcoming work of Ben-Zvi and Safronov will explain, this process may be understood in terms of rotational action of $\mathrm{SU}(2)$ on the little disc operad.

Let us write out what Proposition 60 says in the edge cases:

(a) For $r = 1$, we obtain the equivalence of ∞-categories

$$\mathrm{Crys}_{\mathbb{E}_{n+2}}(X)^{\tau \mathrm{SO}(2)} \simeq \mathrm{Crys}_{\mathbb{E}_n}(X) \otimes_k k((\beta)).$$

(b) For $n = 1$, we obtain the equivalence of ∞-categories

$$\mathrm{Crys}_{\mathbb{E}_{2r+1}}(X)^{\tau \mathrm{SO}(2)^r} \simeq \mathrm{Crys}(X) \otimes_k k((\beta_1,\dots,\beta_r)).$$

(c) For $n = 2$, we obtain the equivalence of ∞-categories

$$\mathrm{Crys}_{\mathbb{E}_{2r+2}}(X)^{\tau \mathrm{SO}(2)^r} \simeq \mathrm{IndCoh}_X(X \times X) \otimes_k k((\beta_1,\dots,\beta_r)).$$

Comparing to (b) and (c), we see a qualitative difference between $\mathbb{E}_r$-crystals for even and for odd values of r. Applying (a) for $n = 1$, we recover the equivalence

$$\mathrm{IndCoh}(\widehat{\mathscr{L}X})^{\tau\mathrm{SO}(2)} \simeq \mathrm{Crys}(X) \otimes_k k((\beta)),$$

the main result of [BZN]. When X is a derived algebraic space, so that $X|_{\mathrm{CAlg}_k^{\heartsuit}}$ takes values in discrete spaces, Proposition 15 shows that we may drop the completion on the left-hand side, obtaining (modulo Tate-periodization) an equivalence between sheaves on the derived free loop space $\mathscr{L}X$, equivariant under the rotation of loops, and D-modules on X.

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