PROJECTIVE EMBEDDING OF PROPER ALGEBRAIC CURVES

ROK GREGORIC

Throughout this note, we are working over an algebraically closed field k. Though we may not always say so explicitly, we are working with a proper algebraic curve X, i.e. a 1-dimensional abstract algebraic variety over k, assumed in addition to be smooth, proper (see Remark 1), connected and irreducible. We use $\mathcal{O}(X)$ to denote the ring of functions on X, sometimes called algebraic or regular functions, and $\mathcal{K}(X)$ to denote the field of rational functions on X.

Remark 1 (On the notion of properness)**.** The precise definition of proneness (as one can find for instance in [Har97, II, Definition before Example 5.6.1, p.100]) is not important for us here; it is a property of algebraic varieties that plays an analogous role to the property of being compact does for manifolds in topology. The reason we can not just use compactness of the underlying Zariski topological space directly, is that the Zariski topology is pathological and compact in the point-set sense much too often; e.g. the affine space $\mathbf{A}^n$ is compact, but not proper. As a guiding example: affine varieties are never proper unless they are a finite collection of points. On the other hand, all projective varieties are proper, including in particular projective space $\mathbf{P}^n$. If we are working over $k = \mathbf{C}$, allowing us to view smooth algebraic variety X as complex manifold X^{an} through the complex-analytic topology, then X is proper if and only if X^{an} is compact.

1. Divisors on an algebraic curve

On a proper algebraic curve X (recall that proper = algebro-geometric speak for "compact"), there are no non-constant regular functions, i.e. $\mathcal{O}(X) = k$. On the other hand, there are very many rational functions, in fact $\dim_k \mathcal{K}(X) = \infty$.

To better control this vastness, we introduce intermediate vector spaces $L(D)$ somewhere between $\mathcal{O}(X)$ and $\mathcal{K}(X)$. Since the difference between the two is that functions in the former are allowed no poles, while in the latter they are allowed arbitrary poles, we achieve this by restricting the allowed behavior at the poles (and, as turns out to be convenient, at zeros as well). For this purpose, it is useful to introduce the kind of object which the collections of poles and zeros of a rational function on X will be an instance of.

Definition 2. A *divisor on a curve* X is a finite formal sum $D = n_1P_1 + \cdots + n_kP_k$ for points $P_i \in X$ and coefficients $n_i \in \mathbf{Z}$.

Example 3. Given a non-zero rational function $f \in \mathcal{K}(X)^\times$ (that is to say, a map $f : X \to \mathbf{P}^1$ which is not constantly equal to either 0 or ∞), we can associate to it a divisor

$$\mathrm{div}(f) = f^{-1}(0) - f^{-1}(\infty)$$

of its zeros minus its poles, both counted with multiplicity.

Let us mention the auxiliary operations with divisors that we will need in our discussion.

- Two divisors can be summed together, i.e. if we write[1] $D = n_1P_1 + \cdots + n_kP_k$ and $D' = n'_1P_1 + \cdots + n'_kP_k$, then their sum is the divisor

$$D + D' = (n_1 + n'_1)P_1 + \cdots + (n_k + n'_k)P_k.$$

Date: October 25, 2021.

University of Texas at Austin.

[1]Here we extend either of the divisors by adding points with coefficient 0 if necessary, so as to ensure that both divisors D and D' are indexed on the same points $P_1, \ldots, P_k \in X$.

- We can similarly compare divisors with a partial order $\geq$. That is, $D \geq D'$ holds for two divisors $D = n_1P_1 + \cdots + n_kP_k$ and $D' = n'_1P_1 + \cdots + n'_kP_k$ if and only if the coefficients satisfy $n_i \geq n'_i$ for all indices $1 \leq i \leq k$.
- Finally, the *degree of a divisor* $D = n_1P_1 + \cdots + n_kP_k$ is defined to be the integer

$$\deg(D) = n_1 + \cdots + n_k.$$

Both the partial ordering on divisors and the degree are compatible with the addition operation, i.e. for instance $\deg(D + D') = \deg(D) + \deg(D')$.

Given a divisor D on X, we associate to it as promised a vector space of rational functions

$$L(D) = \{f \in \mathcal{K}(X)^\times : \operatorname{div}(f) + D \geq 0\} \cup \{0\}.$$

Remark 4. Unpacking, a (non-zero) rational function $f \in \mathcal{K}(X)$ belongs to the linear subspace $L(D) \subseteq \mathcal{K}(X)$ for $D = n_1P_1 + \cdots + n_kP_k$ if and only if the following holds:

- If a point $P \in X$ is not any of the $P_1, \ldots, P_k$, then f is not allowed to have a pole at P, but otherwise no restriction is placed on its behavior at P.
- For the points P_i, the following behavior is demanded of the function f:
  - If $n_i \geq 0$, then f is *allowed to have* a pole of order $\leq n_i$ at the point P_i.
  - If $n_i < 0$, then f is *required to have* a zero of order $\geq -n_i$ at the point P_i.

Examples 5.

- For the divisor $D = 0$, we have $f \in L(0)$ if and only if the rational function f has no poles. In that case, it is an regular algebraic function, i.e. $f \in \mathcal{O}(X)$. Thanks to the assumption that X is proper, this implies that f is constant, and so $L(0) = k$.
- Fix a point $P \in X$ and a positive integer $n > 0$. The vector space $L(-nP)$ consists of regular functions on X with a zero of order $\geq n$ at P. But, since any global function on X is constant, only the zero function satisfies this, and $L(-nP) = 0$.
- With the same notation as in the previous point, $L(nP)$ consists of functions on X which have no poles away from P, but are allowed a pole of order $\leq n$ at P. As such, at least all regular functions will do, and so $L(nP) \supseteq \mathcal{O}(X) = k$.

Since $\deg(nP) = n$, the above examples suggest that the amount of functions in $L(D)$, i.e. the dimension $\ell(D) := \dim_k L(D)$ (which turns out to always be finite), increases with the degree $\deg(D)$. This is indeed the case, in a sense made precise by the following cornerstone result in the theory of algebraic curves.

Theorem 6 (Riemann-Roch)**.** *Let X be a smooth proper algebraic curve of genus g (see Remark 7) over an algebraically closed field k. There exists a divisor K on X with* $\deg(K) = 2g - 2$, *such that for any divisor D on X the following equality holds:*

$$\ell(D) - \ell(K - D) = \deg(D) + 1 - g.$$

We will not say anything about the proof of the Riemann-Roch Theorem here, nor about the divisor K appearing in its statement. For a (hopefully friendly) sketch of both though, see [Gre20].

Remark 7 (The genus of a curve)**.** One thing we should explain though is the notion of the genus of an algebraic curve. It may be defined as $g = \dim_k \Omega^1(X)$, i.e. the dimension of the space of algebraic differential 1-forms on the curve X. If you prefer sheaf cohomology, this is equivalent (via a celebrated theorem called Serre duality) to $g = \dim_k \mathrm{H}^1(X; \mathcal{O}_X)$. When $k = \mathbf{C}$ and algebraic curves may be viewed as instances of Riemann surfaces, then the genus recovers its usual topological meaning, i.e. $g = \frac{1}{2}\dim_{\mathbf{C}} \mathrm{H}^1(X; \mathbf{C})$, and encodes the "number of holes" in the surface.

For our purposes, it will suffice to apply the Riemann-Roch Theorem in the case where the "error term" $-\ell(K - D)$ disappears. This is the content of Corollary 9, for the proof of which we require one auxiliary observation.

Lemma 8. *If* $\deg(D) < 0$*, then* $L(D) = 0$*.*

Proof. It turns out that

$$\deg(\operatorname{div}(f)) = \#f^{-1}(0) - \#f^{-1}(\infty) = 0 \tag{1}$$

for any non-zero rational function $f \in \mathcal{K}(X)^\times$. Indeed, if f is constant, this is clear. On the other hand, if f is not constant, then the map $f : X \to \mathbf{P}^1$ is finite of some degree $d \geq 1$. The preimage $f^{-1}(a)$ of any $a \in \mathbf{P}^1$, if counted with multiplicity, is therefore equal to d, including $a = 0$ and $a = \infty$.

Using (1), let us now suppose that $f \in L(D)$ is non-zero. That would imply that $\operatorname{div}(f) + D \geq 0$, from which we obtain, by passage to degrees, that

$$\deg(D) = \deg(D) + \deg(\operatorname{div}(f)) = \deg(D + \operatorname{div}(f)) \geq 0.$$

This is in contradiction with the assumption that $\deg(D) < 0$, showing that no such f can exist. □

Corollary 9. *If* $\deg(D) \geq 2g - 1$*, then*

$$\ell(D) = \deg(D) + 1 - g.$$

Proof. By the Riemann-Roch theorem, we must show that the given constraint on the degree of the divisor D implies that $\ell(K - D) = 0$. But we know that $\deg(K) = 2g - 2$, so

$$\deg(K - D) = 2g - 2 - \deg(D) \leq 2g - 2 - (2g - 1) = -1.$$

By the preceding lemma, this indeed implies that $\ell(K - D) = 0$ as desired. □

2. Immersions into projective space from divisors

Fix a divisor D on the algebraic curve X, and suppose that the functions $f_0, \dots, f_n \in L(D)$ form a basis for the vector space $L(D)$ (so that in particular $n = \ell(D) - 1$). Then we can try to define a map $\varphi_D : X \to \mathbf{P}^n$ in terms of the homogeneous coordinates on projective space by sending a point $P \in X$ to

$$\varphi_D(P) := [f_0(P) : \dots : f_n(P)].$$

In order for this to be well-defined, it is necessary and sufficient that the functions $f_0, \dots, f_n$ have no common zeros. Since they form a basis for $L(D)$, this can be restated as saying that for any point $P \in X$ there must exist some function $f \in L(D)$ such that $f(P) \neq 0$.

Lemma 10. *The condition on a divisor D in X for the map φ_D to be well-defined is equivalent to demanding that the equality*

$$\ell(D - P) = \ell(D) - 1$$

holds for all points $P \in X$.

Proof. If the point $P \in X$ occurs in the divisor D with multiplicity n, i.e. if $D = nP + \sum_{P' \in X - \{P\}} n_{P'} P'$, then any $f \in L(D)$ may be written, in terms of a local coordinate x on X around the point P (i.e. so that $x(P) = 0$), as $f(x) = \sum_{i \geq -n} a_i x^i$ for some coefficients $a_i \in k$. Similarly therefore, any $f \in L(D - P)$ may be written as $f(x) = \sum_{i \in \geq -n+1} a_i x^i$, showing that $f \in L(D)$ belongs to the linear subspace $L(D - P) \subseteq L(D)$ if and only if it satisfies $a_{-n} = 0$ in terms of the series expansion around P discussed above. That is to say, there is a left exact sequence of vector spaces

$$0 \to L(D - P) \to L(D) \xrightarrow{a_{-n}} k,$$

from which we may infer that either $\ell(D - P) = \ell(D)$ or $\ell(D - P) = \ell(D) - 1$. In the former case, since $L(D - P) \subseteq L(D)$ are equal-finite-dimensional vector spaces, it must be that $L(D - P) = L(D)$, which implies that all functions $f \in L(D)$ have a simultaneous zero at $P \in X$. We saw in the preceding discussion that this never occurring for any $P \in X$ is precisely the condition that φ_D is a well-defined map. □

Remark 11. For those who hate choosing a basis when doing linear algebra, here is a basis-free description of the map φ_D. It is given by

$$\varphi_D(P) := \{f \in L(D) : f(P) = 0\} \subseteq L(D),$$

and the well-definedness assumption amounts to saying that this is a hyperplane (a codimension 1 linear subspace) of $L(D)$. The collection of hyperplanes in a vector space V are collected into the *dual projective space* $\check{\mathbf{P}}(V)$, which is abstractly (but not canonically) isomorphic to the usual projectivization $\mathbf{P}(V)$. The map associated to the divisor D therefore maps $\varphi_D : X \to \check{\mathbf{P}}(L(D))$.

Proposition 12. *A divisor D on a curve X defines a closed immersion $\varphi_D : X \to \mathbf{P}^{\ell(D)-1}$ if and only if both of the following conditions are satisfied:*

(i) For any point $P \in X$, we have $\ell(D - P) = \ell(D) - 1$.
(ii) For any pair of points $P, Q \in X$, we have $\ell(D - P - Q) = \ell(D) - 2$.

Proof sketch. The necessity of condition (i) for the map φ_D to be well-defined was already proved in Lemma 10. On the other hand, we claim that condition (ii) for a pair of distinct points $P \neq Q$ corresponds to injectivity of the map φ_D. Indeed, suppose that $\varphi_D(P) = \varphi_D(Q)$ for some distinct points $P, Q \in X$. That means that Q is the simultaneous zero of all the functions in $L(D - P)$. By Lemma 10, this amounts to requiring that $\ell(D - P - Q) = \ell(D - P) - 1$. Combining this with $\ell(D - P) = \ell(D) - 1$, this indeed shows that $\ell(D - P - Q) = \ell(D) - 2$.

Condition (ii) for a non-distinct pair of points $P = Q$ is slightly more subtle and has to do with tangent behavior, and we will not discuss it in more detail here. See [Har97, IV, Proposition 3.1, p.307] for more details. □

If the conditions of Proposition 12 are met, we may use φ_D to identify X with a closed algebraic curve inside the projective space $\mathbf{P}^n$ for $n = \ell(D) - 1$. Geometrically, any choice of a hyperplane $H \subseteq \mathbf{P}^n$ corresponds to a non-zero function $f \in L(D)$ up to scaling with $k^\times$. The divisor $E = \mathrm{div}(f) + D$ (satisfying $E \geq 0$ by the definition of $L(D)$) is the intersection $\varphi_D(X) \cap H$, counted with multiplicity, of the curve $\varphi_D(X)$ and the hyperplane H inside $\mathbf{P}^n$. That is to say, $\varphi_D(X) \subseteq \mathbf{P}^n$ is a projective variety of degree[2] $\deg(E) = \deg(D)$.

Conclusion 13. If a divisor D on an abstract proper algebraic curve X satisfies the conditions (i) and (ii) of Proposition 12, then it exhibits X as a closed curve of degree D inside $\ell(D) - 1$-dimensional projective space.

3. The main results

To obtain what we are after, we combine the discussion of projective immersions coming from divisors, in particular Proposition 12, with the Riemann-Roch yoga, specifically Corollary 9.

Suppose that all three divisors $D, D - P$ and $D - P - Q$ fall under the paradigm of Corollary 9. That is to say if $\deg(D - P - Q) \geq 2g - 1$, or equivalently $\deg(D) \geq 2g + 1$. In that case we have by Corollary 9

$$\begin{aligned} \ell(D) &= \deg(D) + 1 - g \\ \ell(D - P) &= \deg(D - P) + 1 - g = \deg(D) - g \\ \ell(D - P - Q) &= \deg(D - P - Q) + 1 - g = \deg(D) - 1 - g \end{aligned}$$

and so indeed $\ell(D - P) = \ell(D) - 1$ and $\ell(D - P - Q) = \ell(D) - 2$. That means that D satisfies the conditions of Proposition 12, and as such defines a closed immersion into the projective space of dimension $\ell(D) - 1 = \deg(D) - g$. Altogether, we have now proved:

[2] Here we once again implicity used the observation that $\deg(\mathrm{div}(f)) = 0$ for any non-zero rational function $f \in \mathcal{K}(X)^\times$ from the proof of Lemma 8.

Theorem 14. *Any divisor D of degree $\deg(D) \geq 2g+1$ on a proper algebraic curve X of genus g defines a closed immersion $\varphi_D : X \to \mathbf{P}^n$ onto a projective algebraic curve of degree $\deg(D)$ inside $n = \deg(D) - g$-dimensional projective space.*

Corollary 15. *Any proper algebraic curve of genus g admits a closed immersion into $(g+1)$-dimensional projective space.*

Examples 16.

- For $g = 0$, and for any choice of point $P_0 \in X$, we therefore see that the divisor P_0 induces a closed immersion $X \to \mathbf{P}^1$ onto a degree one subvariety inside the projective line. But since the image of this immersion is also 1-dimensional, it follows that this immersion is in fact an isomorphism. That is to say, a proper algebraic curve X has genus 0 if and only if it is a *rational curve*, i.e. if $X \cong \mathbf{P}^1$.
- For $g = 1$, i.e. when X is an *elliptic curve*, then any choice of base-point $P_0 \in X$ induces through the divisor $3P_0$ a closed immersion $X \to \mathbf{P}^2$ onto a plane curve of degree 3, i.e. a plane cubic curve.

  It is worthwhile to examine this case in more detail, and see how it leads . Note that $L(0) = \mathcal{O}(X) = k$, and so $\ell(0) = 1$. Next, the divisors nP_0 satisfy $\deg(nP_0) = n$, and so fall into the range of Corollay 9 as soon as $n \geq 2g - 1 = 1$. Hence $\ell(nP_0) = n + 1 - g = n$ holds for all $n \geq 1$.

(n =1) This means that $\ell(P_0) = 1$, and so $L(P_0) = L(0) = \mathcal{O}(X) = k$ - there are no functions on X with only a single simple pole at P_0.

(n =2) We have $\ell(2P_0) = 2$, so other than the constant functions, $L(2P_0)$ contains another 1-dimensional subspace. That is to say, there is (up to scaling unique) function $x : X \to \mathbf{P}^1$ which has no poles away from P_0, and a pole of order 2 at P_0. The functions $1, x$ are a basis for $L(2P_0)$.

(n =3) We have $\ell(3P_0) = 3$, so other than 1 and x, there is another function $y : X \to \mathbf{P}^1$, this time with a pole of degree 3 at P_0. As we saw above, the map $P \mapsto [1(P) : x(P) : y(P)]$ is the closed immersion $X \hookrightarrow \mathbf{P}^2$, intersecting the line at infinity with the image of P_0 at the point $[0 : 0 : 1]$.

(n =4) We have $\ell(4P_0) = 4$. From the functions found so far, we indeed have inside it at least the functions $1, x, y$, and x^2, so that is the basis of the four-dimensional $L(4P_0)$.

(n =5) Similarly $\ell(5P_0) = 5$, and the five-dimensional $L(5P_0)$ has the basis $1, x, y, x^2$, and xy.

(n =6) Finally the surprise comes, since $\ell(6P_0) = 6$, but from the functions we have so far we can cook up $1, x, y, x^2, xy, x^3$, and y^2 - seven functions inside the six-dimensional vector space $L(6P_0)$. This means that these functions must satisfy a linear relation; and because x^3 and y^2 have not appeared so far, their coefficients can be set to be 1 (via rescaling - in particular, they must be non-zero). Thus we obtain the equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

for some coefficients[3] $a_1, a_2, a_3, a_4, a_6 \in k$. Voila, the equation of the plane cubic!

For curves of genus $g \geq 2$, the projective embedding result of Corollary 15 can be improved.

Theorem 17. *Any proper algebraic curve admits a closed immersion into $\mathbf{P}^3$.*

[3] This indexing of the coefficients, standard as it is, used to irk me: where is a_5, for instance? But here is how it works: we know that x is supposed to have an pole of order 2 at P_0, so we assign it degree $|x| = 2$. In the same way $|y| = 3$. Now the coefficients a_i in the equation are chosen so that, if the term $a_ix^jy^k$ appears, then $i + j + k = 6$. That isn't too crazy, I think.

Sketch of proof. By Corollary 15, we already know that a curve X may be embedded into *some* projective space $\mathbf{P}^n$. Thus our goal is to start with an projective curve $X \subseteq \mathbf{P}^n$, and try to produce an embedding of it into $\mathbf{P}^{n-1}$. To do that, we employ the classical algebro-geometric method of *projection away from a point.*

Namely, fix a point $P_0 \in \mathbf{P}^n - X$, and a hyperplane $H \subseteq \mathbf{P}^n$ such that $P_0 \notin H$. Projection away from the point P_0 onto the hyperplane H is now the map

$$\pi_{P_0,H} : X - \{P_0\} \to H,$$

defined as follows: it takes a point $P \in X$, $P \neq P_0$, and considers the unique projective line $\overline{P_0P} \subseteq \mathbf{P}^n$, passing through both points $P, P_0 \in L$. Then the intersection $H \cap \overline{P_0P}$ consists of a single point[4], and this is the point we define to be $\pi_{P_0,H}(P) \in H$.

We consider the restriction $\pi := \pi_{P_0,H}|_X : X \to H \cong \mathbf{P}^{n-1}$. Let us analyze the obstruction to it being injective. If $\pi(P) = \pi(Q)$ for two points $P, Q \in X$, then that means that the lines $\overline{P_0P} \subseteq \mathbf{P}^n$ and $\overline{P_0Q} \subseteq \mathbf{P}^n$ each contain both the point P_0 and $\pi(P) = \pi(Q)$, and must therefore coincide. It follows that this line is also the same as the *secant line* $\overline{PQ} \subseteq \mathbf{P}^n$, uniquely determined by passing through the points P and Q. This is therefore the case if and only if the chosen point P_0 lies on the secant line $\overline{PQ}$.

So the map $\pi : X \to \mathbf{P}^{n-1}$ would be injective, were we only to have chosen the point P_0 to project away from in such a way that it lied on no secant line to any pair of points in X. To see if this might be possible, we consider the *secant variety of* X, i.e. the union of all secant lines $\mathrm{Sec}(X) = \bigcup_{P,Q \in X, P \neq Q} \overline{PQ} \subseteq \mathbf{P}^n$. Writing things in equations makes it clear that $\mathrm{Sec}(X)$ is an algebraic subvariety in $\mathbf{P}^n$, so in order to show that it has a non-empty complement, it suffices to show that $\dim(\mathrm{Sec}(X)) < n$. Indeed, note that a point on $\mathrm{Sec}(X)$ is determined by three pieces of data: two points $P, Q \in X$, and a parameter $\lambda \in \overline{PQ} \cong \mathbf{P}^1$. Thus, at least locally (and generically), the secant variety of X is isomorphic to $X \times X \times \mathbf{P}^1$, and so, since $\dim(X) = 1$, it follows that $\dim(\mathrm{Sec}(X)) = 3$.

Thus so long as $n \geq 4$, we will be able to find a point $P_0 \in \mathbf{P}^n - \mathrm{Sec}(X)$, and hence by projecting away from it an injective map $\pi : X \to \mathbf{P}^{n-1}$. We are basically done, as we can repeat this process until we have dropped down to $n = 3$.

To be precise though, we are not quite done yet. Even if we choose the base-point P_0 such that the projection away from it gives an injective map $\pi : X \to \mathbf{P}^{n-1}$, this map might still fail to be a closed immersion. It turns out that we need also to ensure that the map of tangent bundles $d\pi : TX \to T\mathbf{P}^{n-1}$, which is fiber-wise linear, is non-singular. This translates, similarly to the above (see [Har97, IV, Proposition 3.4, p.309] for a rigorous proof), to the requirement that the chosen point P_0 does not lie on any (projective) tangent line $\mathbf{T}_P X \subseteq \mathbf{P}^n$. We once again consider the collection of all of those, the *tangent variety of* X as the union $\mathrm{Tan}(X) = \bigcup_{P \in X} \mathbf{T}_P X \subseteq \mathbf{P}^n$. To find its dimension, we note that a point in $\mathrm{Tan}(X)$ is uniquely specified by two pieces of data: a point $P \in X$, and a parameter $\lambda \in \mathbf{T}_P X \cong \mathbf{P}^1$. Thus $\dim(\mathrm{Tan}(X)) = 2$, and so the subvariety $\mathrm{Sec}(X) \cup \mathrm{Tan}(X) \subseteq \mathbf{P}^n$ still has a non-empty (in fact: dense) complement so long as $n \geq 4$.

Choosing a point $P_0 \in \mathbf{P}^n - (\mathrm{Sec}(X) \cup \mathrm{Tan}(X))$, we finally obtain a closed immersion $\pi : X \to H \cong \mathbf{P}^{n-1}$ as desired. □

Thus any abstract (smooth proper) algebraic curve is in fact isomorphic to a algebraic curve inside projective space $\mathbf{P}^3$.

Remark 18. If one is willing to compromise on smoothness of the image, any algebraic curve X may even be projected as $f : X \to \mathbf{P}^2$ birationally onto a singular plane algebraic curve $f(X) \subseteq \mathbf{P}^2$ with at most finitely many nodal singularities; see [Har97, IV, Corollary 3.11, p.314].

[4]By Bezout's Theorem, if you wish - here it is crucial that we are working in projective space.

References

[Gre20] R. Gregoric, *Baby's First Riemann-Roch*, informal note, February 4, 2020. https://web.ma.utexas.edu/users/gregoric/Baby's%20Riemann-Roch.pdf

[Har97] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics. 52. Springer-Verlag, 1977.