GROTHENDIECK SPECTRAL SEQUENCE IN HIGHER ALGEBRA

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> *"One Ring to rule them all, One Ring to find them,*
> *One Ring to bring them all and in the darkness bind them."*
>
> — J.R.R. Tolkien, *The Lord of the Rings*

Spectral sequences are in many ways like the rings of power from J.R.R Tolkien's magnum opus *The Lord of the Rings.* They are intricate tools of an immense yet subtle power, enabling their users to achieve great feats, while they inspire fear in the hearts of most other beings[1]. They appeared rather suddenly in the development of mathematics, in many different facets with many different names, as gifts from the devil of abstract algebra to many different fields: algebraic topology, algebraic geometry, complex geometry, and representation theory, to just name a few. In each of those fields they allowed those who would dare use them (perhaps most spectacularly Serre) to achieve astounding results and great leaps forward.

And as did the dark lord Sauron forge in secret another ring to control and bring together all the other rings of power, so did Grothendieck in his landmark Tohôku paper construct a then-new spectral sequence, of which a vast majority of spectral sequences in common use at the time were special cases. This was the Grothendieck spectral sequence.

The fall of Mordor marked the end of the Third Age, and likewise has mathematics come a long way since the days of Grothendieck. The ideas that have been revolutionary then have become standard and default today, and especially in the field of homotopy theory, many ideas which were then barely coming into existence, mainly related to spectra and ∞-categories, now take center stage.

In this brave new world, the mantra that the Grothendieck spectral sequence may be used to derive most of the spectral sequences we commonly encounter, is becoming increasingly false. The spectral sequences of Atiyah-Hirzebruch, Adams, and Adams-Novikov, all utilized on a daily basis by most workers in stable homotopy theory, are three prominent examples of spectral sequences which fail to be special cases of Grothendieck's.

But unlike Sauron's power, which all but completely faded from Middle Earth following his fall, the influence of Grothendieck lives on, as powerful today as ever. In this note we strive to further stoke its fire, by exhibiting a generalization of the Grothendieck spectral sequence, which will encompass as special cases some of the more prominent examples of spectral sequences mentioned above that the classical Grothendieck spectral sequence does not.

1. Setting up the spectral sequence

1.1. Indexing of spectral sequences. Let us quickly explain our conventions regarding spectral sequences.

Suppose we are given a collection of objects $\{E_r^{p,q}\}_{p,q,r}$ of an abelian category $\mathcal{A}$ and a collection morphisms $\{d_r^{p,q}\}_{p,q,r}$ in $\mathcal{A}$. We say that these form a *spectral sequence*, and call $E_r^{p,q}$ the *pages* and $d_r^{p,q}$ the *differentials*, if compostable pairs of differentials compose to

Date: February 20, 2020.

[1]Another distinguishing property of the rings of power is that they slowly work to corrupt their users. We leave it to the reader to form their own conclusions as to whether or not spectral sequences are alike in that respect as well.

zero, ther homology of differentials on the r-th page is equivalent to the $(r+1)$-th page, and they are compatible in one of the following ways:

(a) We have $d_r^{p,q} : E_r^{p,q} \mapsto E_r^{p-r,q+r-1}$, in which case we say that the spectral sequence is *homologically-indexed.*

(b) We have $d_r^{p,q} : E_r^{p,q} \mapsto E_r^{p+r,q-r+1}$, in which case we say that the spectral sequence is *cohomologically-indexed.*

(c) We have $d_r^{p,q} : E_r^{p,q} \mapsto E_r^{p+r,q+r-1}$, in which case we say that the spectral sequence is *Adams-indexed.*

These indexing conventions are clearly equivalent, which is to say that a spectral sequence $\{E_r^{p,q}, d_r^{p,q}\}$ indexed according to one of them may be converted into one indexed by another one, simply by appropriately relabelling the indices p and q.

As tradition dictates, we will typically use s an t in place of p and q when dealing with an Adams-indexed spectral sequence. Unlike common practice however, we will follow HA in not switching the subscripts and superscripts in the notation for a page of a homologically-indexed spectral sequence. That is to say, we will write $E_r^{p,q}$ for what might be more commonly denoted $E^r_{p,q}$. To compensate, we will try to be thorough in always clearly specifying which indexing convention we are employing for each specific spectral sequence we shall enounter.

1.2. **t-structures on stable ∞-categories.** We review those basic notions relating to t-structures on stable ∞-categories from §1.2 and §1.3 of HA which will be used extensively throughout this note.

A *t-structure* on a stable ∞-category $\mathcal{C}$ consists of a pair of full subcategories $\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0} \subseteq \mathcal{C}$ satisfying the following properties:

(i) For $X \in \mathcal{C}_{\geq 0}$ and $Y \in \mathcal{C}_{\leq 0}$, the space $\mathrm{Map}_{\mathcal{C}}(X, Y[-1]) \simeq 0$.

(ii) We have inclusions $\mathcal{C}_{\geq 0}[1] \subseteq \mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq 0}[-1] \subseteq \mathcal{C}_{\leq 0}$.

(iii) For any $X \in \mathcal{C}$ there exists a fiber sequence
$$X' \to X \to X''$$
with $X' \in \mathcal{C}_{\geq 0}$ and $X'' \in \mathcal{C}_{\leq 0}[-1]$.

For any $n \in \mathbf{Z}$ we denote $\mathcal{C}_{\geq n} \simeq \mathcal{C}_{\geq 0}[n]$ and $\mathcal{C}_{\leq n} \simeq \mathcal{C}_{\leq 0}[n]$. Inclusions $\mathcal{C}_{\geq n} \subseteq \mathcal{C}$ and $\mathcal{C}_{\leq n} \subseteq \mathcal{C}$ admit a left and a right adjoint respectively, which we denote $\tau_{\geq n} : \mathcal{C} \to \mathcal{C}_{\geq n}$ and $\tau_{\leq n} : \mathcal{C} \to \mathcal{C}_{\leq n}$. Using the fact that $\mathcal{C}_{\geq n} \simeq \mathcal{C}_{\geq 0}[n]$ and $\mathcal{C}_{\leq n} \simeq \mathcal{C}_{\leq 0}[n]$, we may explicitly express these functors as

$$\tau_{\leq n} X \simeq \tau_{\leq 0}(X[-n])[n], \qquad \tau_{\geq n} X \simeq \tau_{\geq 0}(X[-n])[n]. \tag{1}$$

For every object $X \in \mathcal{C}$ the unit and counit of respective adjunctions form a fiber sequence

$$\tau_{\geq n} X \to X \to \tau_{\leq n-1} X. \tag{2}$$

The subcategory $\mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0} \subseteq \mathcal{C}$ is denoted $\mathcal{C}^{\heartsuit}$ and called the *heart* of the t-structure. It is equivalent to an abelian category. The functor $\tau_{\geq 0}\tau_{\leq 0} : \mathcal{C}^{\heartsuit} \to \mathcal{C}$ is denoted π_0, and more generally we define for every $n \in \mathbf{Z}$ the *n-th homotopy group* of an object $X \in \mathcal{C}$ to be $\pi_n X = \pi_0(X[-n])$. Using (1), we find that

$$\tau_{\geq n}\tau_{\leq n} X \simeq \tau_{\geq n}(\tau_{\leq 0}(X[-n])[n]) \simeq (\tau_{\geq 0}\tau_{\leq 0}(X[-n]))[n] \simeq (\pi_n X)[n], \tag{3}$$

relating the n-th homotopy group and the n-truncations.

We shall make use of several conditions one can impose on a t-structure. We say that a t-structure on a stable ∞-category $\mathcal{C}$ is *left complete* if the canonical functor from $\mathcal{C}$ to the tower

$$\cdots \to \mathcal{C}_{\leq 2} \xrightarrow{\tau_{\leq 1}} \mathcal{C}_{\leq 1} \xrightarrow{\tau_{\leq 0}} \mathcal{C}_{\leq 0} \xrightarrow{\tau_{\leq -1}} \mathcal{C}_{\leq -1} \xrightarrow{\tau_{\leq -2}} \mathcal{C}_{\leq -2} \to \dots$$

exibits it as its limit. That implies that every object $X \in \mathcal{C}$ satisfies $X \simeq \varprojlim \tau_{\leq n} X$, and that every such limit of truncations determines an object in $\mathcal{C}$. Left completeness of a t-structure also implies that it is *left separated*, which means that the subcategory $\bigcap \mathcal{C}_{\geq n} \subseteq \mathcal{C}$

contains only the zero object. Dually a t-structure is *right complete* if the canonical functor from the tower

$$\cdots \to \mathcal{C}_{\geq 2} \xrightarrow{\tau_{\geq 1}} \mathcal{C}_{\geq 1} \xrightarrow{\tau_{\geq 0}} \mathcal{C}_{\geq 0} \xrightarrow{\tau_{\geq -1}} \mathcal{C}_{\geq -1} \xrightarrow{\tau_{\geq -2}} \mathcal{C}_{\geq -2} \to \dots$$

to $\mathcal{C}$ exhibits it as the colimit of the tower, and it implies *right separatedness*, i.e. that the subcategory $\bigcup \mathcal{C}_{\leq n} \subseteq \mathcal{C}$ contains only the zero object.

An object $X \in \mathcal{C}$ is *bounded above* if it belongs to the subcategory $\bigcup \mathcal{C}_{\leq n} \subseteq \mathcal{C}$ and *bounded below*[2] if it belongs to the subcategory $\bigcap \mathcal{C}_{\geq n} \subseteq \mathcal{C}$.

Let $\mathcal{C}$ and $\mathcal{D}$ be two stable ∞-categories equipped with t-structures. A functor $F : \mathcal{C} \to \mathcal{D}$ is *left t-exact* if it it exact and satisfies $F(\mathcal{C}_{\leq 0}) \subseteq \mathcal{D}_{\leq 0}$, and *right t-exact* if it is exact and satisfies $F(\mathcal{C}_{\geq 0}) \subseteq \mathcal{D}_{\geq 0}$. Since exact functors commute with suspension, this implies that $F(\mathcal{C}_{\leq n}) \subseteq \mathcal{D}_{\leq n}$ and $F(\mathcal{C}_{\geq n}) \subseteq \mathcal{D}_{\geq n}$ respectively for all $n \in \mathbf{Z}$. A functor which is both left and right t-exact is said to be *t-exact.*

We shall say that a t-structure on a stable ∞-category $\mathcal{C}$ is *compatible with sequential colimits* if $\mathcal{C}$ admits limits of diagrams of the shape $\mathbf{Z}_{\geq 0}$, and the subcategory $\mathcal{C}_{\leq 0} \subseteq \mathcal{C}$ is closed under such colimits.

1.3. **Filtered objects.** Let $\mathcal{C}$ be an ∞-category. A *filtered object* in $\mathcal{C}$ is defined to be a functor $X : \mathbf{Z} \to \mathcal{C}$, where $\mathbf{Z}$ is viewed as a poset and hence as a category. That is to say, a filtered object in $\mathcal{C}$ is a string of composable morphisms

$$\cdots \to X(-2) \to X(-1) \to X(0) \to X(1) \to X(2) \to \dots$$

in $\mathcal{C}$. We adopt suggestive notations $X(-\infty) \simeq \varprojlim X$ and $X(\infty) \simeq \varinjlim X$, provided of course that these (co)limits exist in $\mathcal{C}$.

Proposition 1. *Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure, and let $X : \mathbf{Z} \to \mathcal{C}$ be a filtered object. Assume that the t-structure on $\mathcal{C}$ is compatible with sequential colimits. Then there exists a conditionally convergent homologically-indexed spectral sequence*

$$E_1^{p,q} = \pi_{p+q} \operatorname{cofib}(X(p-1) \to X(p)) \Rightarrow \pi_{p+q} \operatorname{cofib}(X(-\infty) \to X(\infty))$$

with values in the abelian category $\mathcal{C}^{\heartsuit}$. Under the assumption that $X(n) \simeq 0$ for all $n \ll 0$, this spectral sequence converges unconditionally to $\pi_{p+q}X(\infty)$.

Proof. A spectral sequence with the desired first page is constructed in Construction HA.1.2.2.6 for any filtered object X (or more generally any gapped object). This spectral sequence does not change if we replace the filtered object X with the filtered object $\overline{X}$ defined by $\overline{X}(n) \simeq \operatorname{cofib}(X(-\infty) \to X(n))$ by [GP16, Proposition 2.18]. The latter filtered object satisfies $\overline{X}(-\infty) \simeq 0$, the analogue of the Hausdorff condition, and completeness is automatic in our setup, thus [Boa99, Theorem 9.2] ensures conditional convergence of the spectral sequence to $\pi_{p+q}\overline{X}(\infty)$.

Under the additional assumption that $X(n) \simeq 0$ for all $n \ll 0$, we have $\overline{X} \simeq X$ and unconditional convergence follows from Proposition HA.1.2.2.14. □

1.4. **Grothendieck spectral sequence.** We come to our version of the Grothendieck spectral sequence, or perhaps rather the hypercohomology spectral sequence.

Theorem 2. *Let $F : \mathcal{C} \to \mathcal{D}$ be a left t-exact functor between stable ∞-categories equipped with t-structures Assume that the t-structure on $\mathcal{D}$ is right separated and compatible with sequential colimits. Let $X \in \mathcal{C}$ be an object for which the limit $\varprojlim F(\tau_{\leq n}X)$ exists in $\mathcal{D}$. Then there exists a homologically-indexed conditionally convergent spectral sequence*

$$E_2^{p,q} = \pi_p F(\pi_q X) \Rightarrow \pi_{p+q}(\varprojlim F(\tau_{\leq n}X))$$

[2] This is a rare point of departure from the terminology of HA, where the somewhat less intuitive terms left bounded and right bounded are used for what we are referring to as bounded above and bounded below respectively.

with values in the abelian category $\mathcal{D}^\heartsuit$. If X is bounded above, the limit condition is trivially satisfied, and the spectral sequence converges unconditionally to $\pi_{p+q}F(X)$.

Proof. Consider the filtered object $Y : \mathbf{Z} \to \mathcal{D}$ given by $Y(n) \simeq F(\tau_{\geq -n}X)$. We have

$$\operatorname{cofib}(Y(n-1) \to Y(n)) \simeq F(\operatorname{cofib}(\tau_{\geq -n+1}X \to \tau_{\geq -n}X)) \simeq F(\tau_{\geq -n}\tau_{\leq -n}X) \simeq F(\pi_{-n}X)[-n],$$

where we have used that F, being exact, commutes with cofibers and suspension, and the cofiber sequence (2) and equivalence (1) which hold for any t-structure. The canonical maps $\tau_{\geq -n}X \to X$ determine a map $Y(\infty) \simeq \varinjlim F(\tau_{\geq -n}X) \to F(X)$. By the same sort of computations as we had just made, its cofiber is

$$\operatorname{cofib}(Y(\infty) \to F(X)) \simeq \varinjlim \operatorname{cofib}(F(\tau_{\geq -n}X) \to F(X)) \simeq \varinjlim F(\tau_{\leq -n-1}X).$$

Since F is left t-exact, we have $F(\tau_{\leq -n-1}X) \in \mathcal{D}_{\leq -n-1}$, and so (since the t-structure on $\mathcal{D}$ is compatible with sequential colimits) the cofiber of $Y(\infty) \to F(X)$ belongs to $\bigcap \mathcal{D}_{\leq n}$, which is the zero subcategory by virtue of $\mathcal{D}$ being right separated. The ∞-category $\mathcal{D}$ is stable, thus its cofiber being trivial implies the map $Y(\infty) \to F(X)$ to be an eqivalence. Yet another computation of the same sort shows that

$$\operatorname{cofib}(Y(-\infty) \to Y(\infty)) \simeq \varprojlim \operatorname{cofib}(F(\tau_{\geq -n}X) \to F(X)) \simeq \varprojlim F(\tau_{\leq -n-1}X),$$

where the one thing to note is that the limit commutes with the cofiber because every fiber in a stable ∞-category is equivalent to the fiber of the suspension of the same map.

Now we may envoke Proposition 1 to produce a conditionally convergent homologically-indexed spectral sequence

$$E_1^{p,q} = \pi_{2p+q}F(\pi_{-p}X) \Rightarrow \pi_{p+q}(\varprojlim F(\tau_{\leq n}X))$$

with values in $\mathcal{D}^\heartsuit$. Let us introduce new variables $p' = 2p+q$ and $q' = -p$. Since the differential on the r-th page of the spectral sequence acts on the indices by $(p,q) \mapsto (p-r, q+r-1)$, it acts on the new variables by

$$(p', q') \mapsto (2(p-r) + (q+r-1), -(p-r)) = (p' - (r-1), q' + (r-1) - 1).$$

It follows that setting $\overline{E}_r^{p,q} = E_{r-1}^{-q,p+2q}$ produces a new homologically-indexed spectral sequence, existing from the second page onward. It has the desired second page by construction, and the limiting term is also the one we were after, since $p' + q' = p + q$. ☐

Remark 3. From the explicit formula for the spectral sequence of a filtered object given in Construction HA.1.2.2.6, the pages of the spectral sequence of Theorem 2 may be expressed as

$$E_{r+2}^{p,q} = \operatorname{im}(\pi_{p+q}F(\tau_{\geq q}\tau_{\leq q+r}X) \to \pi_{p+q}F(\tau_{\geq q-r}\tau_{\leq q}X)).$$

At some cost of simplicity, it is possible to dispense with many of the assumptions in the statement of Theorem 2.

Variant 4. *Let $F : \mathcal{C} \to \mathcal{D}$ be an exact functor between stable ∞-categories equipped with t-structures. Assume that the t-structure on $\mathcal{D}$ is compatible with sequential colimits. Let $X \in \mathcal{C}$ be an object for which the limit $\varprojlim F(\tau_{\geq -n}X)$ and the colimit $\varinjlim F(\tau_{\geq -n}X)$ both exists in $\mathcal{D}$. Then there exists a homologically-indexed conditionally convergent spectral sequence*

$$E_2^{p,q} = \pi_p F(\pi_q X) \Rightarrow \pi_{p+q} \operatorname{cofib}(\varprojlim F(\tau_{\geq -n}X) \to \varinjlim F(\tau_{\geq -n}X))$$

with values in the abelian category $\mathcal{D}^\heartsuit$.

In the opposite direction, we may obtain a more elegant form of Theorem 2, provided we are willing to accept even stricter assumptions.

Variant 5. *Let $F : \mathcal{C} \to \mathcal{D}$ be a left t-exact functor between stable ∞-categories equipped with t-structures. Assume that the t-structure on $\mathcal{C}$ is left complete, that the t-structure on $\mathcal{D}$ is right separated and compatible with sequential colimits, and that F preserves sequential limits. For every $X \in \mathcal{C}$ there exists a homologically-indexed conditionally convergent spectral sequence*

$$E_2^{p,q} = \pi_p F(\pi_q X) \Rightarrow \pi_{p+q} F(X)$$

with values in the abelian category $\mathcal{D}^\heartsuit$. If X is bounded above, this spectral sequence converges unconditionally.

Remark 6. The Grothendieck spectral sequence is about reconstructing the whole functor $F : \mathcal{C} \to \mathcal{D}$, which we see on the E_∞-page, with the restriction to the heart $F|_{\mathcal{C}^\heartsuit}$ that we see on the E_2-page. This is especially clear in its incarnation as Variant 5. The existence of the other variants attests to how different assumptions on F, $\mathcal{C}$, and $\mathcal{D}$ effect the "best-case hope" for how much of F we can concievably recover from $F|_{\mathcal{C}^\heartsuit}$.

2. Applications

2.1. **Classical Grothendieck spectral sequence.** Let $\mathcal{A}$ be an abelian category with enough injective objects. Its *left bounded derived ∞-category* $\mathcal{D}^+(\mathcal{A})$, as developed in §1.3.2 of HA, is a stable ∞-category equipped with a right complete t-structure and an equivalence of abelian categories $\mathcal{D}^+(\mathcal{A})^\heartsuit \simeq \mathcal{A}$.

By Theorem HA.1.3.3.2, the left bounded derived ∞-category satisfies the following universal property: for any stable ∞-category $\mathcal{C}$ equipped with a right complete t-structure, there is an equivalence between the ∞-category of left t-exact functors $\mathcal{D}^+(\mathcal{A}) \to \mathcal{C}$ and the category of left exact functors $\mathcal{A} \to \mathcal{C}^\heartsuit$, obtained by sending a left t-exact functor functor $F : \mathcal{D}^+(\mathcal{A}) \to \mathcal{C}$ to the composition

$$\mathcal{A} \simeq \mathcal{D}^+(\mathcal{A})^\heartsuit \subseteq \mathcal{D}^+(\mathcal{A}) \xrightarrow{F} \mathcal{C} \xrightarrow{\pi_0} \mathcal{C}^\heartsuit.$$

The inverse map of the stated equivalence sends a left exact functor of abelian categories $F : \mathcal{A} \to \mathcal{C}^\heartsuit$ to a left t-exact functor $RF : \mathcal{D}^+(\mathcal{A}) \to \mathcal{C}$ which we call the *right derived functor of F*. Its homotopy groups are given in terms of classical right derived functors as $\pi_{-i} RF \simeq R^i F$, both viewed as functors $\mathcal{A} \to \mathcal{C}^\heartsuit$, for all $i \in \mathbf{Z}$.

The construction of right derived functors is not functorial in general. That is to say, given abelian categories $\mathcal{A}$ and $\mathcal{B}$ with enough injective objects each, a stable ∞-category $\mathcal{C}$ equipped with a right complete t-structure, and composable functors $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}^\heartsuit$, the left t-exact functors $RG \circ RF$ and $R(G \circ F)$ need not coincide. That is the case under mild assumptions however, for which it is useful to introduce the terminology that *F maps injective objects to G-acyclic objects* if the functor $(RG) \circ F : \mathcal{A} \to \mathcal{C}$ sends the full subcategory spanned by injective objects $\mathcal{A}_{\mathrm{inj}} \subseteq \mathcal{A}$ to $\mathcal{C}^\heartsuit$.

Lemma 7. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories with enough injective objects. Let $\mathcal{C}$ be a stable ∞-category $\mathcal{C}$ equipped with a right complete t-structure, and let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}^\heartsuit$ be left exact composable functors. Suppose that F maps injective objects to G-acyclic objects. Then there is a canonical equivalence $R(G \circ F) \simeq RG \circ RF$.*

Proof. By combining Proposition HA.1.3.3.12 with the universal property of $\mathcal{D}^+(\mathcal{A})$ recalled above, we see that the map $\mathrm{Fun}(\mathcal{D}^+(\mathcal{A}), \mathcal{C}) \to \mathrm{Fun}(\mathcal{A}_{\mathrm{inj}}, \mathcal{C})$, given by restriction to the subcategory $\mathcal{A}_{\mathrm{inj}} \subseteq \mathcal{A} \simeq \mathcal{D}^+(\mathcal{A})^\heartsuit \subseteq \mathcal{D}^+(\mathcal{A})$, induces an equivalence between left t-exact functors $\mathcal{D}^+(\mathcal{A}) \to \mathcal{C}$ and those functors $\mathcal{A}_{\mathrm{inj}} \to \mathcal{C}^\heartsuit$ which preserve finite products. By this equivalence the functor $R(G \circ F)$ belongs to the restriction of $G \circ F$ to injective objects. Hence it suffices to show that the restriction of $RG \circ RF$ to $\mathcal{A}_{\mathrm{inj}}$ agrees with $G \circ F$.

Fix an injective object $A \in \mathcal{A}_{\mathrm{inj}}$. It follows from the argument we just made above that $RF(A)$ belongs to $\mathcal{D}^+(\mathcal{B})^\heartsuit \simeq \mathcal{B}$, and is as such $RF(A) \simeq \pi_0 RF(A) \simeq F(A)$. But by the assumption that F maps injective objects to G-acyclic objects, we also have that $RG(F(A)) \in \mathcal{C}^{\heartsuit}$ and as such

$$RG(RF(A)) \simeq RG(F(A)) \simeq \pi_0 RG(F(A)) \simeq G(F(A)).$$

This is what we needed to show. □

We will need another variation of the derived ∞-category. Given a Grothendieck abelian category $\mathcal{A}$, there exists by §1.3.5 of HA another stable ∞-category equipped with a t-structure, denoted $\mathcal{D}(\mathcal{A})$ and called the *unbounded derived ∞-category of $\mathcal{A}$*. Its t-structure is both left and right complete, compatible with filtered colimits, and there is still an equivalence $\mathcal{D}(\mathcal{A})^{\heartsuit} \simeq \mathcal{A}$. There is also a fully faithful embedding $\mathcal{D}^+(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$.

Proposition 8 (Grothendieck spectral sequence)**.** *Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ be a composable pair of left exact functors of abelian categories. Suppose that $\mathcal{A}$ and $\mathcal{B}$ have enough injective objects, $\mathcal{C}$ is a Grothendieck abelian category, and F maps injective objects into G-acyclic objects. For any $A \in \mathcal{A}$ there exists an unconditionally convergent cohomologically-indexed spectral sequence*

$$E_2^{p,q} \simeq R^p G \circ R^q F(A) \Rightarrow R^{p+q}(G \circ F)(A)$$

with values in $\mathcal{C}$.

Proof. We wish to apply Theorem 2 to the functor $RG : \mathcal{D}^+(\mathcal{B}) \to \mathcal{D}(\mathcal{C})$ evaluated at $RF(A)$. The universal property of the left bounded derived ∞-category ensures that RG is left t-exact, while the t-structure on $\mathcal{D}(\mathcal{C})$ is right complete and compatible with filtered colimits. Thus the assumptions of Theorem 2 are satisfied and it gives rise to an unconditionally convergent homologically-indexed spectral sequence

$$E_2^{p,q} \simeq \pi_p RG(\pi_q RF(A)) \Rightarrow \pi_{p+q} RG(RF(A))$$

with values in $\mathcal{D}^{\heartsuit}$. Here we have used the fact that, since RF is left t-exact, we have $RF(A) \in \mathcal{D}^+(\mathcal{B})_{\leq 0}$, which is in particular bounded above. Reindexing the spectral sequence to the cohomological indexing convention by $p \mapsto -p$ and $q \mapsto -q$, and using Lemma 7 to identify the limiting term, we obtain the desired spectral sequence. □

Remark 9. The above is not the standard derivation of the Grothendieck spectral sequence. Though both may be traced back to an application of the spectral sequence of a filtered object, the filtered objects in question are different. The classical proof chooses a resolution for A and then procedes to use it to construct the spectral sequence. As such, the first page may depend on the arbitrary choice of resolution, and only from the second page onwards does the spectral sequence depend only on the functors F, G, and the object A given. On the other hand, the approach we have taken produces a spectral sequence which only exists from the second page onwards, but we only made use of homotopy-invariant objects throughout.

2.2. **Atiyah-Hirzebruch spectral sequence.** The ∞-category of spectra Sp carries a t-structure in which the subcategory $\mathrm{Sp}_{\geq 0} \simeq \mathrm{Sp}^{\mathrm{cn}} \subseteq \mathrm{Sp}$ is taken to be the full subcategory of connective spectra and $\mathrm{Sp}_{\leq 0} \subseteq \mathrm{Sp}$ is taken to be the full subcategory coconnective spectra. Homotopy groups and truncations with respect to this t-structure recover their usual meanings in stable homotopy theory. The t-structure on spectra is compatible with colimits by Remark HA.1.4.3.5, while Proposition HA.1.4.3.6 ensures that it is both left and right complete, and that there is a canonical equvalence $\mathrm{Sp}^{\heartsuit} \simeq \mathcal{A}\mathrm{b}$.

Proposition 10 (Atiyah-Hirzebruch spectral sequence)**.** *Let X be a connective spectrum. There eixsts for any $E \in \mathrm{Sp}$ a conditionally convergent Adams-indexed spectral sequence*

$$E_2^{s,t} = \mathrm{H}^s(X; \pi_t E) \Rightarrow E^{s-t}(X)$$

with values in abelian groups. If E is eventiually coconnective, this spectral sequence converges unconditionally.

Proof. Consider the functor $F : \mathrm{Sp} \to \mathrm{Sp}$ given by $E \mapsto \underline{\mathrm{Map}}_S(X, E)$, the internal mapping spectrum as discussed in §6.5.3 of SAG. It is defined to be the right adjoint to the functor $E \mapsto X \otimes E$. Since smashing with connective spectra preserves (or increases) connectivity, the latter functor is t-exact. As its right adjoint, F is hence left t-exact and commutes with limits. The stage is set to apply Variant 5 and obtain a homologically-indexed spectral sequence

$$E_2^{p,q} \simeq \pi_p \underline{\mathrm{Map}}_S(X, \pi_q E) \Rightarrow \pi_{p+q} \underline{\mathrm{Map}}_S(X, E).$$

We switch to Adams-indexing by setting $s = -p$ and $t = q$, and note that $E^s(X) \simeq \pi_{-s}\underline{\mathrm{Map}}_S(X, E)$ is the cohomology theory associated to the spectrum E essentially by definition. Since eventually coconnective is precisely the same as bounded above with respect to the t-structure on Sp, the convergence claim also follows from Variant 5. □

Remark 11. The dual version of the Atiyah-Hirzebruch spectral sequence, computing generalized homology, is sometimes preferable to its cohomological cousin from Proposition 10 as it enjoys better convergence properties[3]. One might expect at first glance that we could also deduce it in a similar way to Proposition 10 from a dual version of Theorem 2 or Variant 5. This is however not the case, as the dual version of Theorem 2 would impose the requiremenet that the t-structure be compatible with filtered limits. The standard t-structure on Sp fails to satisfy this condition, and the obstruction to it is measured by the well-known Milnor $\varprojlim^1$ sequence.

2.3. **Fixed-point and Tate spectral sequences.** Let X be a space, which we may equivalently view as an ∞-topos. The functor ∞-category $\mathrm{Fun}(X, \mathrm{Sp})$, which we may view as the ∞-category of *local systems of spectra on* X (also called *parametrized spectra over* X), is stable. The smash product and the t-structure on Sp induce one a symmetric monoidal structure and a t-structure on $\mathrm{Fun}(X, \mathrm{Sp})$ respectively, each by point-wise application of the one on spectra. Thus the fact that the t-structure on Sp is both left and right complete implies the same holds for the one on $\mathrm{Fun}(X, \mathrm{Sp})$. Since colimits in functor ∞-categories are calculated object-wise, it also follows that the t-structure on $\mathrm{Fun}(X, \mathrm{Sp})$ is compatible with colimits. There is an equivalence $\mathrm{Fun}(X, \mathrm{Sp})^\heartsuit \simeq \mathrm{Fun}(X, \mathcal{A}\mathrm{b})$, which is to say that the heart of the t-structure on local systems of spectra is equivalent to the category of local systems of abelian groups[4].

Denote by $p : X \to \{*\}$ the terminal map to a point. Composing with it induces a functor $p^* : \mathrm{Sp} \to \mathrm{Fun}(X, \mathrm{Sp})$ which sends a spectrum to the constant local system on X. It is t-exact, preserves limits and colimits, since all of those are computed point-wise on $\mathrm{Fun}(X, \mathrm{Sp})$. Since the ∞-category $\mathrm{Fun}(X, \mathrm{Sp})$ is presentable, this implies that p^* admits a left and a right adjoint $p_*, p_! : \mathrm{Fun}(X, \mathrm{Sp}) \to \mathrm{Sp}$, which are right and left t-exact respectively. Note that these may be identified with the colimit and limit respectively, i.e. we have $p_*\mathcal{L} \simeq \varinjlim \mathcal{L}$ and $p_!\mathcal{L} \simeq \varprojlim \mathcal{L}$.

By [NS17, Theorem I.4.1] there exists a unique functor $p_*^T : \mathrm{Fun}(X, \mathrm{Sp}) \to \mathrm{Sp}$, called the *Farrell-Tate construction*, fitting into a cofiber sequence

$$p_!(D_X \otimes -) \to p_* \to p_*^T,$$

where the functor $D_X : X \to \mathrm{Sp}$ is given informally for every point $x \in X$ by

$$D_X(x) \simeq \varprojlim_{y \in X} \Sigma_+^\infty \mathrm{Map}_X(x, y). \tag{4}$$

[3]Really the dual Atiyah-Hirzebruch spectral sequence converges dually to the spectral sequence in 5, i.e. for all eventually connective spectra. It is just that we much more often encounter connective spectra than coconnective ones, e.g. suspension spectra of all spaces are connective, while they are quite rarely also coconnective.

[4]There are indeed local systems in the classical sense. Since $\mathcal{A}\mathrm{b}$ is a 1-category, we have $\mathrm{Fun}(X, \mathcal{A}\mathrm{b}) \simeq \mathrm{Fun}(\tau_{\le 1} X, \mathcal{A}\mathrm{b})$, and the 1-truncation $\tau_{\le 1} X$ may be identified with the fundamental groupoid of X, recovering the usual definition (or at least one variant of it) of a local system of abelian groups on X.

Here the morphism space is understood by viewing X as an ∞-groupoid.

Example 12. Let G be a topological group and let the space in question be BG, its classifying space. In that case everything discussed above acquires new familiar names. The ∞-category $\mathrm{Fun}(BG, \mathrm{Sp})$ is variously called *spectra with G-action* and *Borel (or naive) G-equivariant spectra*. For $X \in \mathrm{Fun}(BG, \mathrm{Sp})$ the spectrum $p_* X$ is denoted X_G and called the *coinvariants of X*, while the spectrum $p_! X$ is denoted X^G and called the *fixed-points of X*. We have $D_{BG} \simeq p^*(S[G]^G)$ and the spectrum $p^T_* X$ is denoted X^{tG} and called the *Tate construction*[5] *of X*. The discrete spectra with a G-action are $\mathrm{Fun}(BG, \mathrm{Sp})^\heartsuit \simeq \mathrm{Fun}(BG, \mathcal{A}\mathrm{b})$, the G-modules or representations of G over $\mathbf{Z}$. In that case we recover on the level of homotopy groups

$$\pi_i X_G \simeq \mathrm{H}_i(G; X), \quad \pi_{-i} X^G \simeq \mathrm{H}^i(G; X), \quad \pi_{-i} X^{tG} \simeq \hat{\mathrm{H}}^i(G; X),$$

the usual group homology, group cohomology, and Tate cohomology respectively.

In analogy with Example 12, we will use the notation $\hat{\mathrm{H}}^i(X; \mathcal{L}) \simeq \pi_{-i}(p^T_* \mathcal{L})$ for a local system of abelian groups $\mathcal{L}$ on any space X.

Proposition 13 (Fixed-point spectral sequence)**.** *Let X be a space and $\mathcal{L}$ be a local system of spectra on X There exists an Adams-indexed conditionally convergent spectral sequence*

$$E_2^{s,t} \simeq \mathrm{H}^s(X; \pi_t \mathcal{L}) \Rightarrow \pi_{t-s}(p_* \mathcal{L})$$

with values in abelian groups. If L is bounded below, it converges unconditionally.

Proof. Since we already know that all the required assumptions are satisfied, we can apply Variant 5 to the functor $p_* : \mathrm{Fun}(X, \mathrm{Sp}) \to \mathrm{Sp}$ evaluated at $\mathcal{L}$. This gives rise to the spectral sequence which is the one desired after reindexing to the Adams-indexing convention, and observing that the homotopy group $\pi_{-s}(p_* \mathcal{L})$ for $\mathcal{L} \in \mathrm{Fun}(X, \mathrm{Sp})^\heartsuit \simeq \mathrm{Fun}(X, \mathcal{A}\mathrm{b})$ is the usual cohomology with coefficients in the local system $\mathrm{H}^s(X; \mathcal{L})$. □

Remark 14. Applying Proposition 13 to the constant local system $p^* E$ for any $E \in \mathrm{Sp}$ reproduces the Atiyah-Hirzebruch spectral sequence of Proposition 10 in which the connective spectrum is taken to be the suspension $\Sigma^\infty_+ X$. It is possible to extract the Serre spectral sequence of a fibration in a similar way.

Proposition 15 (Tate spectral sequence)**.** *Let X be a space and $\mathcal{L}$ be a local system of spectra on X. There exists an Adams-indexed conditionally convergent spectral sequence*

$$E_2^{s,t} \simeq \hat{\mathrm{H}}^s(X; \pi_t \mathcal{L}) \Rightarrow \pi_{t-s}(p^T_* \mathcal{L})$$

with values in abelian groups. If L is bounded below, it converges unconditionally.

Proof. We wish to apply Variant 4 to the functor $p^T_* : \mathrm{Fun}(X, \mathrm{Sp}) \to \mathrm{Sp}$, for which we must verify that that the assumptions are satisfied. That is indeed the case because the Farelli-Tate construction is given as the cofiber $p^T_* \simeq \mathrm{cofib}(p_!(D_X \otimes -) \to p_*)$ of two exact functors, and is as such exact itself, and because the ∞-category Sp contains all small limits and colimits, hence in particular also the sequential limit and colimit required in the statement of Variant 4.

Applying said result produces (after reindexing) an Adams-indexed conditionally convergent spectral sequence

$$E_2^{s,t} \simeq \hat{H}^s(X; \pi_t \mathcal{L}) \Rightarrow \pi_{t-s} \,\mathrm{cofib}(\varprojlim p^T_*(\tau_{\geq -n} \mathcal{L}) \to \varinjlim p^T_*(\tau_{\geq -n} \mathcal{L})). \tag{5}$$

To identify the limiting term with the one appearing in the statement of the Proposition, let us compute the two factors of the cofiber separately.

First let us deal with the first factor in the cofiber in (5). Note that the cofiber of a morphism in an ∞-category is equivalent to the fiber of the suspension of the same

[5]For a finite group G this agrees with the Tate construction of Definition HA.6.1.6.21.

morphism. Since limits commute with fibers, it follows that they also commute with cofibers, and we get

$$\varprojlim p_*^T(\tau_{\geq -n}\mathcal{L}) \simeq \mathrm{cofib}(\varprojlim p_!(D_X \otimes \tau_{\geq -n}\mathcal{L}) \to \varprojlim p_*(\tau_{\geq -n}\mathcal{L})).$$

Let us again determine the factors appearing in the cofiber individually. The second factor vanishes because p_*, being a right adjoint, commutes with limits, and the limit of $\tau_{\geq -n}\mathcal{L}$ vanishes due to the t-structure on $\mathrm{Fun}(X, \mathrm{Sp})$ being left separated. For the first factor, note that the explicit formula (4) shows D_X to be connective, we have $D_X \otimes \tau_{\geq -n}\mathcal{L} \in \mathrm{Fun}(X, \mathrm{Sp})_{\geq -n}$. Because the functor $p_!$ is right t-exact, this implies that $p_!(D_X \otimes \tau_{\geq -n}\mathcal{L})$ belongs to $\mathrm{Sp}_{\geq -n}$. Left separatedness of the t-structure on spectra again implies that $\varinjlim p_!(D_X \otimes \tau_{\geq -n}\mathcal{L}) \simeq 0$. Together this implies that the first factor in the cofiber in (5) vanishes.

Now let us examine the second factor in the cofiber in (5). It is given by

$$\varinjlim p_*^T(\tau_{\geq -n}\mathcal{L}) \simeq \mathrm{cofib}(\varinjlim p_!(D_X \otimes \tau_{\geq -n}\mathcal{L}) \to \varinjlim p_*(\tau_{\geq -n}\mathcal{L})),$$

the first factor of which is equivalent to $p_!\mathcal{L}$ since $p_!$ and smashing with D_X both commuting with colimits and the t-structure on Sp is right complete. To determine the second factor, note that the maps $\tau_{\geq -n}\mathcal{L} \to \mathcal{L}$ determine a morphism $\varinjlim p_*(\tau_{\geq -n}\mathcal{L}) \to p_*\mathcal{L}$ the cofiber of which is

$$\mathrm{cofib}(\varinjlim p_*(\tau_{\geq -n}\mathcal{L}) \to p_*\mathcal{L}) \simeq \varinjlim p_*(\mathrm{cofib}(\tau_{\geq -n}\mathcal{L} \to \mathcal{L})) \simeq \varinjlim p_*(\tau_{\leq -n-1}\mathcal{L}),$$

where we have used that p_*, being an exact functor, commutes with cofibers. Since p_* is left t-exact, this colimit belongs to the subcategory $\bigcap \mathrm{Sp}_{\leq -n} \subseteq \mathrm{Sp}$, which contains only the zero object due to the t-structure on Sp being right separated. It follows that the cofiber is zero and so the map $\varinjlim p_*(\tau_{\geq -n}\mathcal{L}) \to p_*\mathcal{L}$ is an equivalence.

Returning to (5), we find that the limiting term of the spectral sequence is given by homotopy groups of $\mathrm{cofib}(p_!(D_X \otimes \mathcal{L}) \to p_*\mathcal{L}) \simeq p_*^T\mathcal{L}$, which is what we wanted to show. □

Remark 16. In the proof of Proposition 15 we made use of Variant 5 instead of Theorem 2. This is because the Farrell-Tate construction, while being an exact functor, in general fails to be either left or right t-exact. Indeed, recall that for $\mathbf{Z}$ equipped with a trivial S^1-action, we have $\hat{\mathrm{H}}^*(S^1; \mathbf{Z}) \simeq \mathbf{Z}[u^{\pm 1}]$ with u a generator in degree 2. The Tate construction $\mathbf{Z}^{tS^1}$ is thus neither connective nor coconnective, despite the fact that $\mathbf{Z} \in \mathcal{A}\mathrm{b} \simeq \mathrm{Sp}^{\heartsuit}$.

Let us spell out the classical versions of the fixed-point and Tate spectral sequences familiar from equivariant homotopy theory. They follow instantly from Proposition 13 and Proposition 15 by applying them to the classifying space BG of a topological group G and using the dictionary of Example 12.

Corollary 17 (Fixed-point spectral sequence)**.** *Let G be a topological group and X a spectrum with a G-action. There exists an Adams-indexed conditionally convergent spectral sequence*

$$E_2^{s,t} \simeq \mathrm{H}^s(G; \pi_t X) \Rightarrow \pi_{t-s}(X^G)$$

with values in abelian groups. If X is bounded below, it converges unconditionally.

Corollary 18 (Tate spectral sequence)**.** *Let G be a topological group and X a spectrum with a G-action. There exists an Adams-indexed conditionally convergent spectral sequence*

$$E_2^{s,t} \simeq \hat{\mathrm{H}}^s(G; \pi_t X) \Rightarrow \pi_{t-s}(X^{tG})$$

with values in abelian groups. If X is bounded below, it converges unconditionally.

Example 19. One application of the Tate spectral sequence, in its classical form of Corollary 18, is to the topological Hochschild homology spectrum $\mathrm{THH}(\mathcal{C})$ of a smooth proper k-linear stable ∞-category $\mathcal{C}$ over a field k, equipped with its usual S^1-action. Recalling that the periodic cyclic homology is defined as $\mathrm{TP}(\mathcal{C}) \simeq \mathrm{THH}(\mathcal{C})^{tS^1}$, the homologically-indexed Tate spectral sequence reads

$$E_2^{*,q} \simeq \mathrm{THH}_q(\mathcal{C})[u^{\pm 1}] \Rightarrow \mathrm{TP}_{*+q}(\mathcal{C}), \tag{6}$$

where we have used the computation of the Tate cohomology of $\mathbf{Z}$ from Remark 16 to identify the second page. This is called the *non-commutative Hodge-to-de-Rham spectral sequence* in [Mat17]. When $\mathcal{C}$ is taken to be the bounded derived ∞-category of coherent sheaves on a smooth proper scheme X over a field k of characteristic zero, the Hochschild-Kostant-Rosenberg Theorem shows that (appropriate base-changes of) $\mathrm{THH}_*(\mathcal{C})$ and $\mathrm{TP}_*(\mathcal{C})$ are equivalent to $\mathrm{H}^*(X;\Omega^*_X)$ and the 2-periodic version of $\mathrm{H}^*_{\mathrm{dR}}(X)$ respectively; see [Mat17] for details. As pointed out there, the spectral sequence (6) is in that case precisely the 2-periodic version of the famous (cohomologically-indexed) *Hodge-to-de-Rham spectral sequence*

$$E_2^{p,q} \simeq \mathrm{H}^p(X;\Omega^q_X) \Rightarrow \mathrm{H}^{p+q}_{\mathrm{dR}}(X).$$

That said, the latter spectral sequence is traditionally obtained as the hypercohomology spectral sequence associated to the algebraic de Rham complex Ω^*_X. Of course the hypercohomology spectral sequence is itself a special case of Theorem 2, applied to the derived ∞-category.

2.4. **Descent spectral sequence.** Let $\mathcal{X}$ be an ∞-topos. A *spectral sheaf on* $\mathcal{X}$ is defined to be a functor $\mathscr{F} : \mathcal{X}^{\mathrm{op}} \to \mathrm{Sp}$ which preserves limits. These form a stable ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ which is complete and cocomplete. By Proposition SAG.1.3.2.7, it carries a t-structure in which $\mathscr{F} \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathscr{F})_{\le n}$ if and only if $\mathscr{F}(U) \in \mathrm{Sp}_{\le n}$ for every $U \in \mathcal{X}$. This t-structure is right complete and compatible with colimits, but it is in general not left complete nor even left separated.

Example 20. Supposing the ∞-topos $\mathcal{X}$ is not hypercomplete, it might contain ∞-connective objects $U \in \mathcal{X}$. That is to say, U might not be contractible despite satisfying $\pi_i U \simeq 0$ for all $i \ge 0$. Its suspension spectrum $\mathscr{F} \simeq \Sigma^\infty_+ U$, viewed as a spectral sheaf on $\mathcal{X}$, is thus n-truncated for every $n \in \mathbf{Z}$ by definition, exhibiting the failure of the t-structure on $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ to be left separated.

Given an ∞-topos $\mathcal{X}$ and a spectral sheaf $\mathscr{F}$ on $\mathcal{X}$, there exists by Remark SAG.1.3.3.5 an essentially unique fiber sequence

$$\widetilde{\mathscr{F}} \to \mathscr{F} \to \widehat{\mathscr{F}}$$

in $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$, in which $\widetilde{\mathscr{F}}$ is ∞-connective and $\widehat{\mathscr{F}}$ is hypercomplete. According to Definition SAG.1.3.3.2 and Proposition SAG.1.3.3.3, that is equivalent to saying that $\pi_i\widetilde{\mathscr{F}} \simeq 0$ for all $i \in \mathbf{Z}$, and that $\Omega^\infty\widehat{\mathscr{F}}$ is a hypercomplete object of $\mathcal{X}$. The essential uniqueness mentioned justifies calling $\widetilde{\mathscr{F}}$ the ∞-*connective cover of* $\mathscr{F}$ and $\widetilde{\mathscr{F}}$ the *hypercompletion of* $\mathscr{F}$.

Lemma 21. *Let $\mathcal{X}$ be an ∞-topos and let $\mathscr{F}$ be a spectral sheaf on $\mathcal{X}$. Then the canonical morphism $\mathscr{F} \to \varprojlim \tau_{\le n}\mathscr{F}$ exhibits the hypercompletion of $\mathscr{F}$ as $\widehat{\mathscr{F}} \simeq \varprojlim \tau_{\le n}\mathscr{F}$.*

Proof. The fiber of the canonical map $\mathscr{F} \to \varprojlim \tau_{\le n}\mathscr{F}$ is equivalent to $\varprojlim \tau_{\ge n}\mathscr{F}$, and we must show that this spectral sheaf is ∞-connective. To determine its homotopy groups, let us make use of the Milnor $\varprojlim^1$ sequence, which is in this incarnation a short exact sequence

$$0 \to \varprojlim{}^1 \pi_k(\tau_{\ge n}\mathscr{F}) \to \pi_k(\varprojlim \tau_{\ge n}\mathscr{F}) \to \varprojlim \pi_k(\tau_{\ge n}\mathscr{F}) \to 0,$$

of sheaves of abelian groups on $\mathcal{X}^\heartsuit$. Since $\pi_k(\tau_{\ge n}\mathscr{F}) \simeq 0$ whenever $k < n$, it follows that for any fixed $k \in \mathbf{Z}$ the sequence $\{\pi_k(\tau_{\ge n}\mathscr{F})\}_{n>0}$ stabilizes at zero as $n \to \infty$ after finitely many steps. Consequently both the first and the last term in the above short exact sequence vanish, forcing the middle term to vanish as well. This shows that $\pi_k(\varprojlim \tau_{\ge n}\mathscr{F}) \simeq 0$ for all $k \in \mathbf{Z}$, hence that $\varprojlim \tau_{\ge n}\mathscr{F}$ is the ∞-connective cover of $\mathscr{F}$. $\square$

There is a canonical equivalence $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})^{\heartsuit} \simeq \mathcal{S}\mathrm{hv}_{\mathcal{A}\mathrm{b}}(\mathcal{X}^{\heartsuit})$ of the heart with the category of sheaves of abelian groups on the underlying ordinary topos $\mathcal{X}^{\heartsuit}$ of $\mathcal{X}$. This induces a fully faithful t-exact embedding of the derived ∞-category $\mathcal{D}(\mathcal{S}\mathrm{hv}_{\mathcal{A}\mathrm{b}}(\mathcal{X}^{\heartsuit})) \to \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$.

Lemma 22. *Let $\mathcal{X}$ be an ordinary topos. The inclusion $\mathcal{D}(\mathcal{S}\mathrm{hv}_{\mathcal{A}\mathrm{b}}(\mathcal{X})) \to \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ is an equivalence of ∞-categories.*

Proof. This follows from Theorem SAG.2.1.2.2, since any ordinary topos is clearly hypercomplete. □

Any geometric morphism of ∞-topoi $f_* : \mathcal{X} \to \mathcal{Y}$ induces by Remark SAG.1.3.2.8 an adjunction $f^* : \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{Y}) \leftrightarrows \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}) : f_*$, in which the pushforward functor is given by point-wise composition with $f^* : \mathcal{Y} \to \mathcal{X}$. The pullback functor f^* is t-exact and so f_* is left t-exact. When $p_* : \mathcal{X} \to \mathcal{S}$ is the terminal geometric morphism, we use the suggestive notation $\Gamma(\mathcal{X}; -) \simeq p_*$ and terminology *global sections functor*. Given a geometric morphism of ∞-topoi $f_* : \mathcal{X} \to \mathcal{Y}$, let us denote by $f_*^{\heartsuit} : \mathcal{X}^{\heartsuit} \to \mathcal{Y}^{\heartsuit}$ the underlying morphism of ordinary topoi. Lemma 22 suggests a relationship between f_* and the derived functor of $f_*^{\heartsuit}$, fleshed out by the following Lemma.

Lemma 23. *Let $f_* : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi, and let $\mathscr{F} \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})^{\heartsuit} \simeq \mathcal{S}\mathrm{hv}_{\mathcal{A}\mathrm{b}}(\mathcal{X}^{\heartsuit})$. Then the homotopy groups of the pushforward along f are given in terms of right derived functors by $\pi_{-i}(f_*\mathscr{F}) \simeq R^i f_*^{\heartsuit}(\mathscr{F})$ for all $i \in \mathbf{Z}$.*

Proof. This is a special case of Theorem SAG.2.1.2.8, which shows more generally that an appropriate restriction of $f_*\mathscr{F}$ is equivalent to the right derived functor $Rf_*^{\heartsuit}$. □

Lemma 24. *Let $\mathcal{X}$ be an ∞-topos and let $\mathscr{F} \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})^{\heartsuit} \simeq \mathcal{S}\mathrm{hv}_{\mathcal{A}\mathrm{b}}(\mathcal{X})$. The homotopy groups of global sections are given in terms of sheaf cohomology as $\pi_{-i}\Gamma(\mathcal{X}; \mathscr{F}) \simeq \mathrm{H}^i(\mathcal{X}^{\heartsuit}; \mathscr{F})$ for all $i \in \mathbf{Z}$.*

Proof. Apply the previous Lemma to the global sections functor $p_* \simeq \Gamma(\mathcal{X}; -)$, and observe that $p_*^{\heartsuit} \simeq \Gamma^{\heartsuit}(\mathcal{X}^{\heartsuit}; -)$ is precisely the usual global sections functor on sheaves of abelian groups. Note that $\mathrm{H}^i(\mathcal{X}^{\heartsuit}; -) \simeq R^i\Gamma^{\heartsuit}(\mathcal{X}^{\heartsuit}; -)$ is just the usual definition of sheaf cohomology as a right derived functor. □

All the pieces are in place to apply Theorem 2 and obtain another spectral sequence.

Proposition 25 (Descent spectral sequence)**.** *Let $\mathcal{X}$ be an ∞-topos and $\mathscr{F}$ a spectral sheaf on $\mathcal{X}$. There exists an Adams-indexed conditionally convergent spectral sequence*

$$E_2^{s,t} \simeq \mathrm{H}^s(\mathcal{X}^{\heartsuit}; \pi_t\mathscr{F}) \Rightarrow \pi_{t-s}\Gamma(\mathcal{X}; \widehat{\mathscr{F}})$$

with values in abelian groups. When $\mathscr{F}$ is bounded above, then this spectral sequence converges unconditionally to $\pi_{t-s}\Gamma(\mathcal{X}; \mathscr{F})$.

Proof. We know that all assumptuons are satisfied, so we may apply Theorem 2 to the functor $\Gamma(\mathcal{X}; -) : \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}) \to \mathrm{Sp}$ to obtain a homologically-indexed conditionally convergent spectral sequence

$$E_2^{p,q} \simeq \pi_p\Gamma(\mathcal{X}; \pi_q\mathscr{F}) \Rightarrow \pi_{p+q}(\varprojlim \Gamma(\mathcal{X}; \tau_{\leq n}\mathscr{F})).$$

Reindexing to the Adams-indexing convention, using Lemma 24 to identify the second page, and the fact that the global sections functor, being a right adjoint, preserves limits, together with Lemma 21, we end up with the desired spectral sequence. The convergence claim follows from Theorem 2. □

Remark 26. By applying Proposition 25 to the presheaf ∞-topos $\mathcal{X} \simeq \mathcal{P}(X)$ for any space X, viewed as an ∞-groupoid, we recover Proposition 13.

With the same proof, only using Lemma 23 instead of its special case Lemma 24, we may obtain a relative version of Proposition 25.

Proposition 27 (Relative descent spectral sequence)**.** *Let $f_* : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi and let $\mathscr{F}$ be a spectral sehaf on $\mathcal{X}$. There exists an Adams-indexed conditionally convergent spectral sequence*

$$E_2^{s,t} \simeq R^s f_*^{\heartsuit}(\pi_t \mathscr{F}) \Rightarrow \pi_{t-s}(f_* \widehat{\mathscr{F}})$$

with values in the abelian category $\mathrm{Shv}_{\mathcal{A}\mathrm{b}}(\mathcal{Y})$. When $\mathscr{F}$ is bounded above, then this spectral sequence converges unconditionally to $\pi_{t-s}(f_* \mathscr{F})$.

Remark 28. One might expect a different relative version of Proposition 25 than Proposition 27, along the lines of the Leray spectral sequence. This should be obtained by applying Proposition 25 to the spectral sheaf $f_*\mathscr{F}$ on an ∞-topos $\mathcal{Y}$ for a geometric morphism $f_* : \mathcal{X} \to \mathcal{Y}$ and a spectral sheaf $\mathscr{F}$ on $\mathcal{X}$. However, this results in the spectral sequence

$$E_2^{s,t} \simeq \mathrm{H}^s(\mathcal{Y}; \pi_t(f_*\mathscr{F})) \Rightarrow \pi_{t-s}\Gamma(\mathcal{Y}; \widehat{f_*\mathscr{F}}),$$

which does not necessarily converge to the homotopy groups of $\Gamma(\mathcal{X}; \mathscr{F})$ as expected from the Leray spectral sequence. That is the case if $\mathscr{F}$ is bounded above, but not in general, since the neither of the functors $\Gamma(\mathcal{Y}; -)$ and f_* needs to commute with hypercompletion.

2.5. **Čech-to-derived spectral sequence.** Let $\mathcal{X}$ be an ∞-topos. In line with Definition SAG.1.2.4.1, we will say that an object $U \in \mathcal{X}$ *covers* $\mathcal{X}$ if the map to the terminal object $U \to \mathbf{1}$ is an effective epimorphism in $\mathcal{X}$. Given such a U, we can form its Čech nerve $U_\bullet$, a simplicial object in $\mathcal{X}$ with geometric realization $|U_\bullet| \simeq \mathbf{1}$. For any spectral presheaf $\mathscr{F} \in \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathrm{Sp})$, we define the *Čech sections of $\mathscr{F}$ over U* to be

$$\check{\Gamma}(U; \mathscr{F}) \simeq \mathrm{Tot}(\mathscr{F}(U_\bullet)),$$

the totalization of the cosimplicial spectrum obtained by composing the contravariant functor $\mathscr{F}$ with the simplicial object $U_\bullet$. As the name suggests, Čech sections are closely related to Čech cohomology.

Lemma 29. *Let $\mathcal{X}$ be an ∞-topos, let $U \in \mathcal{X}$ be an object which covers $\mathcal{X}$, and let $\mathscr{F} \in \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathrm{Sp})^{\heartsuit} \simeq \mathrm{Fun}((\mathcal{X}^{\heartsuit})^{\mathrm{op}}, \mathcal{A}\mathrm{b})$ be a discrete spectral sheaf. Then we have*

$$\pi_{-i}\check{\Gamma}(U; \mathscr{F}) \simeq \check{\mathrm{H}}^i(\tau_{\leq 0}U; \mathscr{F})$$

for all $i \in \mathbf{Z}$.

Proof. Recall that the underlying ordinary topos $\mathcal{X}^{\heartsuit}$ of the ∞-topos may be identified with the subcategory of discrete objects in $\mathcal{X}$. The inclusion $\mathcal{X}^{\heartsuit} \subseteq \mathcal{X}$ admits a left adjoint given by the truncation[6] functor $\tau_{\leq 0} : \mathcal{X} \to \mathcal{X}^{\heartsuit}$. Using j to denote the Yoneda embedding, we find that

$$\begin{aligned} \Omega^{\infty-n}\mathscr{F}(U^k) &\simeq \mathrm{Map}_{\mathcal{P}(\mathcal{X})}(j(U^k), \Omega^{\infty-n}\mathscr{F}) \\ &\simeq \mathrm{Map}_{\mathcal{P}(\mathcal{X}^{\heartsuit})}(j(\tau_{\leq 0}U^k), \Omega^{\infty-n}\mathscr{F}) \\ &\simeq \Omega^{\infty-n}\mathscr{F}((\tau_{\leq 0}U)^k), \end{aligned}$$

where we have used the Yoneda lemma, the fact that $\mathscr{F}$ is discrete, and the fact that $\tau_{\leq 0}$ commutes with finite products. Since $n \in \mathbf{Z}$ was arbitrary above, we obtain an equivalence of spectra $\mathscr{F}(U^k) \simeq \mathscr{F}((\tau_{\leq 0}U)^k)$. Taking the limit over k, we find an equivalence of Čech sections

$$\check{\Gamma}(U; \mathscr{F}) \simeq \check{\Gamma}(\tau_{\leq 0}U; \mathscr{F}),$$

the left taken with respect to the ∞-topos $\mathcal{X}$, and the right with respect to the ordinary topos $\mathcal{X}^{\heartsuit}$. Here note that, since $U \to \mathbf{1}$ is an effective epimorphism in $\mathcal{X}$, so is the morphism

[6]Note that this is not truncation with respect to a t-structure, but the internal truncation inside the ∞-topos $\mathcal{X}$. Of course when considering spectral sheaves on an ∞-topos, the t-structure is defined to interpolate between the standard t-structure on Sp and the internal homotopy theory of the ∞-topos, so the two notions of truncation are not unrelated.

$\tau_{\leq 0}U \to \tau_{\leq 0}\mathbf{1}$ an effective epimorphism in $\mathcal{X}^\heartsuit$, and the left adjoint functor $\tau_{\leq 0}$ preserves terminal objects (the limit of the empty diagram), showing that $\tau_{\leq 0}U$ covers $\mathcal{X}^\heartsuit$. With this, it suffices to assume that $\mathcal{X} \simeq \mathcal{X}^\heartsuit$ is an ordinary topos.

It remains to actually identify homotopy groups of the Čech sections with Čech cohomology. The former are computed as homotopy groups of the spectrum $\check{\Gamma}(U;\mathscr{F})$, which is obtain as totalization inside the ∞-category Sp of the cosimplicial diagram $\mathscr{F}(U_\bullet) : \mathbf{\Delta} \to \mathcal{A}\mathrm{b} \simeq \mathrm{Sp}^\heartsuit \subseteq \mathrm{Sp}$. Though the totalization itself is taken in the ∞-category of spectra, $\mathscr{F}(U_\bullet)$ is cosimplicial abelian group. Thus we may use the Dold-Kan correspondence, in version dual to[7] Lemma HA.1.2.3.13, to identify $\mathrm{Fun}(\mathbf{\Delta}, \mathcal{A}\mathrm{b}) \simeq \mathrm{Ch}(\mathcal{A}\mathrm{b})^{\geq 0}$ the categories of cosimplicial abelian groups and cochain complexes in non-negative degrees. The equivalence is given by sending a cosimplicial abelian group $A_\bullet$ to the normalized cochain complex $N^*(A)$, given explicitly by the dual of Definition HA.1.2.3.9. Using the dual of Remark HA.1.2.3.14, we find that the cohomology group of a chain complex associated to a cosimplicial abelian group is isomorphic to the negative homotopy group of the totalization of the cosimplicial diagram of abelian groups, when viewed as a diagram of discrete spectra. In the case we are interested in, this reduces to

$$\pi_{-i}\check{\Gamma}(U;\mathscr{F}) \simeq \pi_{-i}\,\mathrm{Tot}(\mathscr{F}(U_\bullet)) \simeq \mathrm{H}^i(N^*(\mathscr{F}(U_\bullet))).$$

We may use the dual of (or just application to $\mathcal{A}\mathrm{b}^{\mathrm{op}}$ of) Proposition HA.1.2.3.17 to find that the cohomology groups of normalized cochains $N^*(\mathscr{F}(U_\bullet))$ are equivalent to cohomology groups of unnormalized cochains $C^*(\mathscr{F}(U_\bullet))$. The latter is, accoding to Definition 1.2.3.8, the cochain complex given explicitly as

$$0 \to \mathscr{F}(U) \xrightarrow{d(1)} \mathscr{F}(U^2) \xrightarrow{d(2)} \mathscr{F}(U^3) \xrightarrow{d(3)} \dots,$$

in which the codifferential $d(n) : \mathscr{F}(U^n) \to \mathscr{F}(U^{n+1})$ is given as an alternating sum $d(n) = \sum_{0\leq i\leq n}(-1)^i d^i$ of the coface maps d^i of the cosimplicial abelian group $\mathscr{F}(U_\bullet)$. These coface maps are induced by the face maps of the Čech nerve $U_\bullet$, which are defined in the standard way in terms on projections onto factors in the product. From this, it is clear that the cochain complex described above recovers precisely the classical complex of Čech cochains of $\mathscr{F}$ over U, i.e. $C^*(\mathscr{F}(U_\bullet)) \simeq \check{C}^*(U;\mathscr{F})$, the cohomology groups of which are the Čech cohomology groups $\check{\mathrm{H}}^i(U;\mathscr{F})$ by definition. □

Let $\mathcal{X}$ again be an arbitrary ∞-topos. There is a geometric morphism $j_* : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ from $\mathcal{X}$ to the presheaf ∞-topos $\mathcal{P}(\mathcal{X}) \simeq \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{S})$, given as an adjunction by the Yoneda emedding and sheafification. As we already discussed in 2.4, this geometric morphism induces an adjunction $j^* : \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathrm{Sp}) \simeq \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{P}(\mathcal{X})) \leftrightarrows \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}) : j_*$ on spectral sheaves, with the functor j^* is t-exact and j_* left t-exact. Analogously to the case in 2.3, the t-structure on $\mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathrm{Sp})$ is inherited from the one on spectra. As such, it is both left and right t-complete, unlike the t-strucutre on $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ which is in general only right t-complete. For a spectral sheaf $\mathscr{F}$ on $\mathcal{X}$, let us denote by $\pi_i^{\mathrm{pre}}\mathscr{F}$ the homotopy groups $\pi_i(j_*\mathscr{F})$ for all $i \in \mathbf{Z}$. The spectral presheaf $\pi_i^{\mathrm{pre}}\mathscr{F}$ is explicitly given as $U \mapsto \pi_i(\mathscr{F}(U))$.

Remark 30. The inclusion $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}) \subseteq \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathrm{Sp})$, which may be identified with the functor j_*, is reflexive, admiting a left adjoint in the sheafification functor j^*. This implies that the counit of the adjunction, the natural tranformation $j^*j_* \to \mathrm{id}$, induces an equivalence of spectral sheaves. That is just the standard fact that sheafification is idempotent. In particular, we get using the t-exactness of the pullback functor j^* that the composition

$$j^*\pi_i^{\mathrm{pre}}\mathscr{F} \simeq j^*\pi_i(j_*\mathscr{F}) \simeq \pi_i(j^*j_*\mathscr{F}) \to \pi_i\mathscr{F}$$

is an equivalence in $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})^\heartsuit \simeq \mathcal{S}\mathrm{hv}_{\mathcal{A}\mathrm{b}}(\mathcal{X}^\heartsuit)$, showing that homotopy sheaves $\pi_i\mathscr{F}$ are obtained by sheafifying the homotopy presheaves $\pi_i^{\mathrm{pre}}\mathscr{F}$, generalizing Example SAG.1.3.2.4.

[7]Alternatively, we could use Theorem HA.1.2.3.7 applied to the abelian category $\mathcal{A}\mathrm{b}^{\mathrm{op}}$.

Remark 31. Let $\mathscr{F}$ be a discrete spectral sheaf on $\mathcal{X}$, which is to say that $\mathscr{F} \in \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})^{\heartsuit} \simeq \mathrm{Shv}_{\mathcal{A}\mathrm{b}}(\mathcal{X}^{\heartsuit})$. Then the homotopy presheaves $\pi_i^{\mathrm{pre}}\mathscr{F}$ may be identified with the classical *cohomology presheaves*, traditionally denoted $\mathscr{H}^i(\mathscr{F})$ and given by $U \mapsto \mathrm{H}^i(U;\mathscr{F})$. Together with the previous remark, we recover the classical fact that homotopy sheaves $\mathscr{H}^i(\mathscr{F})$ vanish upon sheafification for all $i \geq 1$.

Proposition 32 (Čech-to-derived spectral sequence)**.** *Let $\mathcal{X}$ be an ∞-topos and $\mathscr{F}$ a spectral sheaf on $\mathcal{X}$. Let $U \in \mathcal{X}$ be an object which covers $\mathcal{X}$. There exists an Adams-indexed conditionally convergent spectral sequence*

$$E_2^{s,t} \simeq \check{\mathrm{H}}^s(\tau_{\leq 0}U;\pi_t^{\mathrm{pre}}\mathscr{F}) \Rightarrow \pi_{t-s}\Gamma(\mathcal{X};\mathscr{F})$$

with values in abelian groups. When $\mathscr{F}$ is bounded above, then this spectral sequence converges unconditionally.

Proof. Writting the functor $\mathscr{G} \mapsto \check{\Gamma}(U;\mathscr{G})$ as the composition

$$\mathrm{Fun}(\mathcal{X}^{\mathrm{op}},\mathrm{Sp}) \xrightarrow{U_\bullet^*} \mathrm{Fun}(\mathbf{\Delta},\mathrm{Sp}) \xrightarrow{\mathrm{Tot}} \mathrm{Sp}$$

of two functors, both of which are evidently left t-exact and commute with limits, it follows that the same holds for the functor of Čech sections over U. Since the t-structure on $\mathrm{Fun}(\mathcal{X}^{\mathrm{op}},\mathrm{Sp})$ is both left as well as right t-complete, we may use Variant 5 to obtain (after reindexing and using Lemma 29 to identify the second page) a conditionally convergent Adams-indexed spectral seuqence

$$E_2^{s,t} \simeq \check{\mathrm{H}}^s(\tau_{\leq 0}U;\pi_t\mathscr{G}) \Rightarrow \pi_{t-s}\check{\Gamma}(U;\mathscr{G})$$

for any spectral presheaf $\mathscr{G}$ on $\mathcal{X}$. Now we set $\mathscr{G} \simeq j_*\mathscr{F}$. To identify the limiting term of this spectral sequence with the desired one, note that fact that $\mathscr{F}$ satisfies descent, which is to say that it commutes with limits as a functor $\mathscr{F} : \mathcal{X}^{\mathrm{op}} \to \mathrm{Sp}$, implies that

$$\check{\Gamma}(U;j_*\mathscr{F}) \simeq \mathrm{Tot}(\mathscr{F}(U_\bullet)) \simeq \mathscr{F}(|U_\bullet|) \simeq \mathscr{F}(\mathbf{1}) \simeq \Gamma(\mathcal{X};\mathscr{F}),$$

where we have also used the assumption that U covers $\mathcal{X}$. □

Corollary 33. *Let $\mathcal{X}$ be a topos and $\mathscr{F}$ a sheaf of abelian groups on $\mathcal{X}$. Let $U \in \mathcal{X}$ be an object which covers $\mathcal{X}$. There exists an cohomologically-indexed unconditionally convergent spectral sequence*

$$E_2^{p,q} \simeq \check{\mathrm{H}}^p(U;\mathscr{H}^q(\mathscr{F})) \Rightarrow \mathrm{H}^{p+q}(\mathcal{X};\mathscr{F})$$

with values in abelian groups.

Remark 34. When $\mathcal{X}$ is the ∞-topos of sheaves on the Cartesian site of smooth manifolds, the spectral sheaves on $\mathcal{X}$ are usually called *smooth spectra*. Restricting to the overtopos $\mathcal{X}_{/M}$ for a compact smooth manifold M, the spectral sequence of Proposition 32 and taking the colimit over covers $U \to M \simeq \mathbf{1}_{\mathcal{X}_{/M}}$ recovers the Atiyah-Hirzebruch spectral sequence for smooth spectra of [GS17, Theorem 11].

2.6. **Adams spectral sequence.** We obtain the Adams and Adams-Novikov spectral sequences as a special case of a descent spectral sequence for quasi-coherent sheaves on a (nice-enough) spectral stack.

2.6.1. *Quasi-coherent sheaves on a functor.* We give an account on how to define the ∞-category of quasi-coherent sheaves on any functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. When X is representable by a nonconnective spectral Deligne-Mumford stack, this recovers its usual meaning from §2.2 of SAG. What we describe is analogous to, but simpler than, the §6.2.2 of SAG, due to the fact that we are ignoring set-theoretical difficulties related to the difference between small and large spaces.

Recall for two ∞-categories $\mathcal{C}$ and $\mathcal{D}$, the latter of which possesses all small colimits, any functor $F_0 : \mathcal{C} \to \mathcal{D}$ extends by Theorem HTT.5.1.5.6 essentially uniquely to a colimit preserving functor $F : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ such that $F \circ j \simeq F_0$, where $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ denotes the Yoneda embedding.

Consider the functor $\mathrm{Mod} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{P}\mathrm{r}^{\mathrm{St}}$, sending a connective $\mathbb{E}_\infty$-ring A to the stable presentable ∞-category Mod_A, and which sends an $\mathbb{E}_\infty$-ring map $A \to B$ to the functor $M \mapsto B \otimes_A M$. Via the recalled universal category of the presheaf ∞-category, there exists a unique colimit-preserving functor $\mathrm{QCoh} : \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S}) \to (\mathcal{P}\mathrm{r}^{\mathrm{St}})^{\mathrm{op}}$, such that $\mathrm{QCoh}(\mathrm{Spec}\, A) \simeq \mathrm{Mod}_A$ for any $A \in \mathrm{CAlg}^{\mathrm{cn}}$. Here we are employing the standard notation $\mathrm{Spec}\, A$ for the Yoneda embedding of A (which is sensible, since it is represented by the eponymous spectral scheme). To any nautral tranformation $f : X \to Y$ in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$, the functor QCoh associates an exact functor $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$. By Proposition SAG.6.2.3.4 the functor f^* prerves colimits, so it admits a right adjoint, which we expectedly denote $f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$.

Remark 35. Note that the above discussion is little more than an elaboration of the usual formula

$$\mathrm{QCoh}(X) \simeq \mathrm{QCoh}\Big(\varinjlim_{\mathrm{Spec}\, A \to X} \mathrm{Spec}\, A\Big) \simeq \varprojlim_{\mathrm{Spec}\, A \to X} \mathrm{Mod}_A,$$

where the colimit and limit are both taken with respect to all natural transformations from representable presheaves into X.

2.6.2. *The flat topology.* Let $f : A \to B$ be a map of $\mathbb{E}_\infty$-rings. Recall from Definitions HA.7.2.2.10 and SAG.B.6.11 that f is said to be *(faithfully) flat* if

(a) The underlying map of commutative rings $\pi_0 A \to \pi_0 B$ is (faithfully) flat.
(b) The evident map
$$\pi_* A \otimes_{\pi_0 A} \pi_0 B \to \pi_* B$$
is an isomorphism of gradec $\pi_0 B$-algebras.

The *fpqc topology* on the ∞-category $\mathrm{CAlg}^{\mathrm{op}}$ is defined by setting a collection of $\mathbb{E}_\infty$-ring maps $\{A \to A_i\}_{i \in I}$ to constitute a covering if there exists a finite subset $I' \subseteq I$ for which the induced map $A \to \prod_{i \in I'} A_i$ is faithfully flat. According to Proposition SAG.B.6.13, this generates a Grothendieck topology, which in particular distinguishes a full subcategory $\mathcal{S}\mathrm{hv}_{\mathrm{fpqc}} \subseteq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ spanned by fpqc sheaves.

The subcategory inclusion $\mathcal{S}\mathrm{hv}_{\mathrm{fpqc}} \subseteq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ admits a left adjoint in the sheafification functor $L : \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S}) \to \mathcal{S}\mathrm{hv}_{\mathrm{fpqc}}$, and by Proposition SAG.6.2.3.1 the unit of the adjunction $X \to LX$ induces an equivalence upon applying QCoh. That is to say, quasi-coherent sheaves are formed on the level of fpqc sheaves. By Proposition SAG.6.2.3.4 the functor $\mathrm{QCoh} : \mathcal{S}\mathrm{hv}_{\mathrm{fpqc}}^{\mathrm{op}} \to \mathcal{P}\mathrm{r}^{\mathrm{St}}$ preserves limits.

2.6.3. *Descent for quasi-coherent sheaves.* Remark SAG.6.2.5.8 shows that $\mathrm{QCoh}(X)$ carries a t-structure with $\mathrm{QCoh}(X)_{\geq 0} \simeq \mathrm{QCoh}(X)^{\mathrm{cn}}$. Note however that, according to Warning SAG.6.2.5.9, this t-structure is in general quite ill-behaved, for instance failing to be compatible with fitered colimits.

Let us say that $X \in \mathcal{S}\mathrm{hv}_{\mathrm{fpqc}}$ is an *admissible spectral stack*[8] if the t-structure on $\mathrm{QCoh}(X)$ is both left and right complete, and compatible with sequential colimits. This includes both nonconnective spectral Deligne-Mumford stacks of Definition SAG.1.4.4.2, as well as quasi-geometric stacks of Definition SAG.9.1.0.1.

[8]This is a compromise notion, weaker than a nonconnective spectral Deligne-Mumford stack or quasi-geometric stack (or even Artin spectral stack, which should include smoothness conditions), yet stronger than just a fpqc sheaf. We have chosen to use it because the mentioned notions, in the form made available by SAG, include various other details not relevant to our discussion that we would rather not get into. Not to say that this extra structure is irrelevant - on the contrary, it is what enables most algebraic geometry. Alas, the theory of descent for quasi-coherent sheaves which we wish to study in this section, works in the more general setting just as well.

Given an admissible spectral stack X, or any functor $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ more generally, we may define its *underlying stack* to be the functor $X^{\heartsuit} : \mathrm{CAlg}^{\heartsuit} \to \mathcal{S}\mathrm{et}$ given by $R \mapsto \pi_0 X(R)$. Then there is an equivalence $\mathrm{QCoh}(X)^{\heartsuit} \simeq \mathrm{QCoh}^{\heartsuit}(X^{\heartsuit})$ between the heart of the t-structure on quasi-coherent sheaves on X, and the abelian category of classical quasi-coherent sheaves on $X^{\heartsuit}$.

Proposition 36 (Descent spectral sequence)**.** *Let X be an admissible spectral stack. Let $\mathscr{F}$ be a quasi-coherent sheaf on X. There exists an Adams-indexed conditionally convergent spectral sequence*

$$E_2^{s,t} \simeq \mathrm{H}^s(X^{\heartsuit}; \pi_t \mathscr{F}) \Rightarrow \pi_{t-s}\Gamma(X; \mathscr{F})$$

with values in the abelian category of $\pi_0\mathscr{O}(X^{\heartsuit})$-modules. When $\mathscr{F}$ is bounded above, then this spectral sequence converges unconditionally to $\pi_{t-s}\Gamma(X;\mathscr{F})$.

Proof. The admissability assumption on X implies that the ∞-category $\mathrm{QCoh}(X)$ with its usual t-structure and the global sections functor $\Gamma(X; -) : \mathrm{QCoh}(X) \to \mathrm{Mod}_{\Gamma(X;\mathscr{O}_X)}$ satisfy the assumptions of Variant 5. ☐

2.6.4. *Adams spectral sequence.* Let A be an $\mathbb{E}_\infty$-ring. The unit map $S \to A$ determines a map of nonconnective spectral sehemes $\operatorname{Spec} A \to \operatorname{Spec} S$, which we may view via their functors of points also as a map of fpqc sheaves. Let $N_\bullet$ denote the Čech nerve of this map, which is to say the simplicial object in $\mathcal{S}\mathrm{hv}_{\mathrm{fpqc}}$ given by

$$[n] \mapsto (\operatorname{Spec} A)^{n+1} \simeq \operatorname{Spec} A^{\otimes(n+1)}$$

with face and degeneracy maps defined by inclusions and projections of factors in products respectively. Define the *stack associated to* A to be $X_A \simeq |N_\bullet|$, with the geometric realization computed in the ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{fpqc}}$. The terminal morphism of fpqc sheaves $p : X_A \to \operatorname{Spec} S$ factors for every $n \geq 0$ as

$$\begin{array}{ccc} \operatorname{Spec} A^{\otimes(n+1)} & \xrightarrow{\quad i_{n+1} \quad} & X_A \\ & {\scriptstyle p_{n+1}} \searrow \quad \swarrow {\scriptstyle p} & \\ & \operatorname{Spec} S, & \end{array}$$

where $i_{n+1} : \operatorname{Spec} A^{\otimes(n+1)} \to X_A$ are the maps exhibiting X_A as the colimit of the functor $N_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{S}\mathrm{hv}_{\mathrm{fpqc}}$. Due to the continuity properties of the functor QCoh reviewed in the last subsection, pullbacks along i_{n+1} exhibit the equivalence of ∞-categories

$$\mathrm{QCoh}(X_A) \simeq \mathrm{Tot}(\mathrm{QCoh}(\operatorname{Spec} A^{\otimes(n+1)})). \tag{7}$$

According to this equivalence, any quasi-coherent sheaf $\mathscr{F}$ on X_A is equivalent to the limit $\mathscr{F} \simeq \varprojlim i_{n*} i_n^* \mathscr{F}$. On the level of global sections, and in view of the above commutative diagram, this gives

$$\Gamma(X_A; \mathscr{F}) \simeq p_* \mathscr{F} \simeq \varprojlim p_* i_{n*} i_n^* \mathscr{F} \simeq \varprojlim p_{n*} i_n^* \mathscr{F} \simeq \varprojlim \Gamma(\operatorname{Spec} A^{\otimes n}, i_n^* \mathscr{F}),$$

since the right adjoint functor p_* commutes with limits.

Given a spectrum $M \in \mathrm{Sp} \simeq \mathrm{QCoh}(\mathrm{Sp})$, we may choose $\mathscr{F}$ to be the constant quasi-coherent sheaf $p^* M$ on the geometric stack X_A. Using once again commutativity of the above diagram, we find that $i_n^* p^* M \simeq p_n^* M$. Under the identification $\mathrm{QCoh}(\operatorname{Spec} A^{\otimes n}) \simeq \mathrm{Mod}_{A^{\otimes n}}$ given by global sections, i.e. by the pushforward functor p_{n*}, its left adjoint $p_n^* : \mathrm{Mod}_{A^{\otimes n}} \to \mathrm{Sp}$ is the left adjoint to the forgetful functor from $A^{\otimes n}$-modules to spectra. Such a left adjoint is given by smashing with $A^{\otimes n}$, so we find that

$$\Gamma(X_A; p^* M) \simeq \varinjlim \ p_{n*} i_n^* p^* M \simeq \varinjlim p_{n*} p_n^* M \simeq \varprojlim A^{\otimes n} \otimes M.$$

Let us denote the right hand side by $M_A^\wedge$. It is the nilpotent completion of Boursfield and Adams (also merely called “derived completion” in [**?**]). A classical result of Bousfield identifies with the Bousfield localization $L_A M$ in many relevant cases:

Lemma 37 (Bousfield)**.** *Let A be connective ring spectrum such that the core of $\pi_0(A)$ is either $\mathbf{Z}[J^{-1}]$ for some set of primes J, or $\mathbf{Z}/n$ for some $n \geq 2$. Then the map $L_A M \to M_A^\wedge$ is an equivalence of spectra for any spectrum M.*

Furthermore we may in that case identify $\mathrm{QCoh}(X_A) \simeq L_A\mathrm{Sp}$, through (7) with the ∞-category of Bousfield A-local spectra, by a descent argument in [**?**]. In particular, it follows that this ∞-category with its usual t-structure satisfies all the requirements for X_A to be an admissible spectral stack.

Proposition 38 (A-based Adams spectral sequence)**.** *Let A be a connective $\mathbb{E}_\infty$-ring, satisfying the conditions o Lemma 37, and such that $\pi_*(A \otimes A)$ is a flat graded π_*A-module. There exists for every spectrum M an Adams-graded conditionally convergent spectral sequence*

$$E_2^{s,t} \simeq \mathrm{Ext}^{s,t}_{\pi_*(A\otimes A)}(\pi_* A, \pi_*(A \otimes M)) \Rightarrow \pi_{t-s} L_A M.$$

When M is bounded above, this spectral sequence converges unconditionally to $\pi_{t-s}L_A M$.

Proof. We apply Proposition 36 to X_A and the constant sheaf $\mathscr{F} = p^* M$. By the discussion preceeding the Proposition, the limiting term is the desired one. It remains to identify the E_2-term.

Since the t-structure on $\mathrm{QCoh}(X_A)$ is compatible with the equivalence (7), homotopy groups with respect to it are given by

$$\pi_k \mathscr{F} \simeq \varprojlim i_{n*} \pi_k(i_n^* \mathscr{F})$$

for all $k \in \mathbf{Z}$. By the discussion preceeding the Proposition we have

$$\pi_k(p^* M) \simeq \varprojlim i_{n*}\pi_k(p_n^* M) \simeq \varprojlim i_{n*}\pi_k(A^{\otimes n} \otimes M).$$

To determine the homotopy groups of the smash product $A^{\otimes n} \otimes M$, note first the evident equivalence

$$A^{\otimes(n+1)} \otimes M \simeq (A \otimes A)^{\otimes_A n} \otimes_A (A \otimes M).$$

The Kunneth spectral sequence of Proposition HA.7.2.1.19

$$E_2^{p,q} = \mathrm{Tor}_p^{\pi_* A}(\pi_*(A \otimes A), \pi_*(A \otimes M))_q \Rightarrow \pi_{p+q}(A \otimes A \otimes M)$$

has vanishing terms on the second page whenever $p \neq 0$ by the flatness assumption. Thus this spectral sequence degenerates, and we may induction on n to obtain an isomorphism

$$\pi_k(A^{\otimes(n+1)} \otimes M) \simeq \big(\pi_*(A \otimes A)^{\otimes n} \otimes_{\pi_* A} \pi_*(A \otimes M)\big)_k,$$

where the tensor product on the right hand side stands for the tensor product of graded modules over the graded ring $\pi_*(A)$, and the subscript k denotes the k-th graded part.

$$\begin{aligned}
\Gamma(X_A; \pi_k(p^* M)) &\simeq p_* \varprojlim i_{n*}\big(\pi_*(A \otimes A)^{\otimes n} \otimes_{\pi_* A} \pi_*(A \otimes M)\big)_k \\
&\simeq \varprojlim p_{n*}\big(\pi_*(A \otimes A)^{\otimes n} \otimes_{\pi_* A} \pi_*(A \otimes M)\big)_k \\
&\simeq \varprojlim \big(\pi_*(A \otimes A)^{\otimes n} \otimes_{\pi_* A} \pi_*(A \otimes M)\big)_k
\end{aligned}$$

The right-hand side may be recognized as (the k-th graded term of) the usual cobar construction for the comodule $\pi_*(A \otimes M)$ over the Hopf algebroid $(\pi_* A, \pi_*(A \otimes A))$. Standard Hopf coalgebroid technology (for instance, Corrolary A1.2.12 from [Rav]) shows that this cobar construction computes Hopf algebroid Ext from the unit, we get[9]

$$\mathrm{H}^s(X_A; \pi_t(p^* M)) \simeq \pi_{-s}\Gamma(X_A; \pi_t(p^* M)) \simeq \mathrm{Ext}^{s,t}_{\pi_*(A\otimes A)}(\pi_* A, \pi_*(A \otimes M))$$

[9] We are playing a little fast-and-loose here: the usual Hopf algebroid nonsense identifies Ext with the homology of the cobar complex, whereas we have homotopy groups of the inverse limit of its sequential form. To translate between the two, one should use Lurie's ∞-categorical Dold-Kan to pass from sequential objects to cosimplicial ones, and a version of the ordinary Dold-Kan to further pass to chain complexes. Keeping track, these equivalences take the limit of the sequential object to the totalization of the cosimplicial object, and homotopy groups of the latter to homology groups of the chain complex.

in the usual notation for bigraded Ext groups of comodules over a Hopf algebroid. □

Thus the A-based Adams spectral sequence is indeed always a descent spectral sequence for the spectral stack X_A associated to A. Applying the Proposition 38 to $A = \mathbf{F}_p$ and $A = \mathrm{MU}$, which both satisfy the required conditions, for $M = S$ the sphere spectrum, we recover the Adams spectral sequence

$$E_2^{s,t} \simeq \mathrm{Ext}^{s,t}_{\mathscr{A}_p^\vee}(\mathbf{F}_p, \mathbf{F}_p) \Rightarrow \pi_{t-s} S_p.$$

and the Adams-Novikov spectral sequence

$$E_2^{s,t} \simeq \mathrm{Ext}^{s,t}_{\pi_*(\mathrm{MU}\otimes\mathrm{MU})}(\pi_*\mathrm{MU}, \pi_*\mathrm{MU}) \Rightarrow \pi_{t-s} S.$$

respectively.

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