Chromatic homotopy theory via spectral algebraic geometry

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ANSS & prehistory of chromatics

- *The noble goal:* determining stable homotopy groups $\pi_*(S)$
- One approach: *the Adams spectral sequence*

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\pi_*(\mathbb{F}_p \otimes_S \mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \pi_{s-t}(S)^\wedge_p$$

- Another approach: ***the Adams-Novikov spectral sequence***

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  $\mathrm{MU} \dots$ the complex bordism spectrum.

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Formal groups

- A ***formal group*** *over a ring* R is a commutative group object $\widehat{\mathbb{G}}$ in 1-dimensional smooth formal R-schemes.
- So locally $\widehat{\mathbb{G}} \simeq \widehat{\mathbb{A}}^1_R = \mathrm{Spf}(R[\![x]\!])$ (choice of a coordinate)
- Group multiplication $\widehat{\mathbb{G}} \times \widehat{\mathbb{G}} \to \widehat{\mathbb{G}}$

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  cont. R-algebra map $R[\![x, y]\!] \leftarrow R[\![x]\!]$

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  $F(x, y) \in R[\![x, y]\!]$ satisfying some axioms

  $\cdots$ a ***formal group law*** *over* R
- Examples :
  - $\widehat{\mathbb{G}}_a = \widehat{\mathbb{A}}^1$ with $F(x, y) = x + y$
  - $\widehat{\mathbb{G}}_m = \widehat{\mathbb{A}}^1$ with $F(x, y) = (x + 1)(y + 1) - 1 = x + y + xy$

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Formal groups from spectra

- A ring spectrum A is ***complex periodic*** if
  1. ring multiplication induces $\pi_*(A) \otimes_{\pi_0(A)} \pi_2(A) \simeq \pi_{*+2}(A)$
  2. A is *complex orientable* (admits Chern classes c_i^A)
- Its *(classical)* ***Quillen formal group*** is

$$\widehat{\mathbb{G}}_A^{\mathcal{Q}_0} := \operatorname{Spf}\big(A^0(\mathrm{BU}(1))\big) \simeq^1 \widehat{\mathbb{A}}^1_{\pi_0(A)},$$

  with formal group law F such that

$$c_1^A(L \otimes L') = F(c_1^A(L), c_1^A(L'))$$

  for all complex line bundles L, L'.
- Examples: $\widehat{\mathbb{G}}^{\mathcal{Q}_0}_{\mathbb{Z}[\beta^{\pm 1}]} \simeq \widehat{\mathbb{G}}_a, \quad \widehat{\mathbb{G}}^{\mathcal{Q}_0}_{\mathrm{KU}} \simeq \widehat{\mathbb{G}}_m$

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Quillen's Theorem

$\mathrm{MP} := \mathrm{MU}[\beta^{\pm 1}] = \bigoplus_{i\in\mathbb{Z}} \Sigma^{2i}(\mathrm{MU})\dots$ ***periodic complex bordism***

Theorem (Quillen; classical version)

The formal group law of $\widehat{\mathbb{G}}^{\mathcal{Q}_0}_{\mathrm{MP}}$ is the universal formal group law.

- Let $\mathscr{M}^{\heartsuit}_{\mathrm{FG}}$ denote the ***moduli stack of formal groups***, i.e.

$$\{R\text{-points of } \mathscr{M}^{\heartsuit}_{\mathrm{FG}}\} \simeq \{\text{formal groups over } R\}$$

Theorem (Quillen; rephrasing)

There is a canonical equivalence of stacks

$$\underbrace{\mathscr{M}^{\heartsuit}_{\mathrm{FG}}}_{groupoid} \simeq \varinjlim \Big(\underbrace{\mathrm{Spec}\big(\pi_0(\mathrm{MP}\otimes_S \mathrm{MP})\big)}_{morphisms} \rightrightarrows \underbrace{\mathrm{Spec}\big(\pi_0(\mathrm{MP})\big)}_{objects} \Big).$$

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The formal group law of $\widehat{\mathbb{G}}^{\mathcal{Q}_0}_{\mathrm{MP}}$ is the universal formal group law.

- Let $\mathscr{M}^{\heartsuit}_{\mathrm{FG}}$ denote the ***moduli stack of formal groups***, i.e.

$$\{R\text{-points of } \mathscr{M}^{\heartsuit}_{\mathrm{FG}}\} \simeq \{\text{formal groups over } R\}$$

Theorem (Quillen; rephrasing)

There is a canonical equivalence of stacks

$$\underbrace{\mathscr{M}^{\heartsuit}_{\mathrm{FG}}}_{\textit{groupoid}} \simeq \varinjlim \Big(\underbrace{\mathrm{Spec}\big(\pi_0(\mathrm{MP}\otimes_S \mathrm{MP})\big)}_{\textit{morphisms}} \rightrightarrows \underbrace{\mathrm{Spec}\big(\pi_0(\mathrm{MP})\big)}_{\textit{objects}} \Big).$$

Quillen's Theorem and ANSS

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- *Dualizing line* of formal group $\widehat{\mathbb{G}}$ over R is the R-module

$$\omega_{\widehat{\mathbb{G}}} := \Omega^1(\widehat{\mathbb{G}})^{\widehat{\mathbb{G}}} \simeq T_{\mathbf{1}}^* \widehat{\mathbb{G}}$$

- *Example*: $\omega_{\widehat{\mathbb{G}}_A^{\mathcal{Q}_0}} \simeq \pi_2(A)$ for any A complex-periodic
- As R and $\widehat{\mathbb{G}}$ vary, get line bundle ω over $\mathscr{M}_{\mathrm{FG}}^{\heartsuit}$

Corollary (of Quillen's Theorem)

The second page of ANSS may be written as sheaf cohomology

$$E_2^{s,t} = \mathrm{H}^s(\mathscr{M}_{\mathrm{FG}}^{\heartsuit}; \omega^{\otimes t}) \implies \pi_{2t-s}(S)$$

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Chromatic homotopy theory

Sp ... stable homotopy category of spectra

Motto

$\{$*algebraic geometry of* $\mathcal{M}_{\mathrm{FG}}^{\heartsuit}\} \leftrightsquigarrow \{$*structural properties of* $\mathrm{Sp}\}$

This has been exploited for various uses:

- "Designer spectra" : $K(n), E(n), \mathrm{tmf}$, etc.
- Bousfield localizations L_n, $L_{K(n)}$
- Nilpotence & Periodicity Theorems
- Thick Subcategory Theorem - the Balmer spectrum of Sp

Goal

Show that not just the E_2-page, but all of the ANSS comes from (a spectral version of) the moduli stack of formal groups.

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1 Intro to chromatics

2 Spectral algebraic geometry

3 Chromatics via SAG

Basic idea of SAG

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Replace commutative rings everywhere with $\mathbb{E}_\infty$-ring spectra.

Algebraic Geometry	$\rightsquigarrow$	*Spectral Algebraic Geometry*
$\mathrm{Aff}^{\mathrm{op}} := \mathrm{CAlg}^{\heartsuit}$	$\rightsquigarrow$	$\mathrm{SpAff}^{\mathrm{op}} := \mathrm{CAlg}$
commutative rings		$\mathbb{E}_\infty$-rings
- Toën-Vezzosi, Lurie [SAG]

1st Approach (*Ringed spaces*)

*A **spectral scheme** is a pair $(X, \mathscr{O}_X)$ of a topological space X and a sheaf of $\mathbb{E}_\infty$-rings $\mathscr{O}_X$ on X, satisfying some axioms.*

- $X^{\heartsuit} := (X, \pi_0(\mathscr{O}_X))$ is the *underlying classical scheme* of X
- $\pi_i(\mathscr{O}_X)$ are quasi-coherent sheaves on $X^{\heartsuit}$

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SAG via Functor of points

2nd Approach (*Functor of points*)

*A **spectral stack** is a functor $X : \mathrm{CAlg} \to \mathcal{S}$ from $\mathbb{E}_\infty$-rings to the ∞-category of spaces $\mathcal{S}$, satisfying flat descent.*

- FOP of a ringed-space-style spectral scheme $(X, \mathcal{O}_X)$ is

$$X(R) := \mathrm{Map}_{\mathrm{SpSch}}(\mathrm{Spec}(R), (X, \mathcal{O}_X))$$

- Spectral affine $\mathrm{Spec}(A)$ via FOP, for an $\mathbb{E}_\infty$-ring A

$$(\mathrm{Spec}(A))(R) \simeq \mathrm{Map}_{\mathrm{CAlg}}(A, R)$$

- Better suited for studying moduli problems:

$$\mathscr{M}(R) = \{\textit{objects in question over } R\}$$

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Geometric spectral stacks

Definition

A spectral stack X is ***geometric*** if[2] it admits a faithfully flat cover $\mathrm{Spec}(A) \to X$ for some $\mathbb{E}_\infty$-ring A.

- It may be written as a geometric realization

$$X \simeq |\mathrm{Spec}(A^\bullet)|$$

for a flat cosimplicial diagram of $\mathbb{E}_\infty$-rings $A^\bullet$

- *Global functions* and the *quasi-coherent sheaves* on X are

$$\mathscr{O}(X) := \mathrm{Tot}(A^\bullet), \qquad \mathrm{QCoh}(X) := \mathrm{Tot}(\mathrm{Mod}_{A^\bullet}),$$

so that $\mathscr{O}(\mathrm{Spec}(A)) \simeq A$ and $\mathrm{QCoh}(\mathrm{Spec}(A)) \simeq \mathrm{Mod}_A$

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- Geometric spectral stack $X \simeq |\mathrm{Spec}(A^\bullet)|$ as before
- Its ***underlying ordinary stack*** $X^\heartsuit := |\mathrm{Spec}(\pi_0(A^\bullet))|$
- Its ***connective cover*** $\tau_{\geq 0}(X) := |\mathrm{Spec}(\tau_{\geq 0}(A^\bullet))|$
- Related as $X \to \tau_{\geq 0}(X) \leftarrow X^\heartsuit$
- Unless X is connective, $X^\heartsuit \neq X|_{\mathrm{CAlg}^\heartsuit}$
- $\mathscr{F} \in \mathrm{QCoh}(X) \rightsquigarrow \pi_i(\mathscr{F}) \in \mathrm{QCoh}(X^\heartsuit)^\heartsuit$
- ***Descent spectral sequence***

$$E_2^{s,t} = \underbrace{\mathrm{H}^s(X^\heartsuit; \pi_t(\mathscr{F}))}_{\text{sheaf cohomology in AG}} \implies \underbrace{\pi_{t-s}(\Gamma(X;\mathscr{F}))}_{\text{global sections in SAG}}$$

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Goal (Refined)

Exhibit the ANSS as the DSS for a geometric spectral stack X.

This spectral stack X will need to satisfy

1. Its underlying ordinary stack is $X^\heartsuit \simeq \mathscr{M}_{\mathrm{FG}}^\heartsuit$
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- Idea: *X will be a moduli stack of formal groups in SAG (with extra structure).*
- ***Formal groups over $\mathbb{E}_\infty$-rings*** indeed make sense in SAG, following Lurie [EII2]. Two caveats:
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Definition (Lurie)

An ***orientation*** on a formal group $\widehat{\mathbb{G}}$ over an $\mathbb{E}_\infty$-ring A is *(an element in $\pi_2(\widehat{\mathbb{G}}(\tau_{\geq 0}(A)))$, which induces by linearization)* an equivalence of A-modules $\omega_{\widehat{\mathbb{G}}} \simeq \Sigma^{-2}(A)$.

- *Example*: ***Quillen formal group*** $\widehat{\mathbb{G}}^{\mathcal{Q}}_A := \mathrm{Spf}\big(C^*(\mathrm{BU}(1); A)\big)$ for any complex-oriented $\mathbb{E}_\infty$-ring A
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The ***moduli stack of oriented formal groups*** $\mathscr{M}^{\mathrm{or}}_{\mathrm{FG}} : \mathrm{CAlg} \to \mathcal{S}$ is the functor sending an $\mathbb{E}_\infty$-ring A to the space of all oriented formal groups over A.

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Oriented formal groups

Definition (Lurie)

An ***orientation*** on a formal group $\widehat{\mathbb{G}}$ over an $\mathbb{E}_\infty$-ring A is *(an element in* $\pi_2(\widehat{\mathbb{G}}(\tau_{\geq 0}(A)))$*, which induces by linearization)* an equivalence of A-modules $\omega_{\widehat{\mathbb{G}}} \simeq \Sigma^{-2}(A)$.

- *Example*: ***Quillen formal group*** $\widehat{\mathbb{G}}^Q_A := \mathrm{Spf}\big(C^*(\mathrm{BU}(1); A)\big)$ for any complex-oriented $\mathbb{E}_\infty$-ring A
- Any oriented formal group is of that form

Definition

The ***moduli stack of oriented formal groups*** $\mathscr{M}^{\mathrm{or}}_{\mathrm{FG}} : \mathrm{CAlg} \to \mathcal{S}$ is the functor sending an $\mathbb{E}_\infty$-ring A to the space of all oriented formal groups over A.

Main result

Theorem ([Gre21])

- *The functor $\mathscr{M}^{\mathrm{or}}_{\mathrm{FG}}$ is a geometric spectral stack.*
- *It satisfies the desired conditions:*
  1. *Its underlying ordinary stack is $(\mathscr{M}^{\mathrm{or}}_{\mathrm{FG}})^{\heartsuit} \simeq \mathscr{M}^{\heartsuit}_{\mathrm{FG}}$*
  2. $\pi_t(\mathscr{O}_{\mathscr{M}^{\mathrm{or}}_{\mathrm{FG}}}) \simeq \begin{cases} \omega^{\otimes t} & t \text{ even} \\ 0 & t \text{ odd} \end{cases}$
  3. *Its $\mathbb{E}_\infty$-ring of global functions is $\mathscr{O}(\mathscr{M}^{\mathrm{or}}_{\mathrm{FG}}) \simeq S$*
- *The DSS* $\boxed{E_2^{s,t} = \mathrm{H}^s(\mathscr{M}^{\heartsuit}_{\mathrm{FG}}; \pi_t(\mathscr{O}_{\mathscr{M}^{\mathrm{or}}_{\mathrm{FG}}})) \implies \pi_{2t-s}(\mathscr{O}(\mathscr{M}^{\mathrm{or}}_{\mathrm{FG}}))}$ *is canonically isomorphic to the ANSS.*
- *There is a canonical equivalence of ∞-categories*

$$\Gamma(\mathscr{M}^{\mathrm{or}}_{\mathrm{FG}}; -) : \mathrm{IndCoh}(\mathscr{M}^{\mathrm{or}}_{\mathrm{FG}}) \simeq \mathrm{Sp} : - \otimes \mathscr{O}_{\mathscr{M}^{\mathrm{or}}_{\mathrm{FG}}}$$

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Idea of proof.

Pick an $\mathbb{E}_\infty$-structure on MP $\rightsquigarrow$ the map $\mathrm{Spec}(\mathrm{MP}) \to \mathscr{M}^{\mathrm{or}}_{\mathrm{FG}}$, classifying the Quillen formal group $\widehat{\mathbb{G}}^{\mathcal{Q}}_{\mathrm{MP}}$, is a flat cover. We obtain a simplicial presentation

$$\mathscr{M}^{\mathrm{or}}_{\mathrm{FG}} \simeq \left|\mathrm{Spec}(\mathrm{MP}^{\otimes \bullet+1})\right|,$$

from which the rest follows via bar construction identifications, e.g.

$$\mathrm{DSS} \simeq \{\text{SS for cosimplicial spectrum } \mathrm{MP}^{\otimes \bullet+1}\} \simeq \mathrm{ANSS}. \qquad \square$$

Motto

The spectral stack $\mathscr{M}^{\mathrm{or}}_{\mathrm{FG}}$ is the witness of the chromatic connection between spectra and formal groups.

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Further results

This motto can be substantiated further:

Theorem ([Gre21b])

There exists a stratification by open spectral substacks

$$\mathscr{M}_{\mathrm{FG}}^{\mathrm{or},\leq 1} \subseteq \mathscr{M}_{\mathrm{FG}}^{\mathrm{or},\leq 2} \subseteq \mathscr{M}_{\mathrm{FG}}^{\mathrm{or},\leq 3} \subseteq \cdots \mathscr{M}_{\mathrm{FG}}^{\mathrm{or}} \otimes_S S_{(p)}$$

which induces on ordinary classical substacks the height filtration

$$\mathscr{M}_{\mathrm{FG}}^{\heartsuit,\leq 1} \subseteq \mathscr{M}_{\mathrm{FG}}^{\heartsuit,\leq 2} \subseteq \mathscr{M}_{\mathrm{FG}}^{\heartsuit,\leq 3} \subseteq \cdots \mathscr{M}_{\mathrm{FG}}^{\mathrm{or}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$$

and on sheaves the chromatic filtration

$$L_1\mathrm{Sp} \leftarrow L_2\mathrm{Sp} \leftarrow L_3\mathrm{Sp} \leftarrow \cdots \leftarrow \mathrm{Sp}_{(p)}.$$

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