Fun classical tidbit via ∞-abstract nonsense

This is extremely well-known and very easy whichever way one chooses to go about it. However, the ∞-categorical proof by abstract nonsense is beautiful in a silly way that warms the cockles of my heart, so let us indulge in it regardless.

Theorem 1 (The monodromy determines the local system)**.** *Let X be a connected space and G a group. There is an equivalence of ∞-categories*

$$\mathrm{Fun}\,(X, \mathrm{B}G) \simeq \mathrm{Hom}_{\mathrm{Grp}}(\pi_1(X), G)/G$$

from the ∞-category of G-local systems on X to the homotopy quotient of the set of group homomorphisms $\pi_1(X) \to G$ under the conjugation action of G.

Recall that $\mathrm{B}G$ is the classifying space of the group G, or equivalently the homotopy quotient $\mathrm{B}G \simeq */G$. It is also called the delooping of G, since the Bordman-Vogt-May Recognition Principle states that it appears in the equivalence of ∞-categories

$$\mathrm{B} : \mathrm{Mon}^{\mathrm{gp}} \simeq \mathcal{S}_*^{\geq 1} : \Omega$$

between grouplike $\mathbb{E}_1$-space (i.e. homotopy coherent groups) and pointed connected spaces. In particular, $\mathrm{B}G$ is the essentially unique pointed connected space satisfying $\Omega\mathrm{B}G \simeq G$.

Before we can give the proof, we need two simple observations.

Lemma 2. *Identifying spaces with ∞-groupoids, the canonical map*

$$\mathrm{Map}_{\mathcal{S}}(X, Y) \to \mathrm{Fun}\,(X, Y)$$

is an equivalence of ∞-categories.

Proof. The fully faithful embedding $\mathcal{S} \to \mathcal{C}\mathrm{at}_\infty$, under which spaces are identified with ∞-groupoids, is the essentially unique colimit preserving functor sending the contractible space $*$ to the terminal ∞-category, abusively also denoted $*$. Note also that it follows from the adjunction equivalence

$$\mathrm{Map}_{\mathcal{C}\mathrm{at}_\infty}(\mathcal{C}, \mathrm{Fun}\,(\mathcal{D}, Y)) \simeq \mathrm{Map}_{\mathcal{C}\mathrm{at}_\infty}(\mathcal{C} \times \mathcal{D}, Y),$$

which holds for any pair of ∞-categories $\mathcal{C}$, $\mathcal{D}$, that the functor $\mathrm{Fun}\,(-, Y) : \mathcal{C}\mathrm{at}_\infty^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ commutes with all small limits. Consequently we have a chain of equivalences in the ∞-category $\mathcal{C}\mathrm{at}_\infty$

$$\mathrm{Fun}\,(X, Y) \simeq \varprojlim_X \mathrm{Fun}\,(*, Y) \simeq \varprojlim_X Y \simeq \varprojlim_X \mathrm{Map}_{\mathcal{S}}(*, Y) \simeq \mathrm{Map}_{\mathcal{S}}(X, Y)$$

as desired. $\square$

Lemma 3. *Let X and Y be pointed spaces. The canonical map* $\mathrm{Map}_{\mathcal{S}_*}(X, Y) \to \mathrm{Map}_{\mathcal{S}}(X, Y)$ *exhibits the homotopy equivalence to the homotopy quotient*

$$\mathrm{Map}_{\mathcal{S}}(X, Y) \simeq \mathrm{Map}_{\mathcal{S}_*}(X, Y)/\Omega Y.$$

Proof. The ∞-category of pointed spaces is by definition the overcategory $\mathcal{S}_* \simeq \mathcal{S}_{*/}$ of spaces over the terminal space $*$. As such, the pointed mapping space can be expressed as the pullback square in the ∞-category of spaces

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{S}_*}(X, Y) & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{S}}(X, Y) & \longrightarrow & Y, \end{array}$$

where the left vertical arrow comes from the forgetful functor $\mathcal{S}_* \to \mathcal{S}$, the right vertical arrow encodes the basepoint of Y, and the lower horizontal arrow is given by composition with the basepoint map $* \to X$.

On the other hand, let G be any grouplike $\mathbb{E}_1$-space and T a space with a G-action. The equivalence of ∞-categories between spaces with a G-action $\mathrm{Mod}_G(\mathcal{S}) \simeq \mathrm{Fun}\,(\mathrm{B}G, \mathcal{S})$ and spaces over the delooping $\mathcal{S}_{/\mathrm{B}G}$ is a simple instance of the straignthening-unstraigthening correspondence. It is in particular given by sending $T \to T/G$, from which it follows that the homotopy quotient is thus determined up to a space contractible choices by fitting into a pullback square

$$\begin{array}{ccc} T & \longrightarrow & * \\ \downarrow & & \downarrow \\ T/G & \longrightarrow & \mathrm{B}G \end{array}$$

in the ∞-category $\mathcal{S}$. Applying this to the case of $T \simeq \mathrm{Map}_{\mathcal{S}_*}(X, Y)$ and $G \simeq \Omega Y$, we obtan the desired identification of the homotopy quotient in light of the previous paragraph. □

Remark 4. The conclusion of Lemma 3 is particularly famous in what it induces on π_0. Then mapping spaces then become identified with Hom sets in the homtopy categories, while $\pi_0(\Omega Y) = \pi_1(Y)$ recovers the fundamental group, and the homtopy quotient reduces to the ordinary quotient. Hence we obtain the bijection

$$\mathrm{Hom}_{\mathrm{h}\mathcal{S}}(X, Y) \simeq \mathrm{Hom}_{\mathrm{h}\mathcal{S}_*}(X, Y)/\pi_1(Y),$$

where we the quotient on the right is the ordinary and not the homotopy quotient. On the level of mapping spaces however, the action need not factor through the truncation map $\Omega Y \to \pi_1(Y)$, and may be much richer.

Proof of Theorem 1. The proof consists of the chain of homotopy equivalences, each of which we will subsequently justify:

$$\begin{aligned}
&(1) & \mathrm{Fun}\,(X, \mathrm{B}G) &\simeq \mathrm{Map}_{\mathcal{S}}(X, \mathrm{B}G) \\
&(2) & &\simeq \mathrm{Map}_{\mathcal{S}_*}(X, \mathrm{B}G)/G \\
&(3) & &\simeq \mathrm{Map}_{\mathrm{Mon}^{\mathrm{gp}}}(\Omega X, G)/G \\
&(4) & &\simeq \mathrm{Hom}_{(\mathrm{Mon}^{\mathrm{gp}})^{\heartsuit}}(\pi_0(\Omega X), G)/G \\
&(5) & &\simeq \mathrm{Hom}_{\mathrm{Grp}}(\pi_1(X), G)/G.
\end{aligned}$$

Equivalences (1) and (2) follow from Lemmas 2 and 3 respectively. Equivalence (3) is an application of the Bordman-Vogt-May Recognition Principle. Equivalene (4) comes from the adjunction $\pi_0 : \mathrm{Mon}^{\mathrm{gp}} \rightleftarrows (\mathrm{Mon}^{\mathrm{gp}})^{\heartsuit}$ whose right adjoint is the inclusion of the discrete grouplike $\mathbb{E}_1$-spaces into arbitrary ones. Equivalence (5) comes from the identification $(\mathrm{Mon}^{\mathrm{gp}})^{\heartsuit} \simeq \mathrm{Grp}$ between the ∞-category of discrete grouplike $\mathbb{E}_1$-spaces and groups and the ordinary category of groups, together with the obvious equality $\pi_0(\Omega X) \simeq \pi_1(X)$. □

Remark 5. For a not-necessarily-connected space X, the conclusion of Theorem 1 would change to predict the homotopy equivalence

$$\mathrm{Fun}\,(X, \mathrm{B}G) \simeq \mathrm{Hom}_{\mathrm{Grpd}}(\pi_{\le 1}(X), G)/G,$$

with the Hom set on the right taken in groupoids, with group G should be considered on the right as a single-object groupoid. Here we would use in place of the Bordman-Vogt-May Recognition Principle the identification between spaces and ∞-groupoids, which recovers ordinary groupoids as the subcategory $\mathrm{Grpd} \simeq \mathcal{S}^{\le 1}$ of 1-truncated objects. In light of this, fundamental groupoid $\pi_{\le 1}(X)$ becomes identified with the 1-truncation $\pi_{\le 1}(X) \simeq \tau_{\le 1} X$.