Tangent Space to a Character Variety

Goal: Determine tangent space to char. variety

$$\boxed{T_\rho\left(\mathrm{Hom}(\pi_1(\Sigma),G)/G\right) \cong H^1(\Sigma;\, \mathfrak{g}_\rho)}$$

ass. bundle to the G-bdle on Σ, corresp to ρ along the adjoint rep $G \circlearrowright \mathfrak{g}$.

Ex: Let Σ be a surface, so that $\Sigma \simeq B\Gamma$ for $\Gamma = \pi_1(\Sigma)$.

Then $\underbrace{\mathrm{VectBun}^b(\Sigma)}_{\text{vect bun + flat conn}} \simeq \underbrace{\mathrm{Rep}}_{\text{linear reps}}(G)$, and $\underbrace{H^i(\Sigma; V)}_{\text{vb cohom}} \cong \underbrace{H^i(\Gamma; V)}_{\text{group cohom}}$

We may view ρ as (conj class) of a homom. $\rho: \Gamma \to G$

and the vect. bdle $\mathfrak{g}_\rho$ corresp. to the linear rep:

$$\Gamma \xrightarrow{\rho} G \xrightarrow{ad} GL(\mathfrak{g})$$

$\Rightarrow$ $H^1(\Sigma; \mathfrak{g}_\rho) \cong \underbrace{H^1(\Gamma; \rho)}_{\text{grp cohom}}$

A bit more generally: consider mapping $\underbrace{\text{"stack"}}_{\text{a red herring:}}$ $\underline{\mathrm{Map}}(\Sigma, X)$

think mfld, or orbifold ... like a mfld, but may have some non-mfldy pts = stacky

Defined so that for any other mfld (or stack) M:

$$\boxed{\underline{\mathrm{Map}}(M, \underline{\mathrm{Map}}(\Sigma, X)) \simeq \underline{\mathrm{Map}}(M \times \Sigma, X)}$$

($\Leftarrow$ may need to be taken in stacks to exist)

Ex: •) For $X = BG$, the classifying stack of (flat) G-bundles, we get:

$$\underline{\mathrm{Map}}(\Sigma, BG) \simeq \underbrace{\mathrm{Bun}^b_G(\Sigma)}_{\text{"space" of flat G-bdls on }\Sigma\text{ = G-local systems on }\Sigma}$$

Special case: $\Sigma = B\Gamma$: $\boxed{\underline{\mathrm{Map}}(B\Gamma, BG) \simeq \mathrm{Hom}_{Grp}(\Gamma, G)/G}$ ... the char. variety!

$\pi_1(B\Gamma) \cong \Gamma$ requires choice of bspt $\hookrightarrow$ conj action

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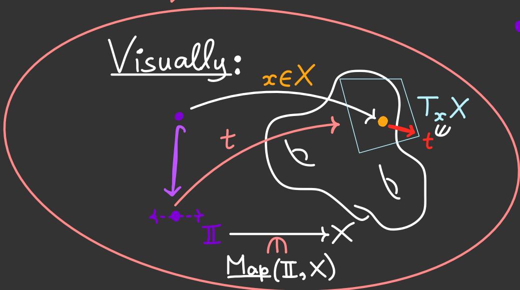
••) For $X = \mathbb{I}$ $:= \underbrace{\mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))}_{\text{red herring AG notation}}$, the "free-flowing tangent vector" or "infinitesimal line segment" looking roughly like ⟵ only one pt + 1-d "stacky fuzz" around it

(functions on $\mathbb{I}$ are $k[\varepsilon]/(\varepsilon^2) \ni a + \varepsilon b$; k = field we are over, e.g. $\mathbb{R}$ or $\mathbb{C}$; a = value at the unique pt; b = 1-st derivative where $\varepsilon^2 = 0$, ε = an infinitesimal)

we get: $\boxed{\underline{\mathrm{Map}}(\mathbb{I}, X) \simeq TX}$ ... the tangent bundle on X.

The inclusion $* \hookrightarrow \mathbb{I}$ induces on mapping stacks

$$\begin{array}{ccc} \underline{\mathrm{Map}}(\mathbb{I}, X) & \longrightarrow & \underline{\mathrm{Map}}(*, X) \\ \wr\wr & & \wr\wr \\ TX & \xrightarrow[\text{bundle proj}]{} & X \end{array}$$



Now we know enough to determine the tangent space to a mapping stack!

Fix a map $f\colon \Sigma \longrightarrow X$, viewing it as a pt $f \in \underline{\mathrm{Map}}(\Sigma, X)$. Then we compute:

$$
\begin{aligned}
\boxed{T_f\big(\underline{\mathrm{Map}}(\Sigma,X)\big)} &\simeq T\big(\underline{\mathrm{Map}}(\Sigma,X)\big) \times_{\underline{\mathrm{Map}}(\Sigma,X)} \{f\} \\
&\simeq \underline{\mathrm{Map}}\big(\mathbb{I}, \underline{\mathrm{Map}}(\Sigma,X)\big) \times_{\underline{\mathrm{Map}}(\Sigma,X)} \{f\} \\
&\simeq \underline{\mathrm{Map}}(\mathbb{I}\times\Sigma, X) \times_{\underline{\mathrm{Map}}(\Sigma,X)} \{f\} \quad \ldots\ldots \\
&\simeq \underline{\mathrm{Map}}\big(\Sigma, \underline{\mathrm{Map}}(\mathbb{I},X)\big) \times_{\underline{\mathrm{Map}}(\Sigma,X)} \{f\} \\
&\simeq \underline{\mathrm{Map}}(\Sigma, TX) \times_{\underline{\mathrm{Map}}(\Sigma,X)} \{f\} \quad \ldots\ldots \\
&\simeq \underline{\mathrm{Map}}(\Sigma, f^*TX) \times_{\underline{\mathrm{Map}}(\Sigma,\Sigma)} \{\mathrm{id}_\Sigma\} \\
&\simeq \boxed{\Gamma(\Sigma; f^*TX)}
\end{aligned}
$$

Diagramatically:

$$
\begin{tikzcd}
\Sigma \arrow[r, hook] \arrow[dr, "f"'] & \mathbb{I}\times\Sigma \arrow[d, dashed] \\
& X
\end{tikzcd}
\qquad \Longleftrightarrow \qquad
\begin{tikzcd}
\Sigma \arrow[r, dashed] \arrow[dr, "f"'] & TX \arrow[d, "pr"] \\
& X
\end{tikzcd}
$$

Remark: We get a bit more from the proof: not just the fiberwise statement, but also $T(\underline{\mathrm{Map}}(\Sigma,X)) \simeq \underline{\mathrm{Map}}(\Sigma,TX)$.

Awesome, we're almost done! We want to apply this to $X = BG$ though.

$\boxed{\text{What is } T(BG)?}$ **Problem:** BG is a stacky stack $\Rightarrow$ its tangent spaces aren't just vector spaces, but can be chain cplxes

Well, it's sthg like a vect. bdle on BG, and those are
$$\mathrm{VectBun}^b(BG) \simeq \underbrace{\mathrm{Rep}(G)}_{\text{linear reps}}$$

turns out: $\boxed{T(BG) \simeq \mathfrak{g}[1]}$ placed into cohomol. degree 1 (~~not~~ 0)

adjoint rep

roughly, use facts: $TG \simeq G \times \mathfrak{g}$ ... trivial bdle

& delooping B

$\Big\downarrow$ pass. to tang. spaces

cohom. shift $[1]$

Altogether we get, for $\rho : \Sigma \longrightarrow BG$

$$\boxed{T_\rho\big(\underline{\mathrm{Map}}(\Sigma, BG)\big) \simeq \Gamma\big(\Sigma ; \rho^*(\mathfrak{g}[1])\big)}$$

Now Γ here is evaluated on a chain cplx of (flat) vect. bundles.
$\Rightarrow$ is actually derived global sections, i.e. for an ordinary fl. VB = local system V on Σ:

$$\underbrace{H^0}_{\text{cohomology}}\big(\underbrace{\Gamma(\Sigma ; V[n])}_{\text{of a chain cplx}}\big) \cong \underbrace{H^n}_{\text{vb/loc sys}}(X ; V)_{\text{cohomology}}.$$

OTOH, identifying ρ with a G-bdle $P \to \Sigma$, then $\rho^*(\mathfrak{g})$ is identified with the associated vector bdle $\rho^*(\mathfrak{g}) \simeq \mathfrak{g}_P := P \times_G \mathfrak{g} \left(= (P\times\mathfrak{g})/G\right)$

Thus:

$$\boxed{\begin{aligned} T_\rho^{class}\big(\underline{\mathrm{Map}}(\Sigma, BG)\big) &= H^0\Big(T_\rho\big(\underline{\mathrm{Map}}(\Sigma, BG)\big)\Big) \\ &= H^0\big(\Gamma(\Sigma ; \mathfrak{g}_P[1])\big) \\ &= H^1(\Sigma ; \mathfrak{g}_P) \end{aligned}}$$

(Also works for "generalized" char. var. $\underline{\mathrm{Map}}(B\Gamma, BG) \simeq \mathrm{Hom}_{Grp}(\Gamma, G)/G$, with group cohom. of lin. Γ-reps on the RHS)

as promised.

Caveat: We have been using the stacky = orbifoldy version of the char. variety throughout

i.e. $\underline{\mathrm{Map}}(\Sigma, BG) \simeq \mathrm{Hom}_{Grp}(\pi_1(\Sigma), G) / G$

stacky = orbifoldy quotient

Now if ρ is such that $\mathrm{Stab}_G(\rho) = \{1\}$, then ρ is a non-stacky pt & it doesn't matter if we take the stacky or ordinary quotient.

$\Longrightarrow$ The above computation only computes T_ρ(usual non-stacky char. var.) for such ρ.

For other ρ, the computation remains valid for the stacky char. var.
(which is then better-behaved anyway.)