Chern Classes and the Chern Character

German Stefanich

Chern Classes

In this talk, all our topological spaces will be paracompact Hausdorff, and our vector bundles will be complex. Let $\mathrm{Bun}_{GL_n(\mathbb{C})}$ be the functor which sends each topological space to the set of isomorphism classes of its n-dimensional vector bundles. It is a basic fact that this functor factors through the homotopy category, so we have a functor

$$\mathrm{Bun}_{GL_n(\mathbb{C})} : \mathrm{HTop}^{\mathrm{op}} \to \mathrm{Set}$$

Observe that ordinary cohomology is another example of a functor between those two categories.

Definition. *A characteristic class for n-dimensional vector bundles is a natural transformation* $\mathrm{Bun}_{GL_n(\mathbb{C})} \implies H^*(-, \mathbb{Z})$

Since $\mathrm{Bun}_{GL_n(\mathbb{C})}$ is represented by $BU(n)$, characteristic classes are in correspondence with cohomology classes in $H^*(BU(n), \mathbb{Z})$. Our first goal will be to understand this ring.

First: $n = 1$

We know that $BU(1) = \mathbb{CP}^\infty$. Its cohomology is easy to compute via its cell decomposition, we have

$$H^*(BU(1), \mathbb{Z}) = \mathbb{Z}[x]$$

where $\deg x = 2$. In particular, we see that all characteristic classes for line bundles are polynomials in x.

Definition. *$c_1 = x$ is called the (universal) first Chern class.*

The first Chern class of a line bundle is then obtained by pullback of the universal one via a classifying map. This implies that c_1 vanishes for trivial line bundles, since the classifying map factors through a point. Conversely, since $BU(1) = K(\mathbb{Z}, 2)$, we see that c_1 is the universal cohomology class of degree 2. Therefore, if a line bundle has vanishing first Chern class, its classifying map has to be constant, and therefore the bundle has to be trivial. Therefore we see that for $n = 1$, vanishing of the Chern classes is equivalent to the bundle being trivial. This won't happen for $n > 1$.

Case $n > 1$

Let $U(1)^n \subset U(n)$ be the diagonal subgroup (one could carry out these arguments for a general group by taking a maximal torus). This inclusion induces a map

$$BU(1)^n \to BU(n)$$

which is invariant under the action of S_n permuting the factors on the left. If we think of $BU(1)^n$ as the classifying space for n-tuples of line bundles, then that map is induced by direct sum. The fiber is $U(n)/U(1)^n$ which is equivalent to the flag variety of $\mathbb{C}^n$ (indeed, the space $BU(1)^n$ may be obtained from $BU(n)$ from the splitting principle applied to the universal bundle).

From the above map we get a map in cohomology

$$\pi^* : H^*(BU(n), \mathbb{Z}) \to H^*(BU(1)^n, \mathbb{Z})^{S_n} = \mathbb{Z}[x_1, \ldots, , x_n]^{S_n} = \mathbb{Z}[e_1, \ldots, e_n]$$

Here x_i are the generators of the cohomology of each $BU(1)$ factor, and they are called the Chern roots. As usual, the e_i are the elementary symmetric polynomials on the x_i.

Theorem. *The map π^* is an isomorphism. In particular, all characteristic classes are polynomials in the c_i, where $\pi^* c_i = e_i$.*

Proof. We shall proceed by induction, splitting one copy of $BU(1)$ at a time. First, from the map $U(1) \times U(n-1) \to U(n)$ (mapping a pair of matrices to a block diagonal matrix) we get

$$BU(1) \times BU(n-1) \to BU(n)$$

The fiber of this map is $\mathbb{CP}^n$, whose cohomology is $\mathbb{Z}[x_1]/x_1^n$. The Leray-Hirsch theorem then tells us that $1, x_1, \ldots, x_1^{n-1}$ is a basis for $H^*(BU(1) \times BU(n-1), \mathbb{Z})$ as a $H^*(BU(n), \mathbb{Z})$-module. Write

$$x_1^n = c_1 x_1^{n-1} - c_2 x_1^{n-2} + c_3 x_1^{n-3} \ldots$$

with $c_i \in H^{2i}(BU(n), \mathbb{Z})$.

In particular, observe that $H^*(BU(n), \mathbb{Z}) \to H^*(BU(1) \times BU(n-1), \mathbb{Z})$ is injective. By an inductive argument, we see that $\pi^* : H^*(BU(n), \mathbb{Z}) \to H^*(BU(1)^n, \mathbb{Z})$ is also injective. From the two equalities

$$x_1^n = \pi^* c_1 x_1^{n-1} - \pi^* c_2 x_1^{n-2} + \pi^* c_3 x_1^{n-3} \ldots$$

and

$$x_1^n = e_1 x_1^{n-1} - e_2 x_1^{n-2} + e_3 x_1^{n-3} \ldots$$

one may conclude that $\pi^* c_i = e_i$. In particular, π^* is also surjective, so it is an isomorphism, as we wanted. □

Definition. *The total Chern class is $c = \sum c_i = 1 + c_1 + c_2 + \ldots$.*

We now take a look at what happens with Chern classes of direct sums of bundles. Denote by $c^{(d)}$ the total Chern class for d-dimensional vector bundles. Consider the map

$$U(k) \times U(l) \to U(k+l)$$
$$(\alpha, \beta) \to \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

It induces a map

$$BU(k) \times BU(l) \to BU(k+l)$$

which is the classifying map for the direct sum of the universal bundles on $BU(k)$ and $BU(l)$.

Theorem (Whitney product formula)**.** *The pullback of $c^{(k+l)}$ along the above map is $c^{(k)} \cup c^{(l)}$. Equivalently, if E, F are vector bundles on a space X, then $c(E \oplus F) = c(E) \cup c(F)$.*

Proof. This is an application of the splitting principle. Consider the diagram

$$\begin{array}{ccc} BU(1)^k \times BU(1)^l & \xrightarrow{=} & BU(1)^{k+l} \\ \downarrow & & \downarrow \\ BU(k) \times BU(l) & \longrightarrow & BU(k+l) \end{array}$$

All the maps in cohomology are injections, and the total Chern classes satisfy

$$c^{(k+l)} = \prod_1^{k+l} (1 + x_i)$$
$$c^{(k)} = \prod_1^{k} (1 + x_i)$$
$$c^{(l)} = \prod_{k+1}^{k+l} (1 + x_i)$$

so the theorem follows. □

Corollary. *Chern classes are stably invariant.*

By the splitting principle, the Chern classes are determined by their values on line bundles, and the Whitney product formula. A common approach to define Chern classes is to take these properties as axioms, and then show that there is a unique collection of classes satisfying those axioms.

Chern Character

Definition. *As before, let $x_1, \ldots, x_n$ be the Chern roots for n-dimensional vector bundles. The (universal) Chern character is defined to be*

$$ch = \exp x_1 + \exp x_2 + \ldots + \exp x_n \in \prod_i H^{2i}(BU(n), \mathbb{Q})$$

Theorem. *Let E, F be two vector bundles over a space X. Then*

i. $ch(E \oplus F) = ch(E) + ch(F)$

ii. $ch(E \otimes F) = ch(E) \cup ch(F)$

Proof. This is again an application of the splitting principle. We may assume E, F to be line bundles. The first item is immediate in this case. The second item reduces to showing that the first Chern class of a product of two line bundles is the sum of the first Chern classes of those bundles.

Consider the following diagram

$$\begin{array}{ccc} BU(1) \times BU(1) & \xrightarrow{\otimes} & BU(1) \\ {\scriptstyle \mathcal{O}(1)\times\mathcal{O}(1)} \uparrow & & \\ \mathbb{CP}^1 \times \mathbb{CP}^1 & & \end{array}$$

The map from projective space to $BU(1)$ is an isomorphism on H^2, so it suffices to show that

$$c_1(\mathcal{O}(1) \otimes \mathcal{O}(1)) = c_1(\mathcal{O}(1)) + c_1(\mathcal{O}(1))$$

Now, $\mathcal{O}(1) \otimes \mathcal{O}(1)$ is the pullback of the twisting sheaf by the Segre embedding. This is a degree two map so the above equality follows. □

From the above theorem we see that the Chern character gives a morphism

$$ch : K^0(X) \to H^{even}(X, \mathbb{Q})$$

Theorem. *Let X be a compact CW-complex. Then*

$$ch : K^0(X) \otimes \mathbb{Q} \to H^{even}(X, \mathbb{Q})$$

is an isomorphism

Remark. *Even if X is non-compact one gets a similar result, although one has to work with rationalized K-theory, which is not the same as $K^0(X) \otimes \mathbb{Q}$.*

Proof. This proof may be seen as a statement about spectra. The rationalized K-theory spectrum has a map to a product of Eilenberg-MacLane spectra, which is an rational isomorphism since both theories agree at the point. Then rational homotopy theory implies that these two spaces are equivalent.

To keep it simple, we shall now give a version of that argument using the language of cohomology theories. We have two cohomology theories

$$K^*_{\mathbb{Q}} = K^* \otimes \mathbb{Q}$$

and

$$\hat{H}^* = \prod_{j \in \mathbb{Z}} H^{*+2j}(-, \mathbb{Q})$$

The Chern character gives a natural transformation $K^0_{\mathbb{Q}} \to \hat{H}^0$. Moreover, from the diagram

$$\begin{array}{ccc} K^{-1}_{\mathbb{Q}}(X) & \overset{ch}{\dashrightarrow} & \hat{H}^{-1}_{\mathbb{Q}}(X) \\ \big\downarrow{\scriptstyle =} & & \big\downarrow{\scriptstyle =} \\ K^0_{\mathbb{Q}}(SX) & \xrightarrow{ch} & \hat{H}^0(SX) \end{array}$$

one is able to define ch in degree -1. From the periodicity that both theories have, ch may then be extended to a natural transformation $K_{\mathbb{Q}} \to \hat{H}$. It may be checked that it gives an isomorphism $K_{\mathbb{Q}}(pt) = \hat{H}(pt)$. Then one concludes that ch is an isomorphism for any compact CW-complex, by induction on the filtration given by the skeleta. □

A nice way of running the last inductive argument is by using the Atiyah-Hirsebruch spectral sequence

Theorem. *Let h be a cohomology theory and X be a finite CW complex. Then we have a spectral sequence*

$$E_2 = H^p(X, h^q(pt)) \implies h^{p+q}(X)$$

The construction of this sequence is similar to the Serre spectral sequence, by considering the filtration by skeleta of the CW. In fact, a similar sequence exists for a fibration $F \to Y \to X$, in which case one has

$$E_2 = H^p(X, h^q(F)) \implies h^{p+q}(Y)$$

This reduces to the Serre spectral sequence in the case that h is ordinary cohomology.

References

[1] `https://ncatlab.org/nlab/show/Chern+class`.

[2] A. Hatcher. Vector bundles and k-theory. `https://www.math.cornell.edu/~hatcher/VBKT/VB.pdf`.