Derived Morita theory and Hochschild Homology and Cohomology of DG Categories

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In this talk we will explore the idea that an algebra A over a field (ring, spectrum) k can be thought of as a way of encoding a category, namely A-mod its category of modules. And anything reasonable we define starting from the algebra (such as Hochschild homology or cohomology) should be defineable just in terms of the category, independent of the presentation. We say two algebras are Morita equivalent if their categories of modules are equivalent. One classical point of view is that Morita theory (and noncommutative geometry) study algebras up to Morita equivalence. In this talk we will take a slightly more modern point of view, and say that the main object of study is categories, which may or may not come from an algebra. Our end goal is to say what it means to take the Hochschild homology and cohomology of a category.

Before going on, we should say what we mean by category. Everything will be as homotopical (derived) as possible, so we will be working with dg-categories over k (or stable categories tensored over k if k is a spectrum, which is needed if one wishes to talk about topological Hochschild homology). So when we say A-mod we mean what is classically called the unbounded derived category of A-modules. Now there is a choice in this subject of whether one wants to work with big categories which have all colimits, or small categories which have small colimits and are idempotent complete. The former is a bit more general than the later, and the two are equivalent if one restricts to compactly generated categories and proper maps. Below we will work in the first context, so for us all categories will be presentable k-linear and cocomplete, unless we say otherwise. The condition of being cocomplete may look like a technical assumption, but it is in reality this what is capturing the Morita invariance. If one has an algebra A one may always form a category BA^{op} with one object with endomorphisms A^{op}. This is not cocomplete, but its free cocompletion is A-mod. So the condition of being cocomplete forces us to collapse Morita equivalence classes of algebras.

Compact generators

The first question we want to answer is, when does a category $\mathcal{C}$ come from an algebra, i.e, when is $\mathcal{C} = A$-mod for some A. The key observation is that A-mod is freely generated under colimits by one object (namely, A), whose endomorphism algebra is A^{op}. Therefore if

$\mathcal{C}$ is to be equivalent to A-mod, it has to have an object with those properties. We capture this in the following definition

Definition. *An object $x \in \mathcal{C}$ is said to be*

- *compact if* $\mathrm{Hom}(x, -)$ *is continuous, that is, it preserves colimits (equivalently, it preserves infinite direct sums).*
- *a generator if* $\mathrm{Hom}(x, -)$ *is conservative, that is,* $\mathrm{Hom}(x, y) = 0$ *if and only if* $y = 0$.

Observe that A is a compact generator of A-mod, because $\mathrm{Hom}(A, -)$ is the forgetful functor from A-mod to k-mod $=$ Vect. Moreover, A^n is also a compact generator. The compact objects in A-mod form the smallest subcategory of A-mod closed under finite colimits and retracts. In other words, they are retracts of finite complexes built out of copies of A. These are called perfect complexes over A. For $A = k$, perfect complexes are finite complexes of finite dimensional vector spaces (i.e, finite sums of shifts of k), which also happen to be generators of k-mod $=$ Vect.

One key result of Morita theory is the following

Theorem. *If $x \in \mathcal{C}$ is a compact generator of $\mathcal{C}$, then $\mathcal{C} = A$*-mod *for $A = \mathrm{End}(x)^{\mathrm{op}}$. The equivalence is given by $y \mapsto \mathrm{Hom}(x, y)$.*

Proof. By formal reasons, the functor $\mathrm{Hom}(x, -)$ has a left adjoint, which we denote by $x \otimes_A -$. It is determined by the fact that it distributes over colimits, and $x \otimes_A A = x$. We first show this is fully faithful. Take two A-modules M, N, and write them as colimits of copies of A:

$$\begin{aligned} M &= \mathrm{colim}_I\, A \\ N &= \mathrm{colim}_J\, A \end{aligned}$$

Then we have

$$\begin{aligned} \mathrm{Hom}(x \otimes_A M, x \otimes_A N) &= \mathrm{Hom}(\mathrm{colim}_I\, x, \mathrm{colim}_J\, x) \\ &= \lim_{I^{\mathrm{op}}} \mathrm{Hom}(x, \mathrm{colim}_J\, x) \\ &= \lim_{I^{op}} \mathrm{colim}_J \mathrm{Hom}(x, x) \\ &= \lim_{I^{op}} \mathrm{colim}_J \mathrm{Hom}(A, A) \\ &= \lim_{I^{op}} \mathrm{Hom}(A, \mathrm{colim}_J\, A) \\ &= \mathrm{Hom}(M, N) \end{aligned}$$

where we used the compactness of x to commute a Hom and a colimit.

Now we show that tensoring with x is surjective. Take $y \in \mathcal{C}$. By tensor-Hom adjunction we have a map

$$x \otimes_A \mathrm{Hom}(x, y) \to y$$

which we claim is an isomorphism. Since x is a generator, one only needs to show that

$$\mathrm{Hom}(x, x \otimes_A \mathrm{Hom}(x, y)) \to \mathrm{Hom}(x, y)$$

is an isomorphism. But the first object is

$$\mathrm{Hom}(x \otimes_A A, x \otimes_A \mathrm{Hom}(x, y)) = \mathrm{Hom}(A, \mathrm{Hom}(x, y)) = \mathrm{Hom}(x, y)$$

We leave it to the reader to verify that the above arrow is the identity of $\mathrm{Hom}(x, y)$, which finishes the proof. □

Example: Let $\mathcal{C} = k\text{-mod} = \mathrm{Vect}$. We already identified all the compact generators of Vect, which happen to be the perfect modules. Therefore we conclude that Vect is equivalent to modules over $\mathrm{End}(V)^{\mathrm{op}} = \mathrm{End}(V^*)$ for any perfect V. In particular when $V = k^n$ we recover the well known equivalence $k\text{-mod} = M_n\text{-mod}$ for M_n the matrix algebra.

We'll now see two (related) ways of think about this Morita equivalence. Let $[n] = \mathrm{Spec}(k^n)$ be a set with n-points, and consider $p : [n] \to \mathrm{pt}$ the projection to a point. Starting from this we can build a groupoid acting on $[n]$ by taking sucessive fiber products:

$$[n] \leftleftarrows [n] \times_{pt} [n] \mathrel{\substack{\leftarrow\\ \leftarrow\\ \leftarrow}} [n] \times_{\mathrm{pt}} [n] \times_{pt} [n] \dots$$

Concretely, the objects of this groupoid are the n-points, and there is one isomorphism for every pair of points. By the way it was constructed, this is equivalent to the trivial groupoid pt. Now we linearize this, that is, we take groupoid algebras. The groupoid algebra of the above groupoid is M_n. And the groupoid algebra of pt is k. So the Morita equivalence between matrix algebras and k can be thought of as a noncommutative version of the mentioned equivalence of groupoids. This gives further motivation for why we want to work Morita invariantly: this is a noncommutative version of requiring that whatever we say about a strict groupoid is invariant under equivalence.

The second and related point of view on this is about descent for the map p. Observe that M_n contains a copy of the algebra k^n, so in particular

$$\mathrm{QCoh}(\mathrm{pt}) = k\text{-mod} = M_n\text{-mod} \hookrightarrow k^n\text{-mod} = \mathrm{QCoh}([n])$$

The above composition is given by pullback along p. Now if we start with a sheaf on $[n]$, showing that it descends to a sheaf on pt amounts to enhancing the k^n-module structure to an M_n-module structure. In that sense, our Morita equivalence can be seen as telling us the descent data for p. In the language of monads, we have a monad p^*p_* acting on $QCoh([n])$. This is the same as an algebra on k^n-bimodules, the algebra being given by $p^*p_*k^n$. This can be computed to be M_n by the following base change diagram:

$$\begin{array}{ccc} [n] \times_{\mathrm{pt}} [n] & \longrightarrow & [n] \\ \downarrow & & \downarrow{\scriptstyle p} \\ [n] & \xrightarrow{\ p\ } & \mathrm{pt} \end{array}$$

So our second point of view on Morita equivalences is that they are descent statements in noncommutative geometry. In fact our theorem that compact generators induce Morita equivalences can be gotten as a corollary of the Barr-Beck-Lurie's monadicity theorem, which is a common ingredient in the proof of descent statements.

Example: Now we'll start with a descent situation and get a Morita equivalence out of it. Consider S^1 as a group in homotopy types, and consider the projection

$$\mathrm{pt} \xrightarrow{p} BS^1$$

Since these are homotopy types, the sheaf theory to consider in this example is local systems. There are two variants of the category of local systems on a given homotopy type X. One may consider local systems with possibly infinite dimensional fibers. We denote this category by $\mathrm{LocSys}(X)$. Or one may consider the ind-completion of the category of local systems with finite dimensional fibers. We denote this by $\mathrm{LocSys}'(X)$. These are the limit and the colimit of Vect over X. It's not hard to show that $\mathrm{LocSys}(X)$ is dual to $\mathrm{LocSys}'(X)$. These are the homotopy-theory version of the categories QCoh and IndCoh from derived algebraic geometry. They are in general different, and have different functoriality. For a map $f : X \to Y$, we have adjoint pairs of continuous functors $(f_!, f^*)$ for LocSys and (f^*, f_*) for LocSys$'$. These satisfy base change, and for $f : X \to \mathrm{pt}$ the projection, one has that $f_!$ and f_* take homology and cohomology, respectively.

Let's consider first the category $\mathrm{LocSys}(BS^1)$. Consider inside it the object $p_!(k_{\mathrm{pt}})$. Since $p_!$ has a continuous right adjoint, it is proper, so $p_!$ is compact. By adjunction, $\mathrm{Hom}(p_!, -)$ takes the fiber at the special point, and since BS^1 is connected this is conservative. Hence

$$\mathrm{LocSys}(BS^1) = Hom(p_!k, p_!k)^{\mathrm{op}}\text{-mod} = (p^*p_!k)^{\mathrm{op}}\text{-mod}$$

By base change, this is computed to be the homology of the circle, with the algebra structure coming from convolution. This is just a free commutative algebra on a generator λ of cohomological degree -1. It is commutative so we can forget the oppossite. Therefore we have the statement

$$\mathrm{LocSys}(BS^1) = H_\bullet(S^1)\text{-mod} = k[\lambda]\text{-mod}$$

Which is the statement of descent for p, and can be seen as arising from the monad $p^*p_!$.

Now consider the local system k_{BS^1} which is a compact object in $\mathrm{LocSys}'(BS^1)$. It is a generator since BS^1 is simply connected, so taking global sections is conservative. Therefore we see

$$\mathrm{LocSys}'(BS^1) = Hom(k_{BS^1}, k_{BS^1})^{\mathrm{op}}\text{-mod}$$

That algebra is $H^\bullet(BS^1) = k[u]$ for u of cohomological degree 2. It is commutative so again equivalent to its opposite. We therefore get the statement

$$\mathrm{LocSys}'(BS^1) = H^\bullet(BS^1)\text{-mod} = k[u]\text{-mod}$$

This can be thought of as saying that BS^1 is affine, and can be seen as arising from the monad $\pi_*\pi^*$ where π is the projection $BS^1 \to \mathrm{pt}$.

Putting everything together, we get a relationship between modules over $k[\lambda]$ and $k[u]$. The most straightforward thing to say is that

$$k[\lambda]\text{-mod}_{perf/k} = k[u]\text{-mod}_{perf}$$

where on the left we have $k[\lambda]$ modules which are perfect as k-modules, and on the right we have perfect $k[u]$-modules. This is a case of Koszul duality. Since S^1 and BS^1 are abelian groups, we can expect there to be a Fourier dual side to the above story. It exists, and happens to be a story about descent in formal derived algebraic geometry. What one considers is the map

$$i : \mathrm{pt} \to \operatorname{Spec} k[\lambda]$$

The $(i_*, i^!)$ adjuction at the level of IndCoh is monadic, so we have that that

$$\mathrm{IndCoh}(k[\lambda]) = \omega_{\mathrm{pt} \times_{\operatorname{Spec} k[\lambda]} \mathrm{pt}}\text{-mod} = k[u]\text{-mod}$$

(equivalently, this comes from looking at the compact generator $i_* k$). On the other hand, $\operatorname{Spec} k[\lambda]$ is obviously affine, so $\mathrm{QCoh}(\operatorname{Spec} k[\lambda]) = k[\lambda]\text{-mod}$. Fourier duality interchanges functions and distributions. In this case, this reads

$$\mathrm{QCoh}(\operatorname{Spec} k[\lambda]) = \mathrm{LocSys}(BS^1)$$

and

$$\mathrm{IndCoh}(\operatorname{Spec} k[\lambda]) = \mathrm{LocSys}'(BS^1)$$

The difference we mentioned before between the two categories of local systems is, on the algebraic geometric side, the difference between perfect sheaves and coherent sheaves. This is measured (in the case of quasi-smooth schemes) by the theory of matrix factorizations/singular support (which can be seen as an application of the calculation of the Hochschild cohomology of the category of coherent sheaves on a scheme).

Bimodules

Let A, B be two algebras. We are interested in continuous (i.e, colimit preserving) functors $F : A\text{-mod} \to B\text{-mod}$. Since A-mod is the free cocompletion of BA^{op}, we see that F determined from the data of $F(A)$, which is an object of B-mod with an A^{op} action. This is an $B - A$-bimodule. One can then recover F from the bimodule as $F(x) = F(A) \otimes_A x$. This proves the following

Theorem. $\mathrm{Funct}_{cont}(A\text{-mod}, B\text{-mod}) = B - A-\mathrm{bimod}$

An important particular case of this is when $A = B$. Then we see that endofunctors of A-mod consist of $A - A$-bimodules. The diagonal bimodule A corresponds to the identity endofunctor.

Example: The Morita equivalence $M_n\text{-mod} = k\text{-mod}$ comes from tensoring with the $k - M_n$-bimodule k^n

Example: The Koszul duality between $k[\lambda]$ and $k[u]$ comes from tensoring with the $k[\lambda] - k[u]$-bimodule k. Here the bimodule structure is not the trivial one, and arises from the joint actions of $H_{\bullet}(S^1)$ and $H^{\bullet}(BS^1)$ on $H^{\bullet}(\mathrm{pt})$.

Hochschild cohomology

Recall that given an algebra A, its Hochschild cohomology (with values in the diagonal bimodule) is $\mathrm{HH}^\bullet(A) = \mathrm{Hom}_{A-A}(A, A)$. By the observations from the previous sections, this coincides with $\mathrm{Hom}_{\mathrm{End}(A\text{-mod})}(\mathrm{id}, \mathrm{id})$. This is manifestly Morita invariant: it only depends on A-mod. We can now implement this definition for an arbitrary dg category

Definition. *Let $\mathcal{C}$ be a dg category. Its Hochschild cohomology (or center) is*

$$\mathrm{HH}^\bullet(\mathcal{C}) = \mathrm{Hom}_{\mathrm{End}(\mathcal{C})}(\mathrm{id}, \mathrm{id})$$

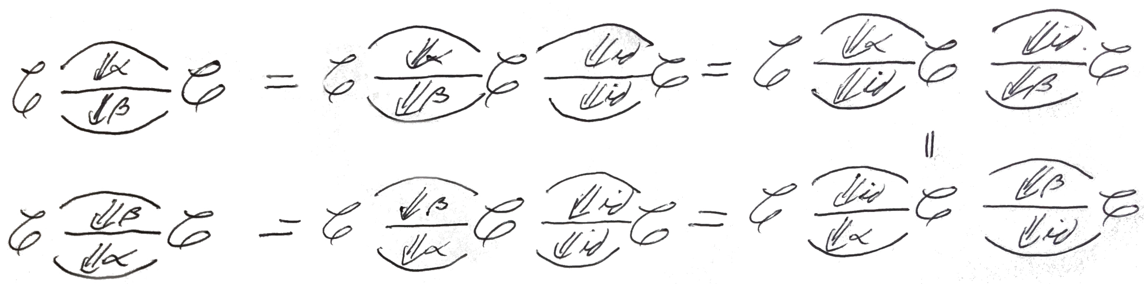
Informally, a cocycle in $\mathrm{HH}^\bullet(\mathcal{C})$ (i.e, a natural transformation from the identity to itself) is the data of an endomorphism $\alpha_x : x \to x$ for every object $x \in \mathcal{C}$, together with the data of commutativity $\alpha_y f = f\alpha_x$ for every arrow $f : x \to y$, plus higher compatibilities for sequences of composable arrows. More formally, this means that the Hochschild cohomology can be computed as the totalization of a certain cosimplicial object

$$\prod_{x\in\mathcal{C}} \mathrm{Hom}(x, x) \rightrightarrows \prod_{x,y\in\mathcal{C}} \mathrm{Hom}(\mathrm{Hom}(x, y), \mathrm{Hom}(x, y)) \overset{\rightarrow}{\rightrightarrows} \prod_{x,y,z\in\mathcal{C}} \mathrm{Hom}(\mathrm{Hom}(x, y)\otimes\mathrm{Hom}(y, z), \mathrm{Hom}(x, z)) \ldots$$

Here the first two arrows are, as we said before, pre and post-composition with the natural transformation, which have to be equalized by the Hochschild cohomology. This construction is an example of categorical end, and inmediately suggests a possible approach for discussing Hochschild homology, via the dual construction (a coend). Observe that if we allowed ourselves to consider the above complex for the category BA^{op}, we would arrive at the usual complex computing the Hochschild cohomology of A. So the above diagram may be though of as a many object version of that (the relevant observation is that for A-mod, the whole center is determined only by what happens on BA^{op}, so the above construction will give the same result as the classical thing one does for an algebra).

Now let's explore a few structures that one has on Hochschild cohomology, from this point of view.

- Cup product: this is just the algebra structure on $\mathrm{HH}^\bullet(\mathcal{C})$ arising from the fact that it is the endomorphisms of an object in a dg category.
- E_2-structure: informally, this structure is the data of an identification $\alpha\beta = \beta\alpha$ for every pair of cocycles $\alpha, \beta \in \mathrm{HH}^\bullet(\mathcal{C})$. We can see this at two levels. Objectwise, $\alpha_x\beta_x = \beta_x\alpha_x$ by applying the fact that α is in the center, so it commutes with all endomorphisms. Another way of thinking about this is that the E_2 structure arises from a manipulation of 2-cells in the category of categories, as follows:



- Action on Hochschild homology. Before we define Hochschild homology, let's indicate why it should be acted on by $\mathrm{HH}^{\bullet}$. Observe that by definition $\mathrm{HH}^{\bullet}\mathcal{C}$ maps to endomorphisms of any object of $\mathcal{C}$. But we can do better: it naturally acts on $\mathrm{Hom}(x, y)$ for any pair of objects x, y, by pre- or post-composition. Therefore the category $\mathcal{C}$ is automatically enriched in $\mathrm{HH}^{\bullet}(\mathcal{C})$-modules. Since Hochschild cohomology is E_2, we can use some commutative intuition to think about it: we can think that $\mathcal{C}$ is a category living over $\mathrm{Spec}(\mathrm{HH}^{\bullet}\mathcal{C})$ (i.e., a sheaf of categories). Morally, the fiber of $\mathcal{C}$ over any point of $\mathrm{Spec}(\mathrm{HH}^{\bullet}\mathcal{C})$ will be a category with trivial center, so this splits the problem of understanding $\mathcal{C}$ into a commutative problem, and a fully noncommutative one.

  The action of $\mathrm{HH}^{\bullet}\mathcal{C}$ on Hochschild homology will be an obvious consequence of the action of $\mathrm{HH}^{\bullet}\mathcal{C}$ on Hom-spaces, together with the fact that Hochschild homology is built out of the Hom-spaces.

Hochschild Homology as a coend

We'll discuss two points of view on Hochschild homology. The first one is as a dual construction to the presentation of Hochschild cohomology. We define, for a compactly generated category $\mathcal{C}$, its Hochschild homology to be the geometric realization of the following simplicial object

$$\bigoplus_{x\in\mathcal{C}^c} \mathrm{Hom}(x,x) \leftleftarrows \bigoplus_{x,y\in\mathcal{C}^c} \mathrm{Hom}(x,y)\otimes\mathrm{Hom}(y,x) \Lleftarrow \bigoplus_{x,y,z\in\mathcal{C}^c} \mathrm{Hom}(x,y)\otimes\mathrm{Hom}(y,z)\otimes\mathrm{Hom}(z,x)\cdots$$

Here the arrows are pairwise compositions. Observe that we are only taking compact objects into account. For Hochschild cohomology this didn't matter, having an endomorphism of the identity endofunctor of $\mathcal{C}^c$ is the same as having an endomorphism of the identity endofunctor of $\mathcal{C}$ (when $\mathcal{C}$ is compactly generated), but for Hochschild homology if we allowed noncompact objects we would get zero (essentially by the same reasons that the K-theory of such a category vanishes: for instance for Vect one has equations like $k \oplus V = V$ for V infinite dimensional, which would show that the class of k is zero).

Observe that if we allowed ourself to work with the category BA^{op} for A an algebra, then the above construction would recover the usual bar complex computing Hochschild homology of A. We can therefore think of the above as being a many object version of the bar complex.

By construction, if we have an endomorphism f of a compact object x, we get a class $\mathrm{tr}(f) \in \mathrm{HH}_{\bullet}\mathcal{C}$ called the trace of f. Moreover, looking at the 1-simplices in the above diagram tells us that if we have $f : x \to y$ and $g : y \to x$ then $\mathrm{tr}(fg) = \mathrm{tr}(gf)$, which is a reasonable property to expect from a trace. We therefore think of Hochschild homology as being the universal recipient for traces of endomorphisms.

Given a compact object x, we can always take our endomorphism to be the identity id_x. In this case we denote $\mathrm{tr}(\mathrm{id}_x) = \mathrm{ch}(x)$, and we call this the character of x. This is a general construction which recovers in special cases many things that we call characters: Chern character, character of a group representation, Dennis trace, etc.

Example: When $\mathcal{C} = \mathrm{Vect} = k\text{-mod}$ the Hochschild homology is k. In this case, the trace of an endomorphism coincides with the usual notion of trace, and in particular the character of a perfect complex V coincides with its dimension (i.e, Euler characteristic).

The Hochschild cohomology is k as well. In this case the enrichement that we mention in the previous section is not interesting, it's just telling us that morphisms between vector spaces are themselves vector spaces.

One advantage of this point of view on Hochschild homology is that it makes evident its functoriality. If $F : \mathcal{C} \to \mathcal{D}$ is a proper map (by which we mean it maps compact objects in $\mathcal{C}$ to compact objects in $\mathcal{D}$), then we get a map between the corresponding simplicial objects, and passing to geometric realization we get a map $F_* : \mathrm{HH}_\bullet(\mathcal{C}) \to \mathrm{HH}_\bullet(\mathcal{D})$. By construction, this obviously commutes with taking characters, so we get the following

Theorem. *For $F : \mathcal{C} \to \mathcal{D}$ proper, and $x \in \mathcal{C}$ compact, we have $\mathrm{ch}(F(x)) = F_* \, \mathrm{ch}(x)$.*

This is a noncommutative version of the Riemann-Roch theorem: if one takes $\mathcal{C}, \mathcal{D}$ to be coherent sheaves on two varieties X, Y, and F to be the pushforward along a proper map $f : X \to Y$, then (after some work) one recovers from this the usual statement of Riemann-Roch.

Example: Let's allow ourselves to step outside the k-linear context for a moment, to get some more intuition as to what the above simplicial object is computing. Let's take G a group, and $\mathcal{C} = BG$ be a strict category with one object and endomorphisms G. Then, in this context, replacing direct sums with coproducts and tensor products with products, the simplicial object becomes

$$G \leftleftarrows G \times G \;\substack{\longleftarrow\\[-2pt]\longleftarrow\\[-2pt]\longleftarrow}\; G \times G \times G \ldots$$

Here the arrows are pairwise multiplication. Looking at the 1-simplices, this is telling us that we have to identify gh with hg, for every pair of elements g, h in G. This is the same as identifying g with hgh^{-1}. In fact, the geometric realization is the adjoint quotient G/G, which is the loop space of BG.

The moral of this is that we may think about our definition of $\mathrm{HH}_\bullet(\mathcal{C})$ for a k-linear category to be a way of implementing loop spaces in noncommutative geometry. The functoriality, from this point of view, is simply pushforward of loops. For categories arising as sheaves on some geometric space, the analogy is even stronger in that the Hochschild homology can usually be shown to be functions on the loop space of the given object.

Hochschild Homology as the dimension of $\mathcal{C}$

We now turn to our second point of view on Hochschild homology. We will argue that it can be considered as the dimension of $\mathcal{C}$. The notion of dimension is something that makes sense in general for any dualizable object in a symmetric monoidal category. In our case, the category we are interested in is the category DGCat_{cont} of k-linear cocomplete categories and continuous functors. It has a monoidal structure $\otimes$. This means, given $\mathcal{C}$, $\mathcal{D}$ two categories, we may form their tensor product $\mathcal{C} \otimes \mathcal{D}$. This is a new category generated by objects of the form $c \otimes d$ with $c \in \mathcal{C}$ and $d \in \mathcal{D}$, with some evident relations so that the functor $\mathcal{C} \times \mathcal{D} \xrightarrow{\otimes} \mathcal{C} \otimes \mathcal{D}$ is continuous in each variable. For our purposes, it will be useful to know that $A\text{-mod} \otimes B\text{-mod} = (A \otimes B)\text{-mod}$, so we can think of $\otimes$ as being a many object generalization of the tensor product of algebras.

One analogy to keep in mind in what follows is that $(\mathrm{DGCat}, \otimes)$ is a categorified version of $(\mathrm{Vect}, \otimes)$. If one replaces categories by vector spaces, our definition of Hochschild homology will recover the dimension of a vector space.

One first observation is that in vector spaces we are not allowed to take dimension of any object, only of finite dimensional ones. In our context, we will restrict to dualizable categories: categories $\mathcal{C}$ for which there is another category $\mathcal{C}^\vee$ (which will happen to be $\mathrm{Funct}(\mathcal{C}, \mathrm{Vect})$) together with maps

$$\mathrm{Vect} \xrightarrow{coev} \mathcal{C} \otimes \mathcal{C}^\vee$$

$$\mathcal{C}^\vee \otimes \mathcal{C} \xrightarrow{ev} \mathrm{Vect}$$

called evaluation and coevaluation, satisfying some standard properties coming from the geometry of 1-dimensional cobordisms. Namely, one requires that the following compositions can be identified with the identities:

$$\mathcal{C} \xrightarrow{\mathrm{id} \otimes coev} \mathcal{C} \otimes \mathcal{C}^\vee \otimes \mathcal{C} \xrightarrow{ev \otimes \mathrm{id}} \mathcal{C}$$

$$\mathcal{C}^\vee \xrightarrow{coev \otimes \mathrm{id}} \mathcal{C}^\vee \otimes \mathcal{C} \otimes \mathcal{C}^\vee \xrightarrow{\mathrm{id} \otimes ev} \mathcal{C}^\vee$$

Most categories one encounters in practice are dualizable. In particular those which are compactly generated, in which case the dual is found as the ind-completion of the opposite of the category of compact objects. A very particular case of this is that for A an algebra, A-mod and A^{op}-mod are dual, the unit and counit being given by the bimodules ${}_{A \otimes A^{\mathrm{op}}}A_k$ and ${}_kA_{A \otimes A\mathrm{op}}$.

For a dualizable category one may identify $\mathcal{C} \otimes \mathcal{C}^\vee$ with $\mathrm{Hom}(\mathcal{C}, \mathcal{C})$ and then the coevaluation is determined by the fact that it maps k to $\mathrm{id}_{\mathcal{C}}$. Moreover, we think about the evaluation as taking trace of an endofunctor. Therefore, a reasonable definition of dimension of $\mathcal{C}$ is obtained by composing the evaluation and coevaluation (the braiding makes their source and target agree).

Definition. *The Hochschild homology of $\mathcal{C}$ is the image of k under the composition*

$$\mathrm{Vect} \xrightarrow{coev} \mathcal{C} \otimes \mathcal{C}^\vee = \mathcal{C}^\vee \otimes \mathcal{C} \xrightarrow{ev} \mathrm{Vect}$$

For example, when $\mathcal{C} = A$-mod, composition arises from tensoring bimodules, so

$$\mathrm{HH}_\bullet\, A\text{-mod} = A \otimes_{A \otimes A^{\mathrm{op}}} A$$

which agrees with the classical definition.

It may be proven that this definition of Hochschild homology agrees with our previous approach through coends. The proof involves some standard manipulations with ends and coends, which we won't discuss.

To finish we will see that this definition of Hochschild homology is not only conceptually useful, but can lead to concrete computations in a straightforward way, at least for categories of sheaves on a geometric object X. In that context, one usually finds that the dual of $\mathcal{C}$ is $\mathcal{C}$ itself, the tensor product $\mathcal{C} \otimes \mathcal{C}^\vee$ is sheaves on $X \times X$, and the unit and counit have to do with taking the structure sheaf of the diagonal, and integrating (taking pushforward) along the diagonal.

Example: Let's first discuss the decategorified version of that idea. Consider the vector space k^n. We will compute its dimension by following the above philosophy. Let $[n] = \operatorname{Spec} k^n$. The relevant geometric diagram is

$$\begin{array}{ccccc} & & [n] & \xrightarrow{p} & \mathrm{pt} \\ & & \downarrow{\scriptstyle \Delta} & & \\ [n] & \xrightarrow{\Delta} & [n] \times [n] & & \\ \downarrow{\scriptstyle p} & & & & \\ \mathrm{pt} & & & & \end{array}$$

The claim is that $(k^n)^\vee = k^n$. To show this one must construct maps

$$k \to k^n \otimes k^n$$

$$k^n \otimes k^n \to k$$

We think of k as being functions on a point, and $k^n \otimes k^n$ as being functions on $[n] \times [n]$. Then we find our unit and counit by doing push-pull along the diagram

$$k \xrightarrow{\Delta_* p^*} k^n \otimes k^n$$

$$k^n \otimes k^n \xrightarrow{p_* \Delta^*} k$$

The composition is $p_* \Delta^* \Delta_* p^*$. The standard way to compute this is via a base change diagram:

$$\begin{array}{ccc} [n] & \xrightarrow{\mathrm{id}} & [n] \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \Delta} \\ [n] & \xrightarrow{\Delta} & [n] \times [n] \end{array}$$

Then the composition just becomes

$$k \xrightarrow{p_* \operatorname{id}_* \operatorname{id}^* p^*} k$$

which is just

$$k \xrightarrow{p_* p^*} k$$

This maps 1 to the integral of the constant function 1 along $[n]$. This shows that the dimension of k^n is n.

One thing we didn't do is to show that our unit and counit satisfy the required axioms. This is in general straightforward and involves some more base change diagrams which we leave for the reader to write down.

Example: As an example of the above strategy at work in the categorified context, let's take $\mathcal{C} = \operatorname{Rep}(G)$ for G a finite group (although this is really general and we could take G to be a group in homotopy types or even an algebraic group).

Then $\mathcal{C}$ is also $\operatorname{LocSys}(BG)$, which is a sheaf theory evaluated on a space, so we expect $\mathcal{C}^\vee = \mathcal{C}$. The unit and counit should come from doing push pull along the diagram

$$\begin{array}{ccccc}
 & & BG & \xrightarrow{p} & \mathrm{pt} \\
 & & \big\downarrow{\scriptstyle \Delta} & & \\
BG & \xrightarrow{\Delta} & BG \times BG & & \\
\big\downarrow{\scriptstyle p} & & & & \\
\mathrm{pt} & & & &
\end{array}$$

Namely, one proves that $\mathcal{C} \otimes \mathcal{C} = \operatorname{LocSys}(BG \times BG)$, and then the unit and counits are

$$\operatorname{Vect} \xrightarrow{\Delta_! p^*} \operatorname{LocSys}(BG \times BG)$$

$$\operatorname{LocSys}(BG \times BG) \xrightarrow{p_! \Delta^*} \operatorname{Vect}$$

At the level of group representations, the unit sends k to the induction of the trivial representation of G to $G \times G$, and the counit is restriction from $G \times G$ to G composed with taking coinvariants. In other words, the Hochschild homology of $\operatorname{Rep} G$ is

$$\operatorname{Ind}_G^1 \operatorname{Res}_{G\times G}^G \operatorname{Ind}_G^{G\times G} \operatorname{Res}_1^G(k)$$

The way one gets a handle on this is by looking at the base change diagram

$$\begin{array}{ccc}
G/G & \xrightarrow{q} & BG \\
\big\downarrow{\scriptstyle q} & & \big\downarrow{\scriptstyle \Delta} \\
BG & \xrightarrow{\Delta} & BG \times BG
\end{array}$$

where G/G is the loopspace of BG, i.e., the adjoint quotient. Then the Hochschild homology of $\mathcal{C}$ becomes the homology of the constant local system $k_{G/G}$, which agrees with the homology of G/G.

When k is a field of characteristic not dividing the cardinality of G, there is no interesting group homology at play here, and the Hochschild homology becomes $k[G/G]$, the space of class functions (concentrated in degree zero). For V a finite dimensional representation of G, its class in $\mathrm{HH}_{\bullet}(\mathrm{Rep}(G))$ coincides with the usual notion of character from the representation theory of finite groups.

As for Hochschild cohomology, we can compute it by observing that $\mathrm{End}(\mathcal{C}) = \mathcal{C}^{\vee} \otimes \mathcal{C}$, and the identity corresponds to the image of k under the unit map. So we have

$$\mathrm{HH}^{\bullet} \operatorname{Rep} G = \mathrm{Hom}(\Delta_! k_{BG}, \Delta_! k_{BG}) = \mathrm{Hom}(k_{BG}, \Delta^* \Delta_! k_{BG}) = \mathrm{Hom}(k_{BG}, q_! k_{G/G})$$

Again, when the characteristic of k does not divide the cardinality of G, there is no group homology to worry about, and we see

$$\mathrm{HH}^{\bullet} \operatorname{Rep} G = k[G/G]$$

Note that we already knew that $HH^0 \operatorname{Rep} G$ had to be $k[G/G]$ since $\operatorname{Rep} G = k[G]$-mod and the center of the group algebra is $k[G/G]$. But now we are able to conclude that the higher cohomologies vanish.