Global existence and uniqueness of solutions to a model of price formation

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Abstract

We study a model, due to J.M. Lasry and P.L. Lions, describing the evolution of a scalar *price* which is realized as a free boundary in a 1D diffusion equation with dynamically evolving, non–standard sources. We establish global existence and uniqueness.

0 Introduction

Here we are concerned with the following PDE:

$$
\text{(P)} \qquad \begin{cases} f_t - f_{xx} = \left[\delta_{p(t)+\underline{a}} - \delta_{p(t)-\underline{a}}\right] f_x(p(t),t) \quad \text{in} \quad (-1,1)\times[0,\infty); \\ f_x(1,t) = f_x(-1,t) = 0. \\ f(x,0) = f_I(x). \end{cases}
$$

where $p(t) = \{x : f(x,t) = 0\}$ presumed, for a.e. t, to be a singleton, and

$$
\underline{a} := \underline{a}(p(t)) = \min\{a, \frac{1}{2}|p(t) \pm 1|\} \text{ with } a < 1. \tag{0.1}
$$

The model with $\underline{a} \equiv a$ was invented in [7] and, as explained therein (see also [5],[6]) is purported to describe the dynamic evolution of a *price* $p(t)$ as influenced by a population of buyers and sellers. In this initial reference, the existence of solutions was discussed, mostly in the context of a non–compact domain.

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While the model on compact domains was featured (strictly speaking, the model on $\mathbb{R}$ does not make economic sense) there was no proviso for the circumstance $|1 \pm p(t)| < a$. Our modification using $\underline{a}$ provides this definition and later (see Corollary 2.3) allows us to ensure that $p(t)$ stays away from the domain boundaries at $x = \pm 1$. In terms of the model, our modification can be viewed as a "rescue plan" to prevent prices from severe deflation or inflation.

Recently, [5] the problem was solved completely for the case of symmetric initial data and in the work [6], global existence, uniqueness and stability was established for initial data sufficiently close (in a certain sense) to the piecewise linear equilibrium solution. [1] Finally, contemporaneous to the present work, the non–compact version of (P) – sometimes with regularization – is investigated in [8]. A complete derivation of uniqueness for short times is presented therein.

Notwithstanding the benign appearance of (P), the system contains intrinsic and convoluted non–linearities. Indeed, the driving term at the sources is the gradient at the dynamically generated zero, $p(t)$. In turn, $p(t)$ controls the location of the sources and therefore constitutes a *free boundary* for the problem. Thus, a central issue is to establish non–degeneracy at this free boundary and thereby aquire some degree of control for its motion. E.g., in this context Hopf's lemma, while useful, is not immediately decisive without some additional regularity information at the free boundary.

We consider the initial data $f_I \in C^2([-1,1])$ satisfying the following:

(i) $\{f_I(x) = 0\} = \{p_I\}$, $f_I(x) > 0$ for $x < p_I$ and $f_I(x) < 0$ for $x > p_I$.

(ii) $\partial_x f_I(-1) = \partial_x f_I(+1) = 0$.

(iii) Given $\lambda_I := -\partial_x f_I(p_I) > 0$, we must have

$$-\partial_x f_I(x) > \frac{3}{4}\lambda_I \quad \text{in } (p_I - a_0, p_I + a_0) \tag{0.2}$$

for some $0 < a_0 \ll a/4$. Note that (iii) is automatic from the regularity of f_I.

It is worthwhile to notice that problem (P) satisfies the following conservation identities

$$\int_{-1}^{p(t)} f(x,t)\,dx \equiv \int_{-1}^{p_I} f_I(x)\,dx =: M_b > 0, \quad \int_{p(t)}^{1} f(x,t)\,dx \equiv \int_{p_I}^{1} f_I(x)\,dx =: -M_p < 0. \tag{0.3}$$

Note also that the free boundary moves with a velocity given by the formula:

$$\dot{p}(t) = -\frac{f_{xx}(p(t),t)}{f_x(p(t),t)},$$

and the flux across the free boundary is given by

$$\lambda(t) := -\partial_x f(p(t),t).$$

The main result in this paper is written below:

[1] It is noted – but not proven – that *existence* might be established via a connection to a stochastic interacting particle model. The utility of this connection is under investigation by the authors, particularly with regards to the question of global stability.

Theorem 0.1. [Global existence and uniqueness of classical solution]

Consider the system described in (P) with initial data f_I satisfying the conditions (i)–(iii) above, and let us define $\Omega := (-1,1)\times(0,\infty)$. Then there is a unique "density" $f(x,t) \in L^\infty(\Omega)$ and a $p : [0,\infty) \to (-1,+1)$, $p(t) \in C^{1/2}([0,\infty))$ that satisfy the following:

(A) For any $0 < t \leq T$ there exists $r = r(T)$ with $r > 0$ such that

(a) f is C^∞ in $\{(x,t) : |x - p(t)| \leq r\}$;

(b) $\lambda(t) > r$;

(c) $p(t) \in (-1 + r, 1 - r)$.

(B) The density f solves the first two equation of (P) in the classical sense (in terms of Duhamel's formula) in Ω, with $f(x,t) > 0$ if $x < p(t)$ and $f(x,t) < 0$ for $x > p(t)$. Moreover $f(x,t)$ uniformly converges to $f_I(x)$ as $t \to 0$.

The organization and summary of what remains is as follows:

• In Section 1 we derive existence and uniqueness of the system (P) for short times; i.e., we prove Theorem 0.1 for positive values of T that are sufficiently small. We use a basic approach deploying, perhaps, more care than is strictly necessary. However this turns out to be important for later developments since it allows us to *directly* tie all sources of potential difficulty to specific attributes of initial conditions.

• In Section 2 we provide the L^1_{loc} property for the flux function $\lambda(t)$. In order to do this, we introduce a *foliated* version of the (P) system (and certain related systems) which is defined on an array of intervals. Both in the context of the foliated models and the "regular" models, we introduce a *zero diffusion* limit. It is our belief that these two auxiliary devices may be of some independent interest. An immediate corollary to the L^1–properties of λ is the fact that whenever a solution exists, the free boundary stays strictly away from the system boundary. The items in this section are serve as the foundational steps for global existence.

• Section 3 represents the substance of this paper and, if the results of the first two sections may could be accepted on faith, may be processed with minimal back reference. All possible conditions under which the derivation in Section 1 might eventually fail are recapitulated and systematically eliminated. For the most part the underlying strategy is primitive: A demonstration that the level of regularity obtained by the finite time solution is of sufficient robustness to preclude all purported obstructions – all of which amount to degeneracy of the zero set. The worst scenario for degeneration is that the behavior of the free boundary is sufficiently wild that it produces a non–trivial interval where it accumulates. To rule out this happenstance, we perform a multi–scale analysis; the subjects of Proposition 3.5 and Lemma 3.6. It is demonstrated that on each interval–scale, sufficient regularity is retained so that, even by the accumulation time, a non–trivial analytic solution can be produced which, necessarily, precludes the possibility of accumulation.

1 Short times

The preliminary results are based on the short–time contraction principle of the following iteration: given $f_n(x,t)$ and $p_n(t)$ such that $p_n(t) := \{x : f_n(\cdot,t) = 0\}$ consists of a unique point for each $t > 0$, consider the function $\lambda_n(t)$ defined by

$$\lambda_n(t) := -(f_n)_x(p_n(t), t).$$

Let $f(x,t)$ solve

$$\begin{aligned} \frac{\partial f}{\partial t} - \frac{\partial^2 f}{\partial x^2} &= \lambda_n(t)\left[\delta_{x=p_n(t)-\underline{a}} - \delta_{x=p_n(t)+\underline{a}}\right], && (1.1)\\ f_x(-1,t) &= f_x(1,t) = 0, && (1.2)\\ f(x,0) &= f_I(x). \end{aligned}$$

Then the solution to the above becomes $f_{n+1}(x,t)$, i.e., serves to define $p_{n+1}(t)$ and $\lambda_{n+1}(t)$; observe that by the maximum principle, $p_{n+1}(t)$ also consists of a single point. As a start-up function $f_1(x,t)$, we may as well use the solution of (P) without sources.

Specifically, we define the map

$$\begin{aligned} \Phi:\ L^\infty((0,t_0);X) &\rightarrow L^\infty((0,t_0);X), \\ f_n &\mapsto f_{n+1}, \end{aligned} \tag{1.3}$$

where $X := L^\infty(p_I - a_0, p_I + a_0)$ for some a_0 sufficiently small.

First let us write the solution $f(x,t)$ of (1.2) using the Duhamel formula:

$$\begin{aligned} f(x,t) = &\int_{-1}^{+1} \Gamma(x,x';t) f_I(x')\,dx' \\ &+ \int_0^t \left[\Gamma(x,(p_n(t')-\underline{a}); t-t') - \Gamma(x,(p_n(t')+\underline{a}); t-t')\right] \lambda_n(t')\,dt' \\ &=: I_1 + I_2, \end{aligned} \tag{1.4}$$

where Γ denotes the fundamental solution appropriate for the domain $(-1,+1)$ with Neumann boundary conditions:

$$\Gamma(x,x';t) = \sum_{k=-\infty}^{\infty} K(x - (2k + [-1]^{|k|}x'), t) \tag{1.5}$$

with

$$K(x,t) = \frac{1}{\sqrt{4\pi t}} \mathrm{e}^{-\frac{x^2}{4t}}.$$

The main result of this section is stated in the following:

Theorem 1.1. *There exists a time t_0 depending only on $\|f_I\|_{L^\infty(-1,1)}$, λ_I and a_0 such that*

$$\sup_{t\in[0,t_0]} \|f-g\|_X \leq \frac{1}{2} \sup_{t\in[0,t_0]} \|f_n - g_n\|_X,$$

where $X := L^\infty(p_I - a_0, p_I + a_0)$ and the functions f, g solve (1.2) with right-hand side, respectively, (f_n, p_n) and (g_n, q_n).

Corollary 1.2. *There exists $t_0 > 0$ depending only on $\|f_I\|_{L^\infty(-1,1)}$ and λ_I such that (P) has a solution f, p for all times $t \in [0, t_0]$. Moreover, f is smooth accross the free boundary, and*

$$\lambda(t_0) \geq \frac{\lambda_I}{4} > 0.$$

The proof of Theorem 1.1 is a consequence of the following series of results:

Lemma 1.3. *For all x and y such that $|x-y| < \alpha/4$, it holds that*

$$\Gamma(x, y\pm\alpha, t) + \; |\Gamma_x(x, y\pm\alpha, t)| + \; |\Gamma_{xx}(x, y\pm\alpha, t)| \leq G(\alpha),$$

where the constant $G(\alpha)$ depends only on α (but diverges as $\alpha \to 0$.)

Proof. It follows from elementary analytical considerations. We show it for Γ_x, the rest are similar. Writing

$$\vartheta_k = \frac{x - (2k + [-1]^{|k|}x')}{2\sqrt{t-t'}}$$

we have

$$\Gamma_x(x, x'; t) = -\frac{1}{\sqrt{4\pi}} \sum_k \frac{1}{t-t'} \vartheta_k \mathrm{e}^{-\vartheta_k^2} = G_1 \sum_k \frac{1}{(x - (2k + [-1]^{|k|}x')^2} \vartheta_k^3 \mathrm{e}^{-\vartheta_k^2} \tag{1.6}$$

with G_1 a constant. Obviously, for any k, we can bound $G_1 \theta_k^3 \mathrm{e}^{-\theta_k^2} \leq G_2$ for another constant G_2, whatever the value of ϑ_k might be.

Next, since for the relevant $x' = y\pm\alpha$ we have that $|x-x'| \geq \alpha/2$ and $|x+x'-2| \geq g_3\alpha$ for a constant g_3, we may sum the series replacing $x \pm x'$ by the relevant worst case scenarios. It is concluded that $|\Gamma_x|$ – unintegrated in the t–variable – is bounded by a finite constant (which depends on α, but not on x, x' or t). ☐

Lemma 1.4. *Consider*

$$\begin{aligned} f_x(x,t) = &\int_{-1}^{1} \Gamma_x(x, x'; t) f_I(x')\, dx' \\ &+ \int_0^t \left[\Gamma_x(x, p_n(t') - \underline{a}; t-t') - \Gamma_x(x, p_n(t') + \underline{a}; t-t')\right] \lambda_n(t')\, dt' \\ &=: \frac{dI_1(x,t)}{dx}(x,t) + \frac{dI_2(x,t)}{dx}(x,t). \end{aligned}$$

If $\bar{t} < (\frac{1}{4G(a_0)})^2$ is a time such that $|p_I - p_n(t)| < a_0$ for all $t < \bar{t}$, then we can estimate

$$\left|\frac{dI_2(x,t)}{dx}\right| \leq G(a_0)\int_0^t |\lambda_n(s)|\, ds,$$

for all $t < \bar{t}$ and $x \in (p_I - a_0, p_I + a_0)$, where $\lambda_I > 0$, p_I are as described in the hypothesis (i) – (iii), and G is as given in Lemma 1.3. Moreover, for

$$x \in (p_I - a_0, p_I + a_0),$$

the linear term can be estimated as

$$\frac{dI_1(x,t)}{dx} \leq -\frac{1}{2}\lambda_I \quad \textit{for all}\, t \leq c_0\sqrt{a_0},$$

where c_0 is a small constant independent of a_0, etc.

Proof. The first estimate follows easily from Lemma 1.3, just taking into account that

$$|x - p_n(t)| \leq |x - p_I| + |p_I - p_n(t)| \leq 2a_0.$$

The second estimate follows from a straightforward estimate using the Green's function formula.

□

Corollary 1.5. *Suppose, in the n^{th} stage of the iteration, that for all $t < \bar{t}$ we have*

$$|\lambda_n(t)| \leq 2\frac{\|f_I\|_{L^\infty(-1,1)}}{\sqrt{t}} \quad \textit{and} \quad |p_I - p_n(t)| < a_0. \tag{1.7}$$

Then we can find the following bound for the next step $f = f_{n+1}$, given in Eq.(1.4): there exists a time $t_0 \leq \bar{t}$, depending on $\|f_I\|_{L^\infty(-1,1)}$ and λ_I, such that

$$\inf_{t,x}[-f_x(x,t)] \geq \frac{\lambda_I}{4},$$

for all $t \leq t_0$ and $|x - p_I| \leq a_0$.

Proof. This is a consequence of the two bounds from Lemma 1.4 and the hypothesis (1.7). Indeed we may write, uniformly for $x \in (p_I - a_0, p_I + a_0)$

$$f_x(x,t) \leq -\frac{1}{2}\lambda_I + 4G(a_0)\|f_I\|_{L^\infty(-1,1)}\sqrt{t}.$$

It is clear that for $t < t_0$ with

$$G(a_0)(1 + 16\frac{\|f_I\|_{L^\infty(-1,1)}}{\lambda_I})\sqrt{t_0} = 1, \tag{1.8}$$

the desired bound will hold. It is noted that the time t_0 does not depend on the iteration coefficient n. □

In the next two lemmas we show a L^∞–bound for f_x and f_{xx} in a neighborhood of the free boundary which is the required input for Corollary 1.5.

Lemma 1.6. *Let $\bar{t}$ be as given above, and suppose that*

$$\sup_x |(f_n)_x(x,t)| \leq 2\frac{\|f_I\|_{L^\infty(-1,1)}}{\sqrt{t}}, \quad 0 < t < \bar{t}, \quad |x - p_I| < a_0, \tag{1.9}$$

then

$$\sup_x |(f_{n+1})_x(x,t)| \leq 2\frac{\|f_I\|_{L^\infty(-1,1)}}{\sqrt{t}}, \quad 0 < t < \bar{t}, \quad |x - p_I| < a_0.$$

Proof. As before, we let

$$\begin{aligned} f_x(x,t) &= \int_{-1}^{1} \Gamma_x(x,x';t) f_I(x')\, dx' \\ &\quad + \int_0^t \left[\Gamma_x(x,(p_n(t') - \underline{a}); t - t') - \Gamma_x(x, p_n(t') + \underline{a}; t - t')\right] \lambda_n(t')\, dt' \\ &=: \frac{dI_1(x,t)}{dx} + \frac{dI_2}{dx}(x,t). \end{aligned}$$

Regularity estimates for caloric functions (see [1], Chapter V, Theorem 8.1) imply

$$\sup_x \left| \frac{dI_1(x,t)}{dx} \right| \leq \frac{\|f_I\|_{L^\infty(-1,1)}}{\sqrt{t}}, \quad \text{for } |x - p_I| < a_0.$$

For the nonlinear part we make use of the estimates for the Green's function given in Lemma 1.3 and the hypothesis (1.9). It holds

$$\sup_x \left| \frac{dI_2(x,t)}{dx} \right| \leq G(a_0) \int_0^t |\lambda_n(s)|\ ds \leq 4G(a_0)\|f_I\|_{L^\infty(-1,1)}\sqrt{t}, \quad |x - p_I| \leq a_0.$$

Since $\sqrt{\bar{t}} < \frac{1}{4G(a_0)}$, we get

$$\sup_x |f_x(x,t)| \leq 2\frac{\|f_I\|_{L^\infty(-1,1)}}{\sqrt{t}}.$$

for all $t \leq \bar{t}$. □

The last bound we need to show is the following local estimate for $\partial_t f$:

Lemma 1.7. *Let $\bar{t}$ be as given above. Then for any $|x - p_I| < a_0$, the following holds:*

$$\sup_x |\partial_t f(x,t)| \leq 2\frac{\|f_I\|_{L^\infty(-1,1)}}{\sqrt{t}}, \quad t < \bar{t}.$$

Proof. We use a very similar argument to the one of the previous lemma. Write the solution as (1.4). For the linear part we use again classical estimates (c.f. [1], Chapter V, Theorem 8.1) that imply

$$|\partial_t I_1| \leq \frac{\|f_I\|_{L^\infty(-1,1)}}{\sqrt{t}}.$$

For the nonlinear part, consider

$$|\partial_t I_2| = \partial_{xx} I_2 \leq G(a_0) \int_0^t |\lambda_n(s)| \leq 4G(a_0) \|f_I\|_{L^\infty(-1,1)} \sqrt{t},$$

where we have used Lemma 1.3 for the estimate of Γ_{xx} and Eq.(1.9).
Choosing $\bar{t}$ as in the previous lemma, it follows that

$$\sup_x |\partial_t f(x,t)| \leq 2 \frac{\|f_I\|_{L^\infty(-1,1)}}{\sqrt{t}} \quad \text{for all } t \leq \bar{t}$$

as claimed. □

The above lemma gives us an estimate in terms of the L^∞–norm of the initial data. Is is also possible to estimate the f_x in terms of the L^1–norm (which is in our case constant in time), at the price of a worse denominator. We will not make use of this lemma in this section, but later in the proof of global existence:

Lemma 1.8. *Let $\bar{t}$ be as given above, and suppose that*

$$\sup_x |(f_n)_x(x,t)| \leq 2 \frac{\|f_I\|_{L^1(-1,1)}}{(\sqrt{t})^3}, \quad 0 < t < \bar{t}, \quad |x - p_I| < a_0. \tag{1.10}$$

Then:

$$\begin{aligned}
\sup_x |(f_{n+1})_x(x,t)| &\leq 2 \frac{\|f_I\|_{L^1(-1,1)}}{(\sqrt{t})^4}, \quad 0 < t < \bar{t}, \quad |x - p_I| < a_0, \\
\sup_x |(f_{n+1})_{xx}(x,t)| &\leq 2 \frac{\|f_I\|_{L^1(-1,1)}}{(\sqrt{t})^5}, \quad 0 < t < \bar{t}, \quad |x - p_I| < a_0.
\end{aligned}$$

Proof. The above estimates can be proven following the same steps as in Lemma 1.6 and 1.7. Instead of using the regularity estimates for the caloric function with the L^∞–norm of the function, we make use now of the L^1–norm. From [4] (Theorem 9, section 2.3.c.) we know that

$$\begin{aligned}
\sup_x \left| \frac{dI_1}{dx} \right| &\leq \frac{\|f\|_{L^1}}{t^2}, \\
\sup_x \left| \frac{d^2 I_1}{dx^2} \right| &\leq \frac{\|f\|_{L^1}}{(\sqrt{t})^5},
\end{aligned}$$

where I_1 is defined as in Lemma 1.6. The rest of the proof follows similarly. □

In the next corollary we show that, for $t \in [0, t_0]$ with t_0 given in Eq.(1.8), the sequence of the free boundary points $\{p_n(t)\}_n$ stays in $(p_I - a_0, p_I + a_0)$, and is uniformly Hölder continuous in time.

Corollary 1.9. *There exists a time $t_0 > 0$, in particular as given in Eq.(1.8), such that the solution $f(\cdot, t)$ constructed in* (1.4) *has a unique zero in $(p_I - a_0, p_I + a_0)$,* *for each fixed time $t < t_0$, which we denote by $p(t)$. Moreover, $p(t)$ satisfies*

$$|p(t) - p_I| \leq \alpha_1 \sqrt{t},$$

where α_1 only depends on the initial data and a_0.

Proof. First, note that Corollary 1.5 implies monotonicity of $f(\cdot, t)$ in $(p_I - a_0, p_I + a_0)$ which assures the existence of at most one zero, $p(t)$. Next, since $f(p(t), t) = 0$ for all t, then

$$\dot{p}(t) = -\frac{f_t(p(t), t)}{f_x(p(t), t)}. \tag{1.11}$$

On the other hand, Lemma 1.7 gives a bound for the velocity $|f_t(x,t)| \leq 2\frac{\|f_I\|_{L^\infty(-1,1)}}{\sqrt{t}}$ in the time interval $(0, \bar{t})$, and Corollary 1.5 bounds the slope $|f_x(x,t)|$ from below in an interval $|x - p_I| < a_0$ for $t \in (0, t_0)$. Existence results for the ODE in Eq. (1.11) implies the existence of this $p(t)$.

As a consequence, the interval $|p(t) - p_I|$ is bounded by

$$|p(t) - p_I| \leq 8\frac{\|f_I\|_{L^\infty(-1,1)}}{\lambda_I}\sqrt{t}, \quad \text{for } 0 < t \leq t_0.$$

The thesis follows because

$$8\frac{\|f_I\|_{L^\infty(-1,1)}}{\lambda_I}\sqrt{t_0} \leq \tilde{c} a_0, \tag{1.12}$$

where $\tilde{c}$ only depends on a_0. □

Now we are ready to show that the map Φ defined in Eq.(1.3) is a contraction in the space $L^\infty((0, t_0); X)$, and consequently to utilize t_0.

Proof of Theorem 1.1:

First note that if $f_n \in L^\infty([0, t_0], X)$ with $f_n(p_n) = 0$ satisfies

$$\text{(K)} \qquad \begin{cases} p_n(t) \in (p_I - a_0, p_I + a_0), \\ \sup_x |f_{nx}(x,t)| \leq 2\frac{\|f_I\|_{L^\infty(-1,1)}}{\sqrt{t}}, \\ f_{nx}(p_n, t) \leq -\frac{\lambda_I}{4}, \end{cases}$$

then the previous results assure that the next step in the iteration $f(x,t)$ has a well defined zero $p(t)$ and satisfies also the same estimates. Note that above estimates hold for $n = 1$: recall that we were using $f_1(x,t)$ as the solution without sources, where all that has preceded holds trivially.

Let $f_n,\ g_n \in L^\infty([0, t_0], X)$ be such that $f_n(p_n) = 0$, $g_n(q_n) = 0$, $\lambda_n = -f_n(p_n)$, $\xi_n = -g_n(q_n)$, and that both satisfy conditions (K). Then we estimate the difference of the images by our mapping Φ: for any $x \in (p_I - a_0, p_I + a_0)$,

$$
\begin{aligned}
f(x,t) - g(x,t) &= \int_0^t (\Gamma(x, p_n - \underline{a}; t-s) - \Gamma(x, p_n + \underline{a}; t-s))\lambda_n(s)\, ds \\
&\quad - \int_0^t (\Gamma(x, q_n - \underline{a}; t-s) - \Gamma(x, q_n + \underline{a}; t-s))\xi_n(s)\, ds \\
&= \int_0^t \left[\Gamma(x, p_n - \underline{a}; t-s) - \Gamma(x, p_n + \underline{a}; t-s)\right](\lambda_n(s) - \xi_n(s))\, ds \\
&\quad + \int_0^t \xi_n(s)[\Gamma(x, p_n - \underline{a}; t-s) - \Gamma(x, q_n - \underline{a}; t-s) \\
&\qquad\qquad + \Gamma(x, p_n + \underline{a}; t-s) - \Gamma(x, q_n + \underline{a}; t-s)]\, ds \\
&=: A_1 + A_2.
\end{aligned}
$$

First note that

$$
|A_1| \le \sup_{0 \le t \le t_0} |\lambda_n(t) - \xi_n(t)| \int_0^t |\Gamma(x, p_n - \underline{a}; t-s) - \Gamma(x, p_n + \underline{a}; t-s)|\, ds.
$$

We now need to estimate the difference $|\lambda_n(t) - \xi_n(t)|$ in terms of the quantity $\|f_n - g_n\|_X$. Therefore

$$
\begin{aligned}
|\lambda_n(t) - \xi_n(t)| &\le |(f_n)_x(p_n(t), t) - (g_n)_x(\xi_n(t), t)| \\
&\le |(f_n)_x(p_n(t), t) - (g_n)_x(p_n(t), t)| + |(g_n)_x(p_n(t), t) - (g_n)_x(\xi_n(t), t)| \\
&\le \|f_n - g_n\|_{C^1(p_I - a_0, p_I + a_0)} + \|(g_n)_{xx}\|_X |p_n(t) - q_n(t)|.
\end{aligned}
$$

Next we estimate and the C^1–norm of $f_n - g_n$ with classical estimates for the caloric functions the term $\|(g_n)_{xx}\|_X$ with Lemma 1.7. Recall that the estimates hold in a neighborhood of the free boundary far away from the source and sink. Therefore

$$
|\lambda_n(t) - \xi_n(t)| \le \frac{\|f_n - g_n\|_X}{\sqrt{t}} + 2\frac{\|f_I\|_{L^\infty(-1,1)}}{\sqrt{t}} |p_n(t) - q_n(t)|,
$$

for each fixed time t. On the other hand, by an elementary geometrical argument, we have

$$
|p_n(t) - q_n(t)| \le \sup_x |f_n(x,t) - g_n(x,t)| \cdot \frac{4}{\lambda_I},
$$

since the slope is bounded by $\frac{\lambda_I}{4} \le |f_{nx}(x,t)| \le 2\frac{\|f_I\|_{L^\infty(-1,1)}}{\sqrt{t}}$. Hence

$$
|\lambda_n(t) - \xi_n(t)| \le \left[1 + \frac{8}{\lambda_I}\|f_I\|_{L^\infty(-1,1)}\right] \frac{\|f_n - g_n\|_X}{\sqrt{t}}. \tag{1.13}
$$

Consequently, the term A_1 can be estimated using Eq.(1.13) and Lemma 1.3 as

$$
|A_1| \le 2\sqrt{t}G(a_0)\left[1 + \frac{8}{\lambda_I}\|f_I\|_{L^\infty(-1,1)}\right] \|f_n - g_n\|_{L^\infty((0,t_0);X)}.
$$

Similar arguments can be applied in the estimates for A_2: the conditions (K) gives that

$$|A_2| \le 2\|f_I\|_{L^\infty(-1,1)} \int_0^t \frac{1}{\sqrt{s}} |\Gamma(x, p_n - \underline{a}; t-s) - \Gamma(x, q_n - \underline{a}; t-s))| \, ds$$
$$+ 2\|f_I\|_{L^\infty(-1,1)} \int_0^t \frac{1}{\sqrt{s}} |\Gamma(x, p_n + \underline{a}; t-s) - \Gamma(x, q_n + \underline{a}; t-s))| \, ds$$

Next, note that $(p_n - \underline{a}, q_n - \underline{a}) \cap (p_I - a_0, p_I + a_0) = \varnothing$. Here we are slightly abusing the notation by assuming that $p_n < q_n$. It holds that, for each $t < t_0$,

$$\begin{aligned}
&\int_0^t \frac{1}{\sqrt{s}} |\Gamma(x, p_n - \underline{a}; t-s) - \Gamma(x, q_n - \underline{a}; t-s))| \, ds \\
&\quad \le \int_0^t \frac{1}{\sqrt{s}} |p_n - q_n| \sup_{y \in (p_n - \underline{a}, q_n - \underline{a})} |\Gamma_y(x, y; t-s)| ds \\
&\quad \le 4tG(a_0) \frac{1}{\lambda_I} \|f_n - g_n\|_{L^\infty((0,t_0);X)},
\end{aligned}$$

In conclusion we have

$$|f(x,t) - g(x,t)| \le c\sqrt{t} \|f_n - g_n\|_{L^\infty((0,t_0);X)},$$

where the constant c is given by

$$c := G(a_0) \left[1 + \frac{8}{\lambda_I} \|f_I\|_{L^\infty(-1,1)} \right].$$

Since

$$G(a_0) \left[1 + \frac{8}{\lambda_I} \|f_I\|_{L^\infty(-1,1)} \right] \sqrt{t_0} < 1,$$

the proposition is proved. □

Remark: We remark that the proof in [6] for short time existence made use of semi–group methods. The derivation here, while admittedly less sophisticated, is certainly more robust and is deliberately tailored to the upcoming developments. In particular, we now have tangible criteria under which short time existence is purported to break down (see also [8]). Indeed, a parallel derivation for the regularized problems allows, at least for the compact case, a straightforward proof of global existence: Given local summability of $\lambda(t)$ – the subject of next section – the asset of regularization easily implies the various derivative bounds which are the central objective of the final section.

2 An L^1 bound on the flux (and the rescue plan at the Neumann boundary)

In this section we show that, as long as the solution of (P) exists, the flux of the solution at the zero set stays bounded. This result will be then used in the next section to provide further estimates on the derivatives of the solutions. As a corollary, we will also show that the zero set of the solution cannot approach the Neumann boundaries too closely.

To facilitate matters, we shall, in essence decouple the positive and negative pieces of f and, in addition, describe problems of this sort on a larger space which restores much of the linearity usually associated with diffusion problems.

Thus, first, we shall define (P′) to be a one–sided version of the system (P), that is to say, (i) The positive part of f is set, identically, to zero (and has no source). (ii) The zero $p(t)$ is predetermined. We shall denote our density by ρ_p, which can be considered as the negative part of f, $-\min(0, f)$. Note that the integral of ρ_p, which is conserved over time, equals M_p. Thus, in principle, in order to recover the system (P), two such (P′) models can be glued together subject to "additional constraints" on their mutual $p(t)$. When two such models are to be used in tandem, the one on the left (now representing the *positive* part of f) will be denoted by ρ_b.

Secondly, we may define these sorts of systems – (P′) will be sufficient – on a *foliated* space. Let $\mathbb{N}$ denote the natural numbers, including zero, and consider $[-1, +1]^{\mathbb{N}}$. We shall refer to the individual elements as *levels* and, denote these, along with various associated quantities with a superscript: $[-1, +1]^{(0)}, [-1, +1]^{(1)}, \ldots$ Let $p(t) : [0, T) \to (-1, +1)$ denote a continuous function. On each level, we have a copy of $p(t)$ (always "located" in the corresponding position) and we consider a sequence of densities $\rho_p^{(0)}(x, t)$, $\rho_p^{(1)}(x, t), \ldots$, with the n^{th} density supported in $[p(t), 1]^{(n)} \subset [-1, 1]^{(n)}$. Initially, $\rho_p^{(n)}(x, 0) \equiv 0$ for $n > 0$ while $\rho_p^{(0)}(x, 0) = \rho_p(x, 0)$. Each of the $\rho_p^{(n)}(x, t)$'s obey the diffusion equation with a source to be described below, Dirichlet boundary conditions at their respective $p(t)$ and Neumann condition at the corresponding $x = 1$. Finally, each $\rho_p^{(n)}(x, t)$, $n > 1$ has a source which is located at its respective $p(t) + \underline{a}$ (where, we remind the reader, $\underline{a} = \min\{a, \frac{1}{2}[1 - p(t)]\}$) and has strength provided by $\lambda^{(n)}(t) = \frac{\partial}{\partial x}\rho_p^{(n-1)}(p(t), t)$. Thus, to be explicit, for $n \geq 1$:

$$
(Y_n) \qquad \begin{cases} \frac{\partial}{\partial t}\rho_p^{(n)} - \frac{\partial^2}{\partial x^2}\rho_p^{(n)} = \frac{\partial}{\partial x}\rho_p^{(n-1)}(p(t), t)\delta_{p(t)+\underline{a}}, & x \in [-1, 1]^{(n)}, \\ \rho_p^{(n)}(x, t) = 0, & x \leq p(t), \\ \frac{\partial}{\partial x}\rho_p^{(n)}(1, t) = 0, \quad \rho_p^{(n)}(x, 0) = 0, \end{cases}
$$

and, for $n = 0$,

$$
(Y_0) \qquad \begin{cases} \frac{\partial}{\partial t}\rho_p^{(0)} - \frac{\partial^2}{\partial x^2}\rho_p^{(0)} = 0, & x \in [-1, 1]^{(0)}, \\ \rho_p^{(0)}(x, t) = 0, & x \leq p(t) \\ \frac{\partial}{\partial x}\rho_p^{(0)}(1, t) = 0, \quad \rho_p^{(0)}(x, 0) = \rho_p(x, 0). \end{cases}
$$

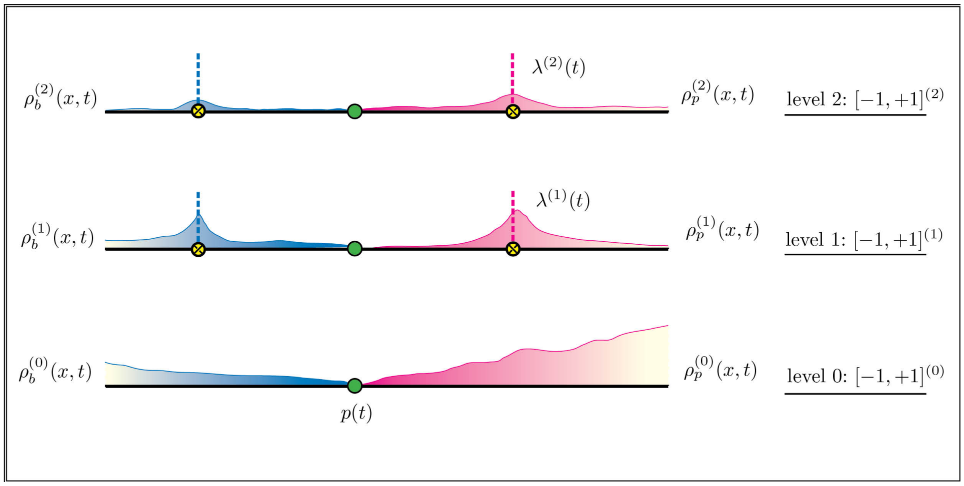

Figure 1: Illustration of the foliated model. Each *level* has its own density which vanishes at $p(t)$. Sources at the $k+1^{\text{st}}$ level – always located at $p(t) \pm \underline{a}$ – have strength that is determined by the flux through $p(t)$ on the k^{th} level. Notice that this completely decouples the flux–source interactions rendering the one–sided problems ($\rho_b \equiv 0$, $p(t)$ determined) an essentially linear problem. However, for the two–sided problems, with $p(t)$ dynamically generated, precise criteria for determination of $p(t)$ without defoliation remains a fully interacting problem.

In the context of the foliated model, we may calculate various quantities for the original model. Of particular relevance, it is seen that:

$$\sum_{n=0}^{\infty} \rho_p^{(n)}(x,t) = \rho_p(x,t) \tag{2.14}$$

and

$$\int_0^t \lambda(t')dt' = \sum_{n=0}^{\infty} n \int_{p(t)}^1 \rho_p^{(n)}(x,t)dx. \tag{2.15}$$

To vindicate the above claims, it is required to demonstrate that the higher levels are sparsely populated and that therein, away from the source, the derivatives of the $\rho^{(j)}$ converge absolutely. Indeed this follows from straightforward computations using Duhamel's formula, from which it is clear that the relevant densities and derivatives decay (at least) geometrically as $j \to \infty$.

Defining

$$\tilde{\rho}_p = \sum_{n=0}^{\infty} \rho_p^{(n)} \tag{2.16}$$

it is clear that $\tilde{\rho}_p$ satisfies the system (P$'$). We claim that this implies $\tilde{\rho}_p(x,t) = \rho_p(x,t)$ for the simple reason that solutions to (P$'$) are unique. Indeed, if we consider, in general, the problem (P$'$) even with density of indefinite sign then, following the contraction arguments of Section 1, we obtain a unique solution for short times. Since $p(t)$ is predetermined, there is no difficulty if $\lambda(t) \to 0$ or even changes sign. Thus, at least for short times, the difference between any solutions with identical initial data stays zero and *this* argument can be iterated for as long as the predetermined $p(t)$ remains well behaved. This establishes Eq.(2.14)

As for Eq.(2.15), we define

$$M_p^{(n)}(t) = \int_{p(t)}^{1} \rho_p^{(n)}(x,t)dx \tag{2.17}$$

it is seen that $\sum_n M_p^{(n)}(t)$ is conserved – and hence identically equal to M_p. Moreover, $M_p - M_p^{(0)}(t)$ is given by

$$M_p - M_p^{(0)}(t) = M_p^{(0)}(0) - M_p^{(0)}(t) = \sum_{n\geq 1} M_p^{(n)}(t) = \int_0^t \lambda^{(1)}(t')dt' \tag{2.18}$$

i.e., the mass lost on the first level is exactly that which fluxed up to the higher levels. Similarly we have for every level, the identity

$$\sum_{n\geq \ell} M_p^{(\ell)} = \int_0^t \lambda^{(\ell)}(t')dt' \tag{2.19}$$

and summing both sides of Eq.(2.19), we obtain Eq.(2.15)

2.1 The zero diffusion limit

This subsections concerns an requisite device which will be used in the proof of Proposition 2.2; in essence, this limit describes the behavior of various relevant systems at short time scales where diffusive effects can be neglected. The pertinent result, Proposition 2.1, is all that will be needed and is, in essence, apparent on physical grounds. Therefore, this subsection may be skimmed on a preliminary reading.

We shall have use for the foliated problem in the absence of diffusion which emerges as a limit of short time problems. As a first step, let us consider (without foliation) a sequence of ordinary Dirichlet – Neumann problems with a density $\sigma^\varepsilon(x,\tau)$ on $[-1,+1]\times[0,\varepsilon]$ where $\varepsilon \ll 1$. The system reads:

$$\begin{aligned} \sigma_t^\varepsilon(x,\tau) - \sigma_{xx}^\varepsilon(x,\tau) = 0; &\quad -1 \leq x \leq +1, \quad 0 \leq \tau \leq \varepsilon \\ \sigma^\varepsilon(x,\tau) = 0, &\quad x \leq p_\varepsilon(\tau) \\ \sigma_x^\varepsilon(1,\tau) = 0, & \end{aligned} \tag{2.20}$$

with $p_\varepsilon(\tau)$ a predetermined function to be described below and a preprescribed initial condition $\sigma(x,0)$ that is independent of ε. We define, as usual,

$$\lambda^{(\varepsilon)}(\tau) = \sigma_x^\varepsilon(p_\varepsilon(\tau),\tau). \tag{2.21}$$

For $0 \leq t \leq 1$, let $p_0(t)$ denote a smooth function with values in $[-1,+1]$. We will take $p_\varepsilon(\tau) = p_0(\varepsilon^{-1}\tau)$. The zero diffusion limit is defined by allowing $\varepsilon \to 0$. First, we define

$$\lambda_\varepsilon(t) := \varepsilon\lambda^{(\varepsilon)}(\varepsilon t) \tag{2.22}$$

as a function on $[0,1]$. With the above in mind, the zero diffusion limit is contained in the following:

Proposition 2.1. *Let σ^ε, λ^ε etc. be as described above with $\sigma(x,0) \in L^1$ non–negative and, for simplicity, it is supposed that p_0 is monotone and differentiable. Then, λ_ε converges, e.g., weakly:*

$$\lambda_\varepsilon(t) \to \lambda_0(t) := \dot{p}_0(t)\sigma(p_0(t), 0). \tag{2.23}$$

Similarly, the limiting density, $\sigma(x,t)$, is given by

$$\sigma(x,t) = \chi_{\{x > p_0(t)\}}\sigma(x,0). \tag{2.24}$$

Proof. Let $\varphi(t)$ denote a smooth test function on $(0,1)$ supported away from the endpoints. Then, by definition,

$$\int_0^1 \varphi(t)\lambda_\varepsilon(t)dt = \int_0^\varepsilon \varphi(\varepsilon^{-1}\tau)\lambda^{(\varepsilon)}(\tau)d\tau. \tag{2.25}$$

We rewrite $\lambda^{(\varepsilon)}$ in terms of the gradient:

$$\int_0^1 \varphi(t)\lambda_\varepsilon(t)dt = \int_0^\varepsilon \varphi\sigma_x^\varepsilon(p_\varepsilon(\tau),\tau)d\tau = -\int_0^\varepsilon \left[\int_{p_\varepsilon(\tau)}^1 \sigma_{xx}^\varepsilon(x,\tau)dx\right]\varphi d\tau \tag{2.26}$$

where the suppressed argument of φ in the last two terms is $\varepsilon^{-1}\tau$ and we have used the Neumann condition in the disintegration step. Using the heat equation we may replace σ_{xx}^ε by σ_t^ε and using the Dirichlet condition at $p_\varepsilon(\tau)$ we arrive at

$$\int_0^1 \varphi(t)\lambda_\varepsilon(t)dt = \int_0^\varepsilon \varepsilon^{-1}\varphi' \left[\int_{p_\varepsilon(\tau)}^1 \sigma^\varepsilon(x,\tau)dx\right] d\tau. \tag{2.27}$$

It is noted that the ε^{-1} is nonsingular since the argument of the φ is tailored for the range $[0,1]$. Next, it is claimed, in the limit $\varepsilon \to 0$, we may replace $\sigma^\varepsilon(x,\tau)$ in the integrand by $\sigma(x,0)$. Indeed, in brief, we add and subtract the desired term after first cutting out the region $p_\varepsilon(\tau) \le x \le p_\varepsilon(\tau) + \Delta$, for $\Delta \ll 1$ at the cost of an error of order $||\sigma(x,0)||_\infty\Delta$. The difference $\int_{p_\varepsilon(\tau)+\Delta}^1 [\sigma_\varepsilon(x,\tau) - \sigma(x,0)]dx$ can be decomposed into the difference of two positive terms each of which has a physical interpretation:

• The negative term represents all mass that was initially in $x > p_\varepsilon(\tau) + \Delta$ which diffused into $p_\varepsilon(\tau) \le x \le p_\varepsilon(\tau) + \Delta$ or into $p_\varepsilon(\tau')$ (with $p_\varepsilon(\tau') < p_\varepsilon(\tau)$) at some time $\tau' < \tau$ during the interval $[0,\tau]$.

• The positive term represents all mass that was initially in $x < p_\varepsilon(\tau) + \Delta$ which diffused into $x > p_\varepsilon(\tau) + \Delta$ (where $p_\varepsilon(\tau') < p_\varepsilon(\tau)$) without encountering the Dirichlet zero at $p_\varepsilon(\tau')$ at any $\tau' < \tau$ during the interval $[0,\tau]$.

Both of these terms are controlled by the mass currently/initially in $p_\varepsilon(\tau) \le x \le p_\varepsilon(\tau) + \Delta$ (which is in turn controlled by $||\sigma(x,0)||_\infty\Delta$) plus mass which, in time less than ε has diffused across a region of size Δ. The diffusive contribution may be bounded by a term of the form $R\exp -(r\Delta^2/\varepsilon)$ with R and r constants of order unity.

In short,

$$\int_0^1 \varphi(t)\lambda_\varepsilon(t)dt = \int_0^\varepsilon \varepsilon^{-1}\varphi'\left[\int_{p_\varepsilon(\tau)}^1 \sigma(x,0)dx\right]d\tau + ||\varphi'||_1 K(\varepsilon) \tag{2.28}$$

with $K \to 0$ as $\varepsilon \to 0$. At this point, the substantive term in the above equation can be reintegrated back with the result

$$\int_0^1 \varphi(t)\lambda_\varepsilon(t)dt = \int_0^1 \varphi(t)\dot{p}_0(t)\sigma(p_0(t),0)dt + ||\varphi'||_1 K(\varepsilon) \tag{2.29}$$

and the first result is established. The result for the density itself can be obtained by a derivation parallel to that which allowed us to replace $\sigma^\varepsilon(x,\tau)$ with $\sigma^\varepsilon(x,0)$ for $x > p_\varepsilon(\tau)$; we omit the details. $\square$

On the basis of the above proposition, we arrive, at least for monotone $p_0(t)$, at the zero diffusion limit: $\sigma_t(x,t) = -\dot{p}_0\sigma(x,0)\delta_{p_0(t)}$ with prescribed initial condition $\sigma(x,0)$. Taking some heed of the fact that the zero diffusion density is usually discontinuous at $x = p_0(t)$ it is noted that, in light of the solution in Eq.(2.24), we may replace the right side in the preceding relation by $-\dot{p_0}\sigma(p_0(t)^+,t)\delta_{p_0(t)}$. (We remark, incidentally, that it is arguable that a "source term" of this type actually belongs on the right side of the first equation in the standard Dirichlet – Neumann systems displayed in various places earlier in this work e.g., in Eq.(2.20), but in these contexts is always suppressed by the Dirichlet condition.) Finally, if $p_0(t)$ is *not* monotone, these result may be derived by first replacing $p_0(t)$ with the function which measures the maximum excursion to the right up till time t. Then it can be checked that $\sigma_t(x,t) = -\dot{p_0}\sigma(p_0(t)^+,t)\delta_{p_0(t)}$ still holds with the original p_0 reinserted.

It is clear that the zero diffusion limit may be transfered directly to the foliated model. Indeed, all that is really needed is the correct identification for the strength of the various source terms which has been provided by Eq.(2.23). Thus, relevant to the problem at hand, we obtain:

$$(Z_n) \qquad \begin{cases} [\rho_p^{(n)}]_t = \lambda^{(n)}(t)\delta_{p(t)+\underline{a}} - \dot{p}(t)\ \rho_p^{(n)}(p(t)^+,t)\delta_{p(t)}; & x \in [-1,1]^{(n)} \\ \rho_p^{(n)}(x,t) = 0; & x < p(t), \\ \rho_p^{(n)}(x,0) = 0 \end{cases}$$

for $n \in \mathbb{N}$ and

$$(Z_0) \qquad \begin{cases} [\rho_p^{(0)}]_t = -\dot{p}(t)\ \rho_p^{(0)}(p(t)^+,t)\delta_{p(t)}; & x \in [-1,1]^{(0)} \\ \rho_p^{(0)}(x,t) = 0; & x < p(t), \\ \rho_p^{(0)}(x,0) = \rho_p(x,0). \end{cases}$$

where in all of the above, $\lambda^{(n)}(t)$ is given by

$$\lambda^{(n)}(t) = \rho_p^{(n-1)}(p(t)^+,t)\dot{p}(t).$$

For future purposes, it noted that in the primary case

$$\int_0^t \lambda^{(1)}(t')\, dt' = M_p - \int_{p(t)}^1 {\rho_p}^{(0)}(x,t)\, dx = M_p - \int_{p(t)}^1 \rho_p(x,0)\, dx = \int_{-1}^{p(t)} \rho(x,0)dx.$$

The results of this subsection will be useful for the analysis of our system under conditions of short time rapid displacement of the zero. Notwithstanding the formidable appearance of the (Z)-system, the mechanics is actually quite simple: mass on each level simply gets displaced, as $p(t)$ sweeps through it, to the next level at the position corresponding to $p(t) + \underline{a}$. As such, is observed that in this limit, the density on each level is supported to the right of the furthest excursion of $p(t)$.

2.2 Proof of summability.

We have assembled the preliminary ingredients necessary for the central result of this section:

Proposition 2.2. *For any $T < \infty$, as long as the solution of (P) exists in $[0, T)$, then*

$$\int_0^T \lambda(t)dt < \infty.$$

Proof. As we shall see, the consequences of a finite time divergence violate sensible notions of the slow scale for the diffusive transport of substantial material over large distances. The pertinent observation is that for small time intervals, there is minimal initial diffusion over any appreciable distance and in the absence of diffusion, the "essential supports" of the p– and b–densities become so widely separated that later diffusion in the time allowed cannot possibly account for *complimentary* transports.

Suppose then that the above display does *not* hold. Then for any δt and K ($\delta t \ll 1$ and $K \gg 1$ to be specified when necessary) there is a $\mathfrak{t}_1$ with $\mathfrak{t}_1 + \delta t < T$ such that

$$\int_{\mathfrak{t}_1}^{\mathfrak{t}_1+\delta t} \lambda(t)dt > K.$$

For simplicity let us reset $\mathfrak{t}_1$ to zero and work with $t \in [0, \delta t]$. The midrange objective is to show that under the stipulation of a large flux in a short time, $p(t)$ must rapidly head towards the boundaries. We consider, for the time being, the one–sided perspective.

Here it is noted that if $a \ll 1$, and $p(t)$ does *not* enter the right rescue zone (namely $x > 1 - 2a$) and we ignore diffusive effects, then there are at most the order of $1/a$ levels that get occupied in the as $p(t)$ sweeps through its range. Thus, in accord with Eq.(2.15), to obtain very large fluxes in very small times, it is indeed necessary to quickly approach the vicinity of the boundary.

To analyze circumstances where diffusion is present and $p(t)$ gets "close" to the boundary, let us consider for $\Delta \in (0, a)$ (where, presumably, Δ/a is "small") the system (P'_Δ) which is defined exactly as the system (P') but with a replaced by Δ. (Also, one may presume, with a corresponding rescue plan, but this shall not enter into our considerations.) Let us contemplate, for identical initial conditions and identical $p(t)$ which remains *outside the* (P'_Δ) *rescue zone*, a comparison between the (P'_Δ) and the (P') systems (with diffusion included). We make two claims both of which are straightforward to verify on the basis of an underlying particle model constructed from non–interacting stochastic elements. The first claim is that the total flux in the Δ–system is not smaller than that of the usual system. The second claim is that in two such (P'_Δ) systems with differing zeros: $p(t) \leq q(t)$ (with both staying out of the rescue zone) the flux in the p–version is not larger than the flux in the q–version.

To establish these claims, let us discuss an underlying particle model for (Y_n): First consider, for fixed $p(t)$ and $\lambda^{(k)}(t)$ a single layer problem. This is a straightforward diffusion problem which, as is well known, can be achieved as the limit under diffusive (parabolic) scaling of a system of independent particles. The particles occupy the sites $[-N, -N+1, \dots N-1, N]$ and, at unit rate each particle randomly selects a neighboring site to which it jumps. Particles which jump into (the discretized version of) $p(t)$ disappear and particles appear at the appropriate location of the source with a rate governed by $\lambda(t)$. Limit results of this sort are readily derived or can be gleaned from standard texts on the subject e.g., [3], [9]. What is only slightly less elementary is the derivation in [2] Lemma 3.2, which allows unambiguous identification of the (limiting) flux through the zero as the (limiting) rate at which particles disappear into the discretized $p(t)$. With this in mind it is seen that, in the foliated problem, we may set up the system in such a way that the particles disappearing at the zero on the k^{th} level reappear at the source on the $k+1^{\text{st}}$ level. We may refer to such events as *promotions.*

We may now consider *coupling* two such systems, (P') and (P'_Δ) with identical initial conditions. Here, each particle is paired with its corresponding partner with the random left–right jumps deemed to be identical. Our first claim concerns a (P'_Δ)–system compared with an ordinary (P') system with a common $p(t)$ under the stipulation that $p(t)$ does *not* enter the rescue zone of the Δ–system. Here, it is seen that each particle in the Δ–system is in the same position, ahead or at a higher level than its corresponding partner. In the latter case, if the corresponding (Δ–system) particle is ℓ levels higher the corresponding position is no further *behind* than ℓ times (the appropriate measure of the length) Δ. This follows easily by induction. If it is true before a jump it is obviously true after the jump when both particles stay on the same level or the Δ–particle gets promoted up one level. On the other hand, if the a–particle receives such a promotion, it will get shunted $\underline{a}$ units (measured appropriately) backwards. Since $p(t)$ never enters the rescue zone determined by the (P'_Δ)–system this is at least Δ in the appropriate units and the desired statement is seen to always hold after the jump.

Similarly consider two (P'_Δ) systems with identical initial conditions but differing zeros – $p(t)$ and $q(t)$ with $q(t) \geq p(t)$ and the stipulation that, at least $q(t)$ stays out of the Δ–rescue zone. Then the same argument applies: particles in the q–system are "ahead" of their partners in the p–system exactly in the sense described above. Indeed, the proof of this assertion follows almost word for word.

In both cases, with the microscopic ordering of the two systems in hand, the ordering of the fluxes now follows immediately. Indeed, the discrete analogs of the right hand side of Eq.(2.15) are ordered as a direct consequence of the microscopic ordering and the left hand side is recovered in the continuum limit.

Returning to the problem at hand, let us suppose that $p(t)$ – in a (P) or (P′) model – is such that on $[0, \delta t]$, (with δt small and still to be specified later)

$$\max_t p(t) < 1 - 4\Delta. \tag{2.30}$$

Then we claim that the flux is limited (e.g., by the order of Δ^{-1}). To prove the claim, we use a comparison to a P'_Δ system with its own $q(t)$. Here $q(0) = p(0)$ and then q (almost) immediately jets out to $x = 1 - 2\Delta$, then backs to $1 - 4\Delta$ and stays there till time δt. The portion of the trajectory that occurs at $t = 0^+$ can be analyzed on the basis of the zero diffusion limit: Letting $N_\Delta \leq 2/\Delta$ denote the number of levels in the initial surge which get occupied. Then, by Eq.(2.15), the initial portion of the flux is not more than $N_\Delta M_p$.

Notice that (still in the q–system with parameter Δ) all the mass on various levels is trapped in the respective regions $\{x \geq 1 - 2\Delta\}$ with the zero – $q(t)$ – a distance 2Δ away. Thus, in the remaining time, the remaining flux is determined by diffusion through the stationary $q(t)$ i.e., how much can flux across various neighborhoods of size 2Δ in time δt. To this end let us now fix δt "small enough". To be specific, $(\delta t)^{1/4} = \gamma\Delta$ for γ sufficiently small ensures that the amplitude for diffusion across 2Δ – a unit mass at $1 - 2\Delta$ and a Dirichlet zero at $1 - 4\Delta$ is less than

$$\varepsilon := \varepsilon(\delta t) = \mathrm{e}^{-1/[\delta t]^{1/2}}.$$

More precisely, consider the problem

$$\begin{aligned} w_t - w_{xx} &= 0, \quad x \in (1 - 4\Delta, 1) \\ w(x, 0) &= \delta_{x=1-2\Delta}, \\ w(1 - 4\Delta, t) &= 0, \quad w_x(1, t) = 0. \end{aligned}$$

The quantity of mass diffuses from $x = 1 - 2\Delta$ to $x = 1 - 4\Delta$ in the interval of time δt is of the order $e^{-(2\Delta)^2/\delta t}$ vindicating the above definition of ε.

We may also interpret ε probabilistically: Starting at the k^{th} level, a fraction (less than) ε of the initial mass makes it to the $k + 1^{\text{st}}$ level in time δt and a fraction (more than) $1 - \varepsilon$ stays behind. Of the former, a fraction (less than) ε gets promoted to the $k + 2^{\text{nd}}$ level etc. In short, the upward distribution of diffused mass after time δt is dominated by the a geometric random variable Y, with parameter ε, and so the total flux coming from a unit mass which at $t = 0^+$ was on the k^{th} level in $[1 - 2\Delta, 1]$ is bounded by

$$\mathbb{E}(\mathrm{Y}) = \varepsilon(1 - \varepsilon) + 2\varepsilon^2(1 - \varepsilon) + \cdots = \frac{\varepsilon}{1 - \varepsilon}. \tag{2.31}$$

This latter quantity – independent of k – must be multiplied by the mass that was on the k^{th} level at $t = 0^+$ and summed. Since the total mass on all levels adds up to M_p this provides a bound on the diffusive contribution $\int \lambda dt$ that is given by

$$\frac{\varepsilon M_p}{1 - \varepsilon}. \tag{2.32}$$

Thus, back in the real problem for fixed $\Delta > 0$ sufficiently small (i.e., compared with the minimum of a and $(1 - a)$ as will be clear below) we may choose δt small enough and take the content of Eq.(2.32) and add this to our $M_p N_\Delta$, and then stipulate the sum of these two to be less than one half of the total K. Under these circumstances, there must be a first time $[\delta t]_p < \delta t$ such that $p([\delta t]_p) > 1 - 4\Delta$ and moreover, at this time $[\delta t]_p$, the total flux is less than $\frac{1}{2}K$.

With the above in hand, we incorporate the constraints of the *full* problem – namely a ρ_p and a ρ_b both in play with fluxes at $p(t)$ that are supposed to match. The first implication is that there is another such time $[\delta t]_b$ at which all of ρ_b is in $[-1, -1+2\Delta]$ and moreover, even at the later of these two times, there is more than half the fluxing left to be done. Let I_b denote the set

$$I_b = \{t \in [[\delta t]_b, \delta t] \mid p(t) < -1 + 4\Delta\} \tag{2.33}$$

and similarly for I_p. (Here, finally, we choose Δ small enough so that the intervals of size 4Δ about $x = \pm 1$ do not intersect and are themselves separated by a gap of at least 4Δ.)

It is clear, from the b–perspective that $\int_{I_b^c} \lambda dt$ is small. Specifically, an estimate of the form in Eq.(2.32) applies. Indeed, at the earlier time all of ρ_b was confined to the region $x < -1 + 2\Delta$ and has less time, for $t \in I_b^c$, than δt to achieve diffusion across the gap of length scale 2Δ. Similarly, from the p–perspective the flux during I_p^c is small. Since $I_b \subset I_p^c$ or vice versa, it is seen that there is no time after $\max\{[\delta t]_p, [\delta t]_b\}$ in which the requisite remaining flux can be achieved. Thus a contradiction is reached if the total flux is as large as was stipulated in the paragraph following Eq.(2.32). $\square$

As an immediate consequence, we may conclude that – at least in any finite time interval – $p(t)$ stays away from a neighborhood of the Neumann boundary.

Corollary 2.3. *Let $T < \infty$ denote any time up to which the solution to (P) exists. Then there is an $\eta > 0$ (depending on T) such that*

$$p(t) \in [-1 + \eta, 1 - \eta] \quad \text{for } 0 \le t \le T.$$

Proof. Suppose not. We may then assume that $p(t)$ approaches -1 as $t \to T$. In particular, there exists a sequence $t_n \to T$ where $\epsilon_n = p(t_n) + 1$ satisfies $\epsilon_{n+1} = \epsilon_n^2$. This means that the new excursion of $p(t)$ went past the previous excursion of the source. We may assume $n \gg 1$ so that $p(t_n) - \underline{a} = \frac{1}{2}\epsilon_n$.

Let $g_{n-1} := \int_{t_{n-1}}^{t_n} \lambda^{(1)}(t)dt$, where $\lambda^{(1)}(t)$ is the flux from the first term in the foliation model, (Y_n) introduced above, with initial data $f(x, t_{n-1})$. (I.e., we defoliate at each t_{k-1} and then run the foliated model till time t_k; recall we have assumed $\epsilon_n \ll 1$ so the rescue plan is in operation.) In the context of the zero diffusion model –(Z_n)– all the mass between $p(t_n)$ and $p(t_{n-1})$ is fluxed through the zero set at least once. On the other hand all the mass removed via all the fluxes at all the zero sets is then deposited to the left side by the source terms and, in the end, resides in $[p(t_n) - \frac{1}{2}\epsilon_n, p(t_n)]$. Therefore we would have

$$g_n = \int_{[p(t_{n+1}), p(t_n)]} \rho_b(x, t_n)dx \ge g_{n-1}$$

which would be an obvious violation of Proposition 2.2.

We shall show, more or less, that the diffusion cannot alter this situation in a significant way, i.e., the above display will be replaced by

$$g_n \geq g_{n-1}[1 - w_0\epsilon_n]$$

with w_0 of order unity. In light of the rapid convergence of the ϵ_n's to zero, this is sufficient.

Consider, then the situation at $t = t_n^-$, the moment before defoliation. Obviously g_n is represented by the total mass in the upper levels which we remind the readers resides in $\cup_{k>1}[-1, p(t_n)]^{(k)}$. All of these masses will be fluxed through $p(t)$ in the time interval $[t_n, t_{n+1}]$ *except* those portions which, by time t_{n+1} have diffused into $[-1, -1+\epsilon_n^2]$. We claim that the *fraction* which manages to do so is of the order ϵ_n. Indeed, consider an "element" of mass deposited on the k^{th} level at time $t_\phi \in [t_{n-1}, t_n]$, at position $p(t_\phi) - \underline{a} > -1 + \frac{1}{2}\epsilon_n$. This element has time $t_n - t_\phi$ to evolve whereupon it experiences an (abrupt) defoliation event bringing it to level zero after which it has an additional increment $t_{n+1} - t_n$ to achieve this task – all the while trying to avoid the Dirichlet zero at $p(t)$. [In $t_\phi \leq t \leq t_n$, this is because otherwise it would be deposited on the $k+1^{\text{st}}$ level whence we will estimate it in *that* context and for $t_n \leq t \leq t_{n+1}$ because then it would indeed contribute to g_{n+1}.] Notwithstanding the defoliation tribulation, this is entirely equivalent to the amplitude of a unit source placed at $p(t_\phi) - \underline{a}$ at time zero diffusing into $[-1, -1+\epsilon_n^2]$ in time $\Delta t = t_{n+1} - t_\phi$ with Dirichlet conditions at the (predetermined) zero $p(t_\phi + \omega)$; $0 \leq \omega \leq \Delta t$ with Neumann conditions at $x = -1$. It is not hard to see that the amplitude is increased by the removal of the Dirichlet boundary condition whereupon it is readily checked that, regardless of Δt, the amplitude is monotone with the distance of the initial position $p(t_\phi) - \underline{a}$. We may therefore uniformly estimate the fraction lost by

$$\epsilon_n^2 \sup_\sigma [G_\sigma(\frac{\epsilon_n}{2} - \epsilon_n^2) + G_\sigma(\frac{\epsilon_n}{2})] \tag{2.34}$$

where $G_\sigma(\cdot)$ is a standard normalized Gaussian of width σ and the two terms represent the contribution from the actual and image source. However, the supremum is of the order ϵ_n^{-1} (which comes about when Δt itself is the order ϵ_n^2) thence the portion of g_{n-1} lost to diffusion is of the order claimed in the display prior to Eq.(2.34). □

3 Global-time existence of the solution

In the previous section we showed that the problem has unique solution for a small time interval t_0. In this section we will show that we can iterate this process to produce the unique solution of our problem for global times.

Let us restart the process at the time t_0. This will lead to the existence (and uniqueness) of a solution in the time interval (t_0, t_1); we shall continue to iterate the process as long as we can; all quantities t_0, a_0, etc. will be indexed according to the iteration stage.

According to Eq.(1.8), the length of the time interval $t_{n+1} - t_n$ in which the contraction can take place is

$$(t_{n+1}-t_n)^{1/2} = \frac{\lambda(t_n)}{G(a_n)[\lambda(t_n)+16\,\|f(\cdot,t_n)\|_{L^\infty(-1,1)}]},$$

We remind the readers that $G(a) \to \infty$ as $a \to 0$.

Suppose there exists a *break-up time* $0 < t^\star < \infty$: it means that we can no longer find a small time interval during which Theorem 1.1 holds. This happens exactly when, for any sequence $t_n \to t^\star$, at least one of the following holds:

(i) $\limsup_{n\to\infty} \|f(\cdot,t_n)\|_{L^\infty(-1,1)} = \infty$ or

(ii) $\liminf_{n\to\infty} \lambda(t_n) \to 0$, or

(iii) $\limsup_{n\to\infty} |f_{xx}|(p(t_n),t_n) \to \infty$.

At the time $t^\star$ two possible configurations may happen: the limit of the free boundary $p(t)$ is an unique point as $t \to t^\star$, or a puddle of zero forms at $t = t^\star$. In both cases we will show that the arguments lead to a contradiction, showing the non-existence of such a $t^\star$. In the non-puddle case (Section 3.1) a contradiction will be yielded by proving that all relevant norms given in (i)-(iii) are bounded up to $t = t^\star$. In the puddle case (Section 3.2) the strategy is a bit different. We first show that (essentially) all derivatives of f are bounded up to $t = t^\star$: we then will show that such regularity result is too strong to hold at a break-up time, therefore concluding that the break-up time $t = t^\star$ does not exist.

We start showing what happens when the free boundary has an unique limit.

3.1 Non-puddle case: $\lim_{t\to t^\star} p(t) = p(t^\star)$

In this case we can show that, due to classical estimates on caloric functions, none of the three breaking factors listed above takes place, therefore yielding a contradiction.

Let $t^\star$ denote the breaking time. When the limit is unique, the source $p(t) - \underline{a}$ and the sink $p(t) + \underline{a}$ also have an unique limit as $t \to t^\star$. Moreover, the locations of the source and sink at the time $t = t^\star$ are $\underline{a}$–away from the point $x = p(t^\star)$.

This allows to find a spatial neighborhood of $p(t^\star)$ and a time interval (before $t^\star$) in which the source and sink never entered in that neighborhood during that time interval.

Let

$$\delta := \frac{\underline{a}(p(t^\star))}{8} > 0$$

and define I_δ as a small neighborhood with radius δ of the point $p(t^\star)$, i.e., $I_\delta := (p(t^\star)-\delta, p(t^\star)+\delta)$. Then there exists $0 < t_\delta < t^\star$ such that $I_\delta \times [t_\delta, t^\star]$ is far from (the trajectory of the) sink or source.

In terms of estimates, the fact that sink and source are far away from I_δ implies that we can use classical estimates for the homogeneous heat equation, as in the proofs Lemma 1.7 and 1.6.

Let us start the problem (P) with initial datum $f_I := f(x, t_\delta)$. Following the same contraction argument for the local existence, we can show that there exists an unique solution $f(x,t)$ in the time interval $(t_\delta, t_\delta + \Delta t_1)$. Iterating the process another time, we can extend the solution to a bigger time interval $(t_\delta, t_\delta + \Delta t_1 + \Delta t_2)$, with $\Delta t_2 > 0$. The extension process could continue by iteration. The following is due to local estimates for solutions of the heat equation:

Lemma 3.1. *Let $f(x,t)$ solve (P) in the time interval $(t_\delta, t_\delta + \Delta t)$. Then*

$$|f^{(k)}(x, t_\delta + \Delta t)| \leq C \frac{\|f(x,\cdot)\|_{L^1}}{[\Delta t]^{k+1/2}} \text{ in } I_\delta.$$

where $f^{(k)}$ denotes the k-th spatial derivative of f. The constant $C > 0$ does not depend on Δt nor the data at $t = t_\delta$.

Proof. We omit a formal proof; the result follows from the splitting as in Eq.(1.4). We use standard caloric estimates for the linear part and, for the non–linear part, we may isolate the sources use L^∞ norms on the fundamental solutions along with the L^1–properties for $\lambda(t)$ (as proved in Section 2). $\square$

The above lemma implies that in I_δ, f and all its derivatives are uniformly bounded up to $t = t^\star$:

$$|f^{(k)}(x, t_\delta + \tau)| \leq C(M_p + M_b)(\Delta t)^{-(k+1/2)} \quad \text{for } 0 < \Delta t < t^\star - t_\delta.$$

Therefore case (i) and (iii) are eliminated.

Moreover f_t is also uniformly bounded near $t = t^\star$ in I_δ, and thus $f(x,t)$ uniformly converges to a C^∞–function $F(x)$ as $t \to t^*$. Let us call this function $f(x, t^\star)$. Then $f(x,t)$ solves the heat equation in $I_\delta \times [t_\delta, t^\star]$. Below we show that $F(x)$ is non-degenerate at $x = p(t)$, thus eliminating (ii).

In order to establish this non–degeneracy, we first establish analyticity (and some sensible properties) of F in I_δ:

Lemma 3.2. *For every $t_\delta < t \leq t^\star$ fixed, the function $f(\cdot, t)$ is analytic in I_δ. Moreover the function $f(\cdot, t^\star)$ is positive to the left of $p(t^\star)$, and negative to the right of $p(t^\star)$.*

Proof. We write

$$\begin{aligned} f(x,t) = & \int_{-1}^{+1} \Gamma(x, x'; t) f_I(x')\, dx' \\ & + \int_0^t \left[\Gamma(x, (p(t') - a); t - t') - \Gamma(x, (p(t') + a); t - t')\right] \lambda(t')\, dt' \end{aligned}$$

The part of the solution coming from the linear term is analytic (see [4] Section 2.3.c). For the nonlinear part, since $x \in I_\delta$, there is no immediate singularity from the sources. We may replace $x \to z = x + iy$ with $|y| \lesssim 1$ which makes f a well-defined function of z and t. Moreover $f(\cdot, t)$ is complex differentiable i.e., satisfies the equation

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$$

and is therefore analytic. As for $t = t^*$, note that the formula still holds for $F(x) = f(x, t^\star)$.

For the second claim observe that $f(x,t)$ solves the heat equation in the domain $I_\delta \times [t_\delta, t^\star)$ with continuous zero set $p(t)$, and with initial data $f(x, t_\delta)$ which was positive to the left of $p(t_\delta)$ and negative to the right of $p(t_\delta)$. Furthermore on the lateral boundary of I_δ, $f(x,t) > 0$ on the right and $f(x,t) < 0$ on the left, for $t < t^\star$. Therefore the claim follows directly from one–sided comparisons. $\square$

We now turn to the possibility of (ii). Assume that

$$f_x(p(t^\star), t^\star) = 0.$$

The following holds

Proposition 3.3. *Under the assumptions of the non–puddle case, let* $t^\star$ *the first time at which the gradient at the free boundary degenerates i.e.,* $p(t^\star)$ *is well–defined and* $f_x(p(t^\star), t^\star) = 0$*. Then, also,*

$$f_{xx}(p(t^\star), t^\star) = 0.$$

Proof. The proof follows from a very simple observation: Suppose that $f_{xx}(p(t^\star), t^\star) \neq 0$. This fact cannot coexist with the properties of the function f: The function f is positive for $x < p(t^*)$ and negative for $x > p(t^*)$ with purportedly vanishing derivative and very smooth behavior across the free boundary (indeed f is spatially analytical in space in a neighborhood of the free boundary, as we have just shown). But $f_{xx}(p(t^\star), t^\star) \neq 0$ implies that in a sufficiently small deleted neighborhood of $p(t^\star)$, f is non–zero and of the same sign. $\square$

The question is now how to define the velocity of the free boundary at the time $t^\star$. Note that as long as $f_x(p(t), t)$ stays away from zero, the speed of $p(t)$ is given by

$$\dot{p}(t) = -\frac{f_{xx}(p(t), t)}{f_x(p(t), t)},$$

but, as we have just shown, this is an indefinite form if the first derivative degenerates at $x = p(t)$.

So, in order to find the speed of $p(t)$ at this critical time $t^\star$, let us take the Taylor expansion of the equation $f(p + \Delta p, t + \Delta t) = 0$. Then, the following holds:

$$\begin{aligned} 0 =& f(p + \Delta p, t + \Delta t) = f(p,t) + f_x(p,t)\Delta p + f_t(p,t)\Delta t \\ &+ \frac{1}{2!}[f_{xx}[\Delta p]^2 + 2f_{xt}\Delta p \Delta t + f_{tt}[\Delta t]^2] \\ &+ \frac{1}{3!}[f_{xxx}[\Delta p]^3 + 3f_{xxt}[\Delta p]^2\Delta t + 3f_{xtt}\Delta p[\Delta t]^2 + f_{ttt}[\Delta t]^3] + ... \end{aligned}$$

The third and higher order terms are ostensibly negligible so, focusing attention on the second order terms and using the diffusive relation between spatial and temporal derivatives, we find

$$\dot{p} = -\frac{f_{xxxx}}{2f_{xxx}}.$$

We would like now do determine if this speed in finite or infinite: from the Hopf Lemma we know that in a parabolic problem, as long as the free boundary has finite speed, the derivative at the boundary is strictly positive (or negative) for all time. This implies that in our case, at the time $t^\star$ the speed of p is not finite. Consequently the value $f_{xxx}(p(t^\star), t^\star)$ has to be zero (because f_{xxxx} *cannot* be infinite). Now we can repeat Lemma 3.3 for the derivatives f_{xxx} and f_{xxxx} and conclude that at $(p(t^\star), t^\star)$, both of these derivatives are zero. It is clear that the procedure continues indefinitely:

Lemma 3.4. *Under the assumptions of Proposition 3.3 all the spatial derivatives of f evaluated at $(p(t^\star), t^\star)$ vanish.*

Proof. We proceed by induction. Suppose that the first $2(n-1)$ spatial derivatives of f at $(p(t^\star), t^\star)$ vanish (which by the above is known for n small enough). Then, the content of the $(n-1)^{\text{st}}$ order in the Taylor expansion is vacuous and we turn to the n^{th} order. (Note that writing $\Delta p \approx \dot{p}\Delta t$, this may be viewed as the n^{th} order in Δt.) We obtain

$$0 = \sum_{m=0}^{n} \binom{n}{m} [\dot{p}]^m \partial_x^m \partial_t^{n-m} f\,|_{(p(t^\star),t^\star)}. \tag{3.1}$$

We exchange temporal for spatial derivatives: $\partial_t^{n-m} \to \partial_x^{2n-2m}$ and, on the basis of the inductive assumption, notice than only the $m = 0$ and $m = 1$ terms survive whence

$$\dot{p} = -\frac{1}{n}\frac{f^{(2n)}}{f^{(2n-1)}}, \tag{3.2}$$

where the superscript denotes the order of the spatial derivative and these quantities to be evaluated at $(p(t^\star), t^\star)$. Repeating the argument prior to this lemma, the Hopf Lemma and regular behavior at the free boundary necessitates that the odd derivative, the $2n-1^{\text{st}}$, vanish. But this term was purported to be the coefficient of the leading order in the spatial Taylor expansion of $f(x, t^\star)$ about $x = p(t^\star)$. By invoking even non–negativity/non–positivity in the neighborhood of $p(t^\star)$ this implies the vanishing of the $2n^{\text{th}}$ derivative as well. □

We now easily finish off case (ii). The preceding implies that all the derivatives of $f(x, t^\star)$ are zero at $x = p(t^\star)$. Analyticity, one subject matter of Lemma 3.2, then dictates that $f(x, t^\star)$ is identically zero in the entire neighborhood I_δ. This is a contradiction since we saw in the other part of Lemma 3.2 that any solution at the time $t^\star$ is strictly positive for $x < p(t^\star)$ and negative for $x > p(t^\star)$. Therefore the gradient of the function at the free boundary $p(t)$ stays strictly negative for any time $t \geq 0$. We are done if there are no puddles.

Remark As it turns out, the techniques of this subsection actually apply, almost without modification, to the case of a sufficiently small puddle; in particular, when the diameter of the puddle does not exceed $\underline{a}$ as measured from the closest endpoint to the boundary. However this observation does not provide any simplification for the forthcoming: We provide a treatment for purported puddles of any non–zero size.

3.2 The puddle case: $\mathcal{L} := \liminf_{t\to t^\star} p(t) < \mathcal{R} := \limsup_{t\to t^\star} p(t)$

Here the situation requires more careful analysis: first we claim that if a puddle forms at $t = t^\star$ then all spatial derivatives of f are bounded in the whole domain, at least along a sequence of times converging to $t^\star$. (For precise statement see Proposition 3.5 below). This result, along with another analyticity argument, will immediately yield a contradiction. It is re-emphasized that, in the regularized case, such results follow essentially from quadrature. However, these estimates deteriorate as the regularization is removed so we shall not pursue this venue.

If $\mathcal{L}$ and $\mathcal{R}$ are defined as in the heading of this subsection it is clear that

$$\int_{\mathcal{L}}^{\mathcal{R}} |f|(x,t)dt \to 0 \text{ as } t \uparrow (t^\star)$$

(This is because the zero travels between the endpoints of the puddle so fast that there is not enough time for mass to diffuse into or flux through the zero set as $t \to t^\star$; c.f. the proof of Proposition 2.2.)

In what is to follow, we will consider some $t_0 \leq \mathfrak{t}_0 < t^\star$ which is adjustable and of no actual significance. Roughly speaking $\mathfrak{t}_0$ is the purported time when "things begin to go wrong". In particular, let us choose $\mathfrak{t}_0$ close enough to $t^\star$ such that

$$\text{(BH)} \qquad \sup_{\mathfrak{t}_0 \leq t < t^\star} \int_{\mathcal{L}}^{\mathcal{R}} |f|(x,t)dx \leq \frac{1}{10}\min(M_b, M_p).$$

3.2.1 Derivative bounds

Proposition 3.5. *Suppose*

$$\liminf_{t\to t^\star} p(t) = \mathcal{L} < \mathcal{R} = \limsup_{t\to t^\star} p(t).$$

Then for any $\ell \in \mathcal{N}$, there exists a constant C_ℓ such if $\beta > 0$ is sufficiently small then the following holds: for any sequence $t_n \to t^\star$ such that $p(t_n) \to \mathcal{R}$,

$$\limsup_{n\to\infty} |f^{(\ell)}(x,t_n)| \leq C_\ell \text{ for any } x \in [\mathcal{R}-\beta, \mathcal{R}+\beta].$$

A parallel result holds for $\mathcal{L}$.

Proof. Since $p(t_n) \to \mathcal{R}$, we can choose β so that $\underline{a}(\mathcal{R}) > 3\beta$ and also $\mathcal{R} - \mathcal{L} > \beta$. This means that for all n sufficiently large,

$$p(t_n) + \underline{a}(p(t_n)) > \mathcal{R} + 2\beta \tag{3.3}$$

i.e., at the times $t = t_n$ the sink is well outside the interval under consideration. Moreover at some point $\tilde{t}$ in $[\mathfrak{t}_0, t^\star]$,

$$p(t) - \underline{a}(p(t)) < \mathcal{R} - 2\beta \text{ for } t \geq \tilde{t}. \tag{3.4}$$

Thus, for $t > \tilde{t}$ and for any $x_0 \in [\mathcal{R} - \beta, \mathcal{R} + \beta]$, the source only represents a distant agitation. With this in mind, let us reset $\mathfrak{t}_0$ such that Eq.(3.3) holds for all n and Eq.(3.4) holds for all times greater than $\mathfrak{t}_0$. Due to an upcoming plethora of indices, we might as well define $t_n := t^{\#}$, with (only) the property of Eq.(3.3) to be reserved for later.

Our goal is to estimate magnitude of the ℓ^{th} derivative of $f(x_0, t^{\#})$ for $x_0 \in [\mathcal{R} - \beta, \mathcal{R} + \beta]$. It will prove convenient to work with the auxiliary variable which measures the time remaining:

$$\theta := \theta(t) = t^{\#} - t \tag{3.5}$$

for $t \in [\mathfrak{t}_0, t^{\#}]$. Let us define a slowly diverging function $B(\theta)$:

$$B(\theta) := \sqrt{B_0} |\log \theta|^{1/2}, \tag{3.6}$$

with B_0 to be chosen in the next paragraph.

We denote by $X_0 := x_0 - \underline{a}$ which locates $p(t)$ when the sink directly contributes to $f(x_0, \cdot)$ and its derivatives. If we denote by $C(t)$ the distance between $p(t)$ to X_0 then by the Green's function formula

$$|f^{(\ell)}(x_0, t^{\#})| \leq \int_{\mathfrak{t}_0}^{t^{\#}} \frac{\lambda(t)}{\theta^L} \mathrm{e}^{-C(t)^2/\theta} dt \tag{3.7}$$

for some $L = L(\ell)$. We start by defining the set $\mathbb{H} \subset [\mathfrak{t}_0, t^{\#}]$ via

$$\mathbb{H} = \{t \mid C(t) < B(\theta)\sqrt{\theta}\}. \tag{3.8}$$

Our first claim is that $\mathbb{H}$ is the only important set for the ostensible development of singularities in $f(x_0, \cdot)$. Indeed

$$\int_{\mathbb{H}^c} \frac{\lambda(t)}{\theta^L} \mathrm{e}^{-C(t)^2/\theta} dt \leq \int_{\mathbb{H}^c} \frac{\lambda(t)}{\theta^L} \mathrm{e}^{-B^2} dt \leq \int_{\mathfrak{t}_0}^{t^{\#}} \lambda(t) \theta^{B_0 - L} dt \tag{3.9}$$

which converges and is small independent of $t^{\#} < t^{\star}$ for $B_0 > L$. Henceforth we may focus on events that take place when $t \in \mathbb{H}$ where, for all intents and purposes, there is no help from the exponential factors (But, on the positive side, $\mathbb{H}$ is disjoint from the tail end of $[\mathfrak{t}_0, t^{\#}]$).

To aid with our objectives, it will be convenient to divide $[\mathfrak{t}_0, t^{\#}]$ into disjoint regions that are of equal size on a logarithmic scale: Let $H \gg 1$ denote a sufficiently large number the precise (minimum) value of which will be determined in what is to follow. Roughly speaking, we wish the k^{th} region to be of size H^{-1} of the $k - 1^{\text{st}}$. Specifically, we may proceed as follows: The k^{th} region will be denoted by g_k, $k = 0, 1, \dots$ and the size,

$|g_k|$, will satisfy $|g_k| = H^{-k}|g_0|$. (Thus $|g_0| = \frac{H-1}{H}(t^{\#} - \mathfrak{t}_0)$). Therefore

$$g_0 = [\mathfrak{t}_0, \mathfrak{t}_0 + |g_0|), \ \dots \ , g_k = [\mathfrak{t}_0 + |g_0| + \dots + |g_{k-1}|, \mathfrak{t}_0 + |g_0| + \dots + |g_k|]. \tag{3.10}$$

Note that the value of θ at the right end of g_k is a constant (that is very near one) times the size of g_{k+1}.

We shall also need two spatial regions: $\mathbb{A}_2$, $\mathbb{A}_3$ which are given by

$$\mathbb{A}_2 = (X_0 + b, X_0 + 2b), \quad \mathbb{A}_3 = (X_0 + 2b, X_0 + 3b)$$

with $b > 0$ chosen small enough so that these sets are inside $[\mathcal{L}, \mathcal{R}]$ lying well (on the scale of b) to the left of $\mathcal{R}$. (see Figure 2 below.) Moreover b is large enough (or $\mathfrak{t}_0$ late enough) so that

$$b >> B(\theta(\mathfrak{t}_0))\sqrt{\theta(\mathfrak{t}_0)}.$$

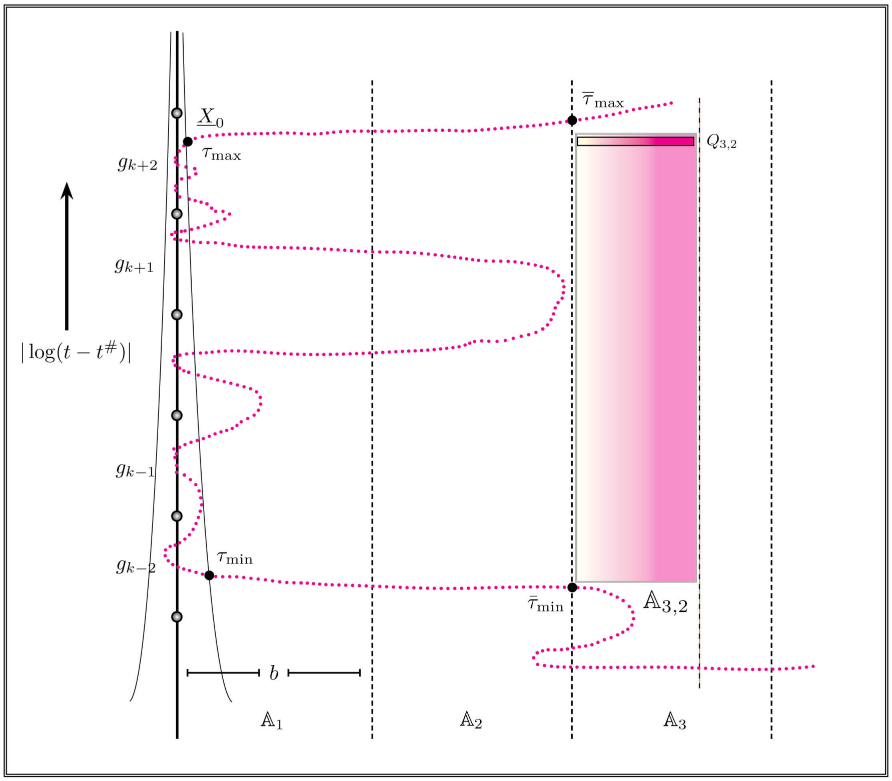

Figure 2: An *epoch* spread out over several (logarithmic) scales. The density ρ_p is non–zero the right while the density ρ_b is non–zero to the left of the dotted line which marks the trajectory of $p(t)$. The buildup of ρ_p in the region $\mathbb{A}_{3,2}$ (to a total amount of $Q_{3,2}$) takes place throughout the epoch. The content of Proposition 3.5 and Lemma 3.6 is that epochs are confined to regions of a single scale–size.

Next we define *epochs* that are punctuated by certain exits from and entrances to the region $\mathbb{A}_3$. The beginning of an epoch, denoted by $\overline{\tau}_{\min}$ is when $p(t)$ enters $\mathbb{A}_2$ from $\mathbb{A}_3$ and will not revisit $\mathbb{A}_3$ before first having touched the appropriate $B\sqrt{\theta}$ neighborhood of X_0. In particular, $p(\overline{\tau}_{\min}) = X_0 + 2b$. The time $\tau_{\max}$ is when $p(t)$

leaves the $B\sqrt{\theta}$ neighborhood of X_0 and does not touch these neighborhood types till another visit to $\mathbb{A}_3$. To be specific, $\tau_{\max}$ will be the moment of this departure, so $p(\tau_{\max}) = X_0 + [B\sqrt{\theta}]|_{\tau_{\max}}$. For future purposes, we shall define $\underline{X}_0$ as the point of departure: $\underline{X}_0 =: p(\tau_{\max})$. There is a third time, namely $\tau_{\min}$ when $p(t)$ actually enters the appropriate $B\sqrt{\theta}$ neighborhood but this time does not play a major role.

Our first claim is that if epochs are localized e.g., to a single g_k then their contribution to $||f^{(l)}(x_0, \cdot)||$ is tractable. Indeed, assume for simplicity that times

$$\overline{\tau}_{\min}^{[1]} < \tau_{\min}^{[1]} < \tau_{\max}^{[1]} < \cdots < \overline{\tau}_{\min}^{[J]} < \tau_{\min}^{[J]} < \tau_{\max}^{[J]}$$

are the only punctuation marks in g_k. Then

$$\int_{\tau_{\min}^{[1]}}^{\tau_{\max}^{[J]}} \frac{\lambda(t)}{\theta^L} \mathrm{e}^{-C(t)/\theta^2} \chi_{\mathbb{H}} dt \leq \frac{1}{\theta_{k+1}^L} \int_{g_k \cap \mathbb{H}} \lambda(t) dt \tag{3.11}$$

where $\theta_{k+1} := t^{\#} - (t_0 + ... + |g_k|) \approx |g_{k+1}|$ is the time remaining by the end of g_k.

Next we show that the integral to be done is actually of the order of $\exp[-1/\theta_k]$. This will be facilitated by the perspective of ρ_p: Note that at $t = \overline{\tau}_{\min}^{[1]}$, the density $\rho_p(x, \overline{\tau}_{\min}^{[1]})$ vanishes for $x < X_0 + 2b$ and has some positive profile for $x > X_0 + 2b$. Starting from this profile, we are supposed to compute the flux of ρ_p through $p(t)$ while $t \in \mathbb{H}$. It is not hard to see that this is less than the total flux through (the vicinity of) X_0 with Dirichlet boundary conditions at $x = X_0 + B\sqrt{\theta}$, and an initial profile of a delta mass at $X_0 + 2b$. The strength of the δ–mass might initially taken to be the total of M_p. But to account for the possibility of reflux from the p–source (sink) – which is much further away, let us add to this the quantity

$$\Lambda := \int_0^{t^\star} \lambda dt < \infty$$

which estimated all that ever has fluxed and all that ever will flux. This leads to the estimate

$$\int_{g_k \cap \mathbb{H}} \lambda(t) dt \ \leq \ C_2 \mathrm{e}^{-c_2/\theta_k} \tag{3.12}$$

where $C_2 < \infty$ and $c_2 > 0$ are constants which do not depend on k, or the end time $t^{\#}$.

The key issue therefore is to show that the epochs do not extend over many g–scales. Indeed, it is remarked that, an extension of the above reasoning tracking a single epoch through scales $K_1 \leq k \leq K_2$ would yield a bound of the form

$$\frac{\tilde{C}_1}{(\theta_{K_2})^L} \mathrm{e}^{-c_1/\theta_{K_1}} \tag{3.13}$$

replacing the corresponding right hand side of Eq.(3.11). And, as unlikely as it may seem, if $K_1 \gg K_2$ this could be large. We further remark that some positive powers of $C(t) = |X_0 - p(t)|$ originating from various places are available (inside the integrand) for the estimate. But even in the best case scenario – namely the estimate for the norm of $f(x_0, t)$ itself – there is still a logarithmic divergence ($\propto K_2 - K_1$) weighing in against the exponential prefactor. Finally, it might be *technically* noted that since $t^{\#} < t^{\star}$ it must be the case that all but a finite number of the g_k are devoid of epochs. However the technicality is of no practical significance since we seek bounds that are uniform in $t^{\#} < t^{\star}$. We turn to the task at hand.

Consider an epoch defined by the times $\overline{\tau}_{\min}$ and $\tau_{\max}$. Let $\mathbb{A}_{3,2}$ denote the left and middle third of the $\mathbb{A}_3$ region namely

$$\mathbb{A}_{3,2} = \{x \mid X_0 + 2b \le x \le X_0 + \frac{8}{3}b\} \tag{3.14}$$

and let us estimate the accumulation of mass in the region $\mathbb{A}_{3,2}$ at time $t = \tau_{\max}$. Note that in addition to the flux from the outside of the region, there is the initial mass that was in $\mathbb{A}_3$ at time $t = \overline{\tau}_{\min}$. This is actually of *negative* utility. We cannot rely on it staying in the region and we do not wish to account for where it might go during $\overline{\tau}_{\min} < t < \tau_{\max}$. However, in accord with condition (BH), this could only account for 1/10 of M_p. Then by mass conservation there is a significant portion of M_p outside the puddle, on the right of $\mathcal{R}$.

Thus, we shall rely on this material that diffuses in from the right. Here, it is worthwhile to recollect that the underlying condition of the epoch is that $p(t)$ stay out of $\mathbb{A}_3$. So, placing the guaranteed fraction of M_p at the extreme right – $x = +1$ – and placing Dirichlet boundary conditions at $x = X_0 + 2b$ we obtain the estimate

$$Q_{3,2} = \int_{\mathbb{A}_{3,2}} \rho_p(x, \tau_{\max}) dx \ge C_{3,2} \mathrm{e}^{-c_{3,2}/[\theta(\overline{\tau}_{\min}) - \theta(\tau_{\max})]} \tag{3.15}$$

where $C_{3,2} > 0$ and $c_{3,2} < \infty$ are constants independent of the parameters of the epoch.

Now just about all of $Q_{3,2}$ and more will be swept into $p(t)$ between time $\tau_{\max}$ and the next time that $p(t)$ crosses all the way to the right side of the region $\mathbb{A}_3$. This may happen immediately but it might incur the passage of other possible (long) epochs. However, from the stipulations about the time sequence (t_n) it is inevitable that this will happen before $t = t^{\#}$.

Our final substantive claim is that $Q_{3,2}$ is essentially dominated by the flux through the left boundary of $\mathbb{A}_2$ in the time interval $[\tau_{\max}, t^{\#}]$ by a delta–source of strength $M_p + M_b + \Lambda$ placed at $\underline{X}_0 = X_0 + B\sqrt{\theta}|_{\tau_{\max}}$ at time $\tau_{\max}$. The implication provides an upper bound which we state as a separate lemma:

Lemma 3.6. *Let $Q_{3,2}$ be as described. Then*

$$Q_{3,2} \le C_4 e^{-c_4/\theta(\tau_{max})} \tag{3.16}$$

where c_4 and C_4 do not depend on the parameters of the epoch nor on the time $t^{\#}$.

On the basis of Lemma 3.6 and Eq.(3.15), we may conclude that

$$\theta(\tau_{\max}) \ge c_1 \theta(\overline{\tau}_{\min}) \tag{3.17}$$

with $c_1 > 0$ independent of the parameters of the epoch and the time $t^{\#}$. Thus consider H to be sufficiently large ($H > 1/c_1$) and let us double cover $[\mathfrak{t}_0, t^{\#}]$ with two overlapping partitions of the general type described – both with parameter H – in such a way that (on the logarithmic scale) the midpoints of one partition form the endpoints of the other.

Then according to Eq.(3.17), all epochs are caught in individual elements of one or the other partition. This enables us to sum (twice) the analogue of Eq.(3.13) with matching k's all the way to $k = \infty$ which demonstrates the desired result. $\square$

Proof of Lemma 3.6.

Consider the situation at $t = \tau_{\max}$ where there is non–zero ρ_b for $x < \underline{X}_0$ and non–zero ρ_p for $x > \underline{X}_0$. The initial densities (and ensuing fluxes) are, evidently such that $p(t)$ will exit from the neighborhood of X_0 and, soon enough, cross all the way into $\mathbb{A}_3$. Later – maybe much later on the log–scale – it will cross all the way *through* $\mathbb{A}_3$.

Now consider $\tilde{\rho}(x,t)$ defined in the domain $\Sigma^{\#} := [-1, X_0+3b] \times [\tau_{\max}, t^{\#}]$ as follows:

(a) $(\partial_t - \Delta)\tilde{\rho} = \lambda(t)\delta_{x=p(t)-\underline{a}}$;

(b) $\partial_x \tilde{\rho}(-1,t) = 0$;

(c) $\tilde{\rho}(X_0 + 3b, t) \equiv 0$;

(d) $\tilde{\rho}(x, \tau_{\max}) = \rho_b(\cdot, \tau_{\max}) + \varphi(x) - \rho_p(x,t)\chi_{(X_0+2b, X_0+\frac{8}{3}b)}(x)$;

where φ is the mirror image of $\rho_p(\cdot, \tau_{\max})$ in $\mathbb{A}_3$ reflected about $x = X_0 + b$, i.e.,

$$\varphi(x) := \rho_p(2(X_0 + b) - x, \tau_{\max})\chi_{(X_0-b, X_0)}(x).$$

We denote $\alpha(t) := \{x : \tilde{\rho}(x,t) = 0\}$ and denote by $t^{\ddagger}$ be the first time when $\alpha(t)$ touches $X_0 + 3b$. There are two observations:

• First, by maximum principle for caloric functions it follows that $\rho \le \tilde{\rho}$ in $\Sigma^{\#}$, and thus

$$p(t) \le \alpha(t) \quad \text{ for } \tau_{\max} \le t \le t^{\ddagger} \tag{3.18}$$

This, of course, places $t^{\ddagger} \le t^{\#}$.

• Secondly, observe that $\tilde{\rho}(x,t) \ge \tilde{\rho}(2(X_0 + b) - x, t)$ in Σ, again by the maximum principle for caloric functions. In particular,

$$\alpha(t) \in [X_0 + b, X_0 + 3b] = \mathbb{A}_2 \cup \mathbb{A}_3; \quad \tau_{\max} \le t \le t^{\ddagger}. \tag{3.19}$$

Since $\int_{\alpha(\tau_{max})}^{X_0+3b} \tilde{\rho}(x, \tau_{\max})dx = Q_{2,3}$ and $\int_{\alpha(t^{\ddagger})}^{X_0+3b} \tilde{\rho}(x, t^{\ddagger})dx = 0$, it follows that

$$Q_{2,3} = \int_{\tau_{\max}}^{t^{\ddagger}} (-\tilde{\rho}_x)(\alpha(t'), t')dt' + \int_{\tau_{\max}}^{t^{\ddagger}} \tilde{\rho}_x(X_0 + 3b, t')dt'. \tag{3.20}$$

Moreover since we know $\alpha_t \in \mathbb{A}_3 \cup \mathbb{A}_2$ for all $t \in [\tau_{\max}, t^{\ddagger}]$, by obvious dominance the second term in Eq. (3.20) is bounded by the flux of $h(x,t)$ through $x = X_0 + 3b$, where $h(x,t)$ solves the heat equation in the region to the right of $X_0 + b$ during the times $[\tau_{\max}, t^\ddagger]$ with initial data the same as $\tilde{\rho}$ at $t = \tau_{\max}$ and Dirichlet conditions both sides of $\mathbb{A}_2 \cup \mathbb{A}_3$. Since $t^\ddagger < t^\#$ we may write that the *fraction* of $Q_{3,2}$ that is lost via the right boundary is less than $C_5 \mathrm{e}^{-c_5/\theta(\tau_{\max})}$ where the C_5 and c_5 only depends on b. (So that the *total* which is lost is no more than $Q_{2,3} C_5 \mathrm{e}^{-c_5/\theta(\tau_{\max})}$. As far as we are concerned it is sufficient that this is less than half of $Q_{2,3}$.)

It remains to estimate the first term in Eq.(3.20). For this purpose, let us consider $g(x,t)$ solving the same problem $(a) - (d)$ in Σ as $\tilde{\rho}$, except that $g(x,t)$ has Dirichlet boundary conditions at $x = X_0 + b$. By caloric inequalities the flux of $\tilde{\rho}$ through $\alpha(t)$ (where $\alpha(t) \geq X_0 + b$) in the interval $[\tau_{\max}, t^\ddagger]$ is, from the perspective of the left, less than the flux of g through the line $x = X_0 + b$. Finally we may further modify g to a $\tilde{g}$ which has the Dirichlet condition at $x = X_0 + b$, but has all conceivably available mass – namely $M_p + M_b$ – placed at $\underline{X}_0$. Moreover, the source term is placed as far forward as possible and allowed (more than) all of its available flux as soon as possible. This amounts to adding Λ to the M_p and M_b which are in the mass at $x = \underline{X}_0$. Thence

$$\begin{aligned} \int_{\tau_{max}}^{t^\ddagger} (-\tilde{\rho}_x)(\alpha(t'), t') dt' &\leq \int_{\tau_{max}}^{t^\ddagger} (-\tilde{g}_x)(X_0 + 2b, t') dt' \\ &\leq C_4 e^{-c_4/\theta(\tau_{max})} \end{aligned}$$

where c_4 and C_4 only depends on b, M_p, etc. □

Now may now we finish the proof of Theorem 0.1.

Corollary 3.7. *There does not exist a finite break-up time $t^\star$.*

Proof. We already eliminated the non-puddle case in Section 3.1, so let us discuss the puddle case. Let $\mathcal{L}$ and $\mathcal{R}$ be as above. Let $t_n \to t^\star$ be such that $p(t_n) \to \mathcal{R}$. In particular, choose n large enough such that $|p(t_n) - \mathcal{R}| < \epsilon \ll \frac{1}{8}\underline{a}$. Here let us define $f_n(x) = f(x, t_n)$. The function $f_n(x)$ has a unique zero at $p(t_n)$. In Lemma 3.2 it was proven that (for n sufficiently large) f_n is an analytic function in any subset of the open interval $(p(t_n) - \frac{1}{8}\underline{a}, p(t_n) + \frac{1}{8}\underline{a})$. This implies that $f_n(x)$ is analytic in the interval $(\mathcal{R} - \beta, \mathcal{R} + \beta)$ for large enough n.

Moreover due to Proposition 3.5 the sequence $\{f_n\}$ is uniformly bounded and uniformly Lipschitz continuous in the 2β neighborhood of $\mathcal{R}$. Hence Ascoli-Arzela Theorem ensures the existence of a subsequence f_{n_k} such that $f_{n_k} \to \phi(x)$ as $t_n \to t^\star$ uniformly in $(\mathcal{R} - \beta, \mathcal{R} + \beta)$.

Now, since the sequence of analytic functions f_{n_k} converges uniformly, the limiting function ϕ is also analytic in $(\mathcal{R} - \beta, \mathcal{R} + \beta)$. Let us observe the profile of ϕ near $\mathcal{R}$. From the choice of the sequence t_n, it is clear that $\phi(x)$ is positive to the right of $x = \mathcal{R}$. (Indeed for any $\epsilon > 0$, for sufficiently large n depending on ϵ the function $f(x,t)$ solves the heat equation with source term in $\Sigma_\epsilon := [-\mathcal{R} + \epsilon, +1] \times [t_n, t^\star)$ with positive boundary data at $\mathcal{R} + \epsilon$ and $t = t_n$ and Neumann boundary data at $x = 1$. Therefore $f(x,t)$ stays strictly positive in Σ_ϵ, staying uniformly away from zero as $t \to t^\star$.) On the other hand ϕ is identically zero to the left of $\mathcal{R}$. These two facts interdict the possibility of analyticity for the function ϕ. □

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