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Exponential Decay in Time of Solutions of the Viscous Quantum Hydrodynamic Equations

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Abstract—The long-time asymptotics of solutions of the viscous quantum hydrodynamic model in one space dimension is studied. This model consists of continuity equations for the particle density and the current density, coupled to the Poisson equation for the electrostatic potential. The equations are a dispersive and viscous regularization of the Euler equations. It is shown that the solutions converge exponentially fast to the (unique) thermal equilibrium state as the time tends to infinity. For the proof, we employ the entropy dissipation method, applied for the first time to a third-order differential equation.

Keywords—Quantum hydrodynamics, Wigner-Fokker-Planck equation, Long-time behavior of solutions, Entropy dissipation method.

1. INTRODUCTION

It is well known that quantum models, like the Schrödinger equation or the Wigner equation, allow for a fluid dynamical description in terms of macroscopic quantities, like the particle density or the current density, satisfying the so-called Madelung or quantum hydrodynamic equations [1]. These equations do not include a model of collisions. Collisions of the particles with an oscillator bath can be modeled by the Wigner-Fokker-Plank equation [2,3]. From this equation, the macroscopic viscous quantum hydrodynamic model can be derived, using a moment method as in [4]. This model consists of the continuity equations for the particle density $n(x,t)$ and the current density $J(x,t)$, coupled to the Poisson equation for the electrostatic potential $V(x,t)$. The scaled

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equations in one space dimension read as follows:

$$n_t + J_x = \nu n_{xx}, \tag{1}$$

$$J_t + \left(\frac{J^2}{n} + Tn\right)_x - nV_x - \frac{\varepsilon^2}{2} n \left(\frac{\sqrt{n}_{xx}}{\sqrt{n}}\right)_x = \nu J_{xx} - \frac{J}{\tau}, \tag{2}$$

$$\lambda^2 V_{xx} = n - 1, \qquad x \in \Omega, \quad t > 0, \tag{3}$$

where $\Omega = (0,1)$, with initial conditions

$$n(\cdot, 0) = n_I, \quad J(\cdot, 0) = J_I, \qquad \text{in } \Omega. \tag{4}$$

The (scaled) physical parameters are the temperature constant $T > 0$, the Planck constant $\varepsilon > 0$, the momentum relaxation time constant $\tau > 0$, and the Debye length $\lambda > 0$. The viscosity $\nu > 0$ models the strength of interaction of the particles with the oscillators. Equations (1),(2) can be derived exactly as in [4], where the (linear) Fokker-Planck term gives the viscous contributions νn_{xx} and νJ_{xx} to the right-hand sides of (1),(2). In the Poisson equation, we have prescribed a constant concentration of fixed background charges. For the choice of boundary conditions, see below.

For $\varepsilon = 0$ and $\nu = 0$, system (1)–(3) is the hydrodynamic model for an ensemble of charged particles, for instance, electrons moving in a semiconductor crystal [5]. For $\varepsilon > 0$, $\nu = 0$ (and $T = 0$, $1/\tau = 0$), equations (1)–(3) are the Madelung equations used in the modeling of superfluids [6]. With this choice of parameters, (1)–(3) are formally equivalent to the Schrödinger-Poisson system. Finally, equations (1)–(3) with $\varepsilon > 0$, $\nu = 0$ (and $T > 0$, $1/\tau > 0$) are known as the quantum hydrodynamic equations which have been used to model quantum semiconductor devices [4,5]. No results are available in the mathematical literature for (1)–(3) with $\varepsilon > 0$ and $\nu > 0$.

In this paper, we study the long-time asymptotics of the solutions of (1)–(3) towards the so-called thermal equilibrium state (no current flow). A special thermal equilibrium state is given by $J = 0$, $n = 1$, and $V = 0$ in Ω. We assume that the boundary conditions are in that thermal equilibrium state

$$n = 1, \quad n_x = 0, \quad V = 0, \qquad \text{on } \partial\Omega \times (0, \infty), \tag{5}$$

$$\int_{\partial\Omega} J \left[J_x \left(\frac{\varepsilon^2}{4\nu} + \nu\right) - \frac{1}{2} J^2 \right] (\cdot, t) \, ds = 0, \qquad t > 0. \tag{6}$$

The boundary condition (6) can be interpreted as a generalized thermal equilibrium condition for the current density (see Remark 1). We prove that any strong solution of (1)–(6) converges exponentially fast to the (unique) thermal equilibrium state $(n, J, V) = (1, 0, 0)$. The rate of convergence for $n(\cdot, t)$ and $V(\cdot, t)$ depends on the viscosity constant $\nu > 0$. If $\nu = 0$, no convergence rate can be obtained. The proof is based on the entropy dissipation method. This method has been used to derive explicit convergence rates of *second-* and *fourth-*order equations [7,8]. Here we apply the method for the first time to a *third-*order equation.

The entropy dissipation method is based on *a priori* estimates for the entropy (or, more precisely, free energy) functional

$$E(t) = \int_0^1 \left[\frac{\varepsilon^2}{2} \left(\sqrt{n}\right)_x^2 + T(n(\log n - 1) + 1) + \frac{\lambda^2}{2} V_x^2 + \frac{J^2}{2n} \right] (x, t) \, dx \geq 0 \tag{7}$$

consisting of the quantum energy, thermodynamical entropy, electric energy, and kinetic energy of the system. The idea is to derive an inequality of the form

$$E(t) + \int_0^t \int_0^1 P(x, t) \, dx \, ds \leq E(0),$$

where the entropy dissipation rate $P(x,t) \geq 0$ depends on the variables and their derivatives. We show that

$$\int_0^1 P(x,t)\,dx \geq \gamma E(t)$$

for some $\gamma > 0$, and thus, Gronwall's lemma implies

$$E(t) \leq E(0)e^{-\gamma t}, \qquad t \geq 0.$$

More precisely, our main result reads as follows.

THEOREM 1. *Let $n \in H^1(0,T^*;L^2(\Omega)) \cap L^2(0,T^*;H^3(\Omega))$, $J \in H^1(0,T^*;L^2(\Omega)) \cap L^2(0,T^*;H^2(\Omega))$, $V \in L^2(0,T^*;H^2(\Omega))$ be a solution to (1)–(3) for any $T^* > 0$ such that $n > 0$ in $\Omega \times (0,T^*)$ and let $n_I \in H^1(\Omega)$, $J_I \in L^2(\Omega)$ such that $n_I > 0$ in Ω. Then*

$$\varepsilon^2\|n(\cdot,t)-1\|^2_{L^\infty(\Omega)} \leq 2E(0)e^{-4\nu(1+2T/\varepsilon^2)t},$$
$$\lambda^2\|V(\cdot,t)\|^2_{L^\infty(\Omega)} \leq 2E(0)e^{-4\nu t},$$
$$\left\|\frac{J(\cdot,t)}{\sqrt{n(\cdot,t)}}\right\|^2_{L^2(\Omega)} \leq 2E(0)e^{-2t/\tau}, \qquad t > 0.$$

REMARK 1.

(i) The global-in-time existence for (1)–(3) will be studied in [9].
(ii) The boundary condition (6) is needed for technical reasons. It is a weaker condition than the (physically reasonable, but mathematically overdetermining) boundary conditions $J(0,t) = J(1,t) = 0$, $t > 0$.
(iii) As expected, no convergence rate for n and V can be expected if $\nu = 0$. However, the kinetic energy $J^2(\cdot,t)/n(\cdot,t)$ converges to zero exponentially in the $L^1(\Omega)$ norm as $t \to \infty$ with a decay rate $1/\tau$. This is physically reasonable since τ models the momentum relaxation time.
(iv) Exponential decay rates for solutions of the Wigner-Fokker-Planck equation (from which system (1)–(3) has been derived) towards the thermal equilibrium state are obtained in [10]. The decay rates of [10] are different from ours since in our system, the electrostatic potential is given self-consistently, whereas in [10], the potential is a given function not depending on the particle density.

2. PROOF OF THEOREM 1

Let $T^* > 0$ and fix $t \in (0,T^*)$. Multiply (2) by J/n, integrate over $\Omega = (0,1)$, and integrate by parts

$$\begin{aligned}\int_0^1 J_t\frac{J}{n}\,dx &= -\int_0^1\left(\frac{J^2}{n}\right)_x\frac{J}{n}\,dx - T\int_0^1 J\frac{n_x}{n}\,dx + \int_0^1 V_xJ\,dx - \frac{\varepsilon^2}{2}\int_0^1 J_x\frac{(\sqrt{n})_{xx}}{\sqrt{n}}\,dx\\ &\quad + \frac{\varepsilon^2}{2}\int_{\partial\Omega} J\frac{(\sqrt{n})_{xx}}{\sqrt{n}}\,ds - \nu\int_0^1 J_x\left(\frac{J}{n}\right)_x\,dx + \nu\int_{\partial\Omega} J_x\frac{J}{n}\,ds - \frac{1}{\tau}\int_0^1\frac{J^2}{n}\,dx\\ &= A_1 + \cdots + A_8.\end{aligned} \tag{8}$$

Multiply (2) by the function $T\log n - J^2/2n^2 - V - \varepsilon^2(\sqrt{n})_{xx}/2\sqrt{n}$, integrate over Ω, and integrate by parts

$$\begin{aligned}\int_0^1 n_t\left(T\log n-\frac{J^2}{2n^2}-V-\varepsilon^2\frac{(\sqrt{n})_{xx}}{2\sqrt{n}}\right)dx &= -\nu T\int_0^1\frac{n_x^2}{n}\,dx+\nu\int_0^1\left(\frac{J^2}{2n^2}\right)_x n_x\,dx\\ &\quad+\nu\int_0^1 V_x n_x\,dx-\nu\frac{\varepsilon^2}{2}\int_0^1 n_{xx}\frac{(\sqrt{n})_{xx}}{\sqrt{n}}\,dx\\ &\quad+T\int_0^1 J(\log n)_x\,dx+\int_0^1\frac{J^2J_x}{2n^2}\,dx\\ &\quad-\int_0^1 V_xJ\,dx+\frac{\varepsilon^2}{2}\int_0^1 J_x\frac{(\sqrt{n})_{xx}}{\sqrt{n}}\,dx\\ &= B_1+\cdots+B_8.\end{aligned} \tag{9}$$

Here, we have used that $n_x=0$ and $V=0$ on $\partial\Omega$.

First, we consider the terms on the left-hand side of (8) and (9). Since $n_x=0$ and $V=0$ on $\partial\Omega$, it holds that

$$\int_0^1\left(\frac{JJ_t}{n}-\frac{J^2n_t}{2n^2}\right)dx=\partial_t\int_0^1\frac{J^2}{2n}\,dx,$$

$$\int_0^1 n_t\log n\,dx=\partial_t\int_0^1(n(\log n-1)+1)\,dx,$$

$$-\int_0^1 Vn_t\,dx=-\lambda^2\int_0^1 V_{xxt}V\,dx=\lambda^2\int_0^1 V_{xt}V_x\,dx=\frac{\lambda^2}{2}\partial_t\int_0^1 V_x^2\,dx,$$

$$-\frac{\varepsilon^2}{2}\int_0^1 n_t\left(\frac{(\sqrt{n})_{xx}}{\sqrt{n}}\right)dx=-\varepsilon^2\int_0^1(\sqrt{n})_t(\sqrt{n})_{xx}\,dx=\frac{\varepsilon^2}{2}\partial_t\int_0^1(\sqrt{n})_x^2\,dx.$$

Hence, the sum of the left-hand sides of (8) and (9) is equal to $\partial_tE(t)$ where $E(t)$ is defined in (7).

Now, we compute the right-hand sides of (8) and (9). Notice that $A_2+B_5=0$, $A_3+B_7=0$, $A_4+B_8=0$. Using $n=1$ on $\partial\Omega$, we obtain

$$A_1+B_6=\int_0^1\left(\frac{n_xJ^3}{n^3}-\frac{3}{2}\frac{J^2J_x}{n^2}\right)dx=-\frac{1}{2}\int_0^1\left(\frac{J^3}{n^2}\right)_x dx=\frac{1}{2}J^3(0)-\frac{1}{2}J^3(1).$$

A computation gives

$$A_6+B_2=\nu\int_0^1\left(-\frac{J_x^2}{n}+\frac{2JJ_xn_x}{n^2}-\frac{J^2n_x^2}{n^3}\right)dx=-\nu\int_0^1\left(\frac{J_x}{\sqrt{n}}-\frac{n_xJ}{\sqrt{n}^3}\right)^2dx.$$

By integration by parts and $n_x=0$ on $\partial\Omega$, we obtain

$$\int_0^1\frac{n_x^2n_{xx}}{n^2}\,dx=\frac{2}{3}\int_0^1\frac{n_x^4}{n^3}\,dx,$$

and therefore,

$$\begin{aligned}B_4&=-\nu\frac{\varepsilon^2}{2}\int_0^1 n_{xx}\frac{(\sqrt{n})_{xx}}{\sqrt{n}}\,dx=\varepsilon^2\nu\int_0^1\left(\frac{1}{8}\frac{n_x^2n_{xx}}{n^2}-\frac{1}{4}\frac{n_{xx}^2}{n}\right)dx\\ &=-\varepsilon^2\nu\int_0^1\left(\left(\sqrt{n}\right)_{xx}^2+\frac{1}{48}\frac{n_x^4}{n^3}\right)dx.\end{aligned}$$

Finally, the boundary condition $n=1$ on $\partial\Omega$ implies

$$B_3=\nu\int_0^1 V_x(n-1)_x\,dx=-\frac{\nu}{\lambda^2}\int_0^1(n-1)^2\,dx. \tag{10}$$

Here we need that $n=\text{const}$ on $\partial\Omega$. The above computations show that the sum of (8) and (9), integrated over $(0,t)$, can be written as

$$E(t)-E(0)=-\int_0^t\int_0^1\left[\varepsilon^2\nu\left(\sqrt{n}\right)_{xx}^2+\varepsilon^2\nu\frac{1}{48}\frac{n_x^4}{n^3}+4\nu T\left(\sqrt{n}\right)_x^2+\frac{\nu}{\lambda^2}(n-1)^2\right.$$
$$\left.+\frac{\nu}{n^3}(nJ_x-n_xJ)^2+\frac{1}{\tau}\frac{J^2}{n}\right]dx\,dt+\int_0^t\int_{\partial\Omega}\left[\frac{\varepsilon^2}{2}\frac{(\sqrt{n})_{xx}}{\sqrt{n}}J+\nu JJ_x-\frac{1}{2}J^3\right]ds\,dt.$$

The first integral on the right-hand side is the entropy dissipation rate. Since $n_x=0$ and $n=1$ on $\partial\Omega$, we can write the boundary integral as

$$\int_0^t\int_{\partial\Omega}\left[\frac{\varepsilon^2}{4}n_{xx}J+\nu JJ_x-\frac{1}{2}J^3\right]ds\,dt.$$

The boundary condition $n(\cdot,t)=1$ on $\partial\Omega$ implies $\nu n_{xx}(\cdot,t)-J_x(\cdot,t)=n_t(\cdot,t)=0$ on $\partial\Omega$, and thus, using (6),

$$\int_0^t\int_{\partial\Omega}\left[\frac{\varepsilon^2}{4}n_{xx}J+\nu JJ_x-\frac{1}{2}J^3\right]ds\,dt=\int_0^t\int_{\partial\Omega}\left[\frac{\varepsilon^2}{4\nu}JJ_x+\nu JJ_x-\frac{1}{2}J^3\right]ds\,dt=0.$$

We apply the Poincaré inequality

$$\|u\|_{L^2(\Omega)}\le\frac{1}{\sqrt{2}}\|u_x\|_{L^2(\Omega)},\qquad\forall u\in H_0^1(\Omega),$$

to $u=(\sqrt{n})_x$ to obtain

$$\frac{\varepsilon^2}{2}\int_0^1\left(\sqrt{n}\right)_x^2(x,t)\,dx\le E(t)\le E(0)-\left(2\nu\varepsilon^2+4\nu T\right)\int_0^1\int_0^t\left(\sqrt{n}\right)_x^2(x,t)\,dx\,dt.$$

Gronwall's lemma implies

$$\varepsilon^2\left\|\left(\sqrt{n}\right)_x(\cdot,t)\right\|_{L^2(\Omega)}^2\le 2E(0)e^{-4\nu\left(1+2T/\varepsilon^2\right)t},\qquad t\ge 0.$$

The Sobolev-Poincaré inequality

$$\|u-1\|_{L^\infty(\Omega)}\le\|u_x\|_{L^2(\Omega)},\qquad\forall u-1\in H_0^1(\Omega),$$

then gives

$$\varepsilon^2\left\|\sqrt{n}(\cdot,t)-1\right\|_{L^\infty(\Omega)}^2\le 2E(0)e^{-4\nu\left(1+2T/\varepsilon^2\right)t},\qquad t\ge 0.$$

Furthermore, we conclude from

$$\frac{1}{2}\int_0^1\frac{J^2}{n}\,dx\le E(t)\le E(0)-\frac{1}{\tau}\int_0^1\int_0^t\frac{J^2}{n}\,dx\,dt$$

the estimate

$$\left\|\frac{J(\cdot,t)}{\sqrt{n(\cdot,t)}}\right\|_{L^2(\Omega)}^2\le 2E(0)e^{-2t/\tau},\qquad t\ge 0.$$

Finally, from the elliptic estimate

$$\sqrt{2}\lambda^2\|V_x\|_{L^2(\Omega)}\le\|n-1\|_{L^2(\Omega)},$$

we infer

$$\frac{\lambda^2}{2}\int_0^1\int_0^t V_x^2\,dx\,dt\le E(t)\le E(0)-2\nu\lambda^2\int_0^1\int_0^t V_x^2\,dx\,dt,$$

and hence,

$$\lambda^2\|V(\cdot,t)\|_{L^\infty(\Omega)}^2\le\lambda^2\|V_x(\cdot,t)\|_{L^2(\Omega)}^2\le 2E(0)e^{-4\nu t},\qquad t\ge 0.$$

This proves the theorem.

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