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A NOTE ON THE TIME DECAY OF SOLUTIONS FOR THE LINEARIZED WIGNER-POISSON SYSTEM

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Abstract. We consider the one-dimensional Wigner-Poisson system of plasma physics, linearized around a (spatially homogeneous) Lorentzian distribution and prove that the solution of the corresponding linearized problem decays to zero in time. We also give an explicit algebraic decay rate.

Dedicated to the memory of Marcello Anile

1. Introduction.

1.1. Physical motivation.

We consider in a semiclassical framework the dynamics of N electrons under the influence of a constant ionic background and their self-consistently generated electrostatic field accounting for mean field effects in the study of quantum plasmas models near admissible stationary states. We are motivated by the study of stability properties and decay rates to those stationary states. In fact, the study of quantum plasmas has been gaining more scientific interest, reflected in the recent works [1, 3, 7] and references given therein. To this end we shall rely on the so-called *Wigner-transformed* picture of quantum mechanics: Consider the N-particle density matrix describing the state of N electrons, each of which is given by a single wave function $\psi_j \in L^2(\mathbb{R}^d)$, i.e.

$$\Psi(t,x,y) := \sum_{j=1}^{N} \psi_j(t,x)\overline{\psi_j(t,y)},$$

and transform it according to

$$W(t,x,v) := \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} \Psi\left(t, x + \frac{\hbar}{2}\eta\ ,\ x - \frac{\hbar}{2}\eta\right) \mathrm{e}^{\mathrm{i}v\cdot\eta}\,\mathrm{d}y. \tag{1.1}$$

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Here we included Planck's constant $\hbar$ in order to be able to compare our calculations to the classical case, which is obtained in the (semi-)*classical limit* $\hbar \to 0$. For the purpose of our study we set all other physical constants, like the electron mass, its charge, etc., to be equal to one without loss of generality. By definition, $W(t,\cdot,\cdot) \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$ is *real-valued*. Therefore the Wigner function $W(t,x,v)$ can be considered a quantum mechanical analog to the classical phase-space distribution. However, in contrast to classical distribution, the Wigner function $W(t,x,\xi)$ in general takes also negative values. The time-evolution of $W(t,x,v)$ is governed by the *Wigner-Poisson system* (WP) [8]:

$$\begin{cases} \partial_t W + v \cdot \nabla_x \, W + \Theta[V] W = 0, \quad x, v \in \mathbb{R}^d, t > 0, \\ -\Delta V = \rho(t,x) - n, \end{cases} \tag{1.2}$$

which (up to an inverse Wigner transformation) is *equivalent* to the Schrödinger-Poisson system and reminiscent of the classical Vlasov-Poisson system (1.4) below. In (1.2), $n > 0$ describes the constant background density of the ions. Moreover, formally it holds [8]:

$$\rho(t,x) = \int_{\mathbb{R}^d} W(t,x,v) \, \mathrm{d}v \equiv \sum_{j=1}^N |\psi(t,x)|^2,$$

which is again reminiscent of the classical definition (1.5). In (1.2), the self-consistent potential $V = V(t,x)$ enters via the pseudo-differential operator

$$(\Theta[V]f)(x,\xi) := -\frac{\mathrm{i}}{\hbar(2\pi)^d} \iint_{\mathbb{R}^{2d}} \delta V_\hbar(x,\eta) W(t,x,v') \, \mathrm{e}^{\mathrm{i}\eta\cdot(v-v')} \, \mathrm{d}v' \, \mathrm{d}\eta, \tag{1.3}$$

where the symbol $\delta V_\hbar$ is given by

$$\delta V_\hbar(x,\eta) := V\left(x + \frac{\hbar}{2}\eta\right) - V\left(x - \frac{\hbar}{2}\eta\right).$$

The aim of the paper is study the behavior of solutions to (1.2) for $d = 1$, after linearization around a spatially homogeneous equilibrium state $w_\infty(v)$. We expect that as $t \to +\infty$ the solution to the linearized equation $w(t,x,v)$ decays in some sense (to be made precise in the following) to zero.

In classical kinetic theory, the description of collisionless plasmas relies on the already mentioned *Vlasov-Poisson system* (VP), governing the dynamics of phase-space densities $f_j(t,x,v) \geq 0$, for $j = 1, \dots, N$, via

$$\begin{cases} \partial_t f_j + v \cdot \nabla_x \, f_j - \nabla_x V(t,x) \cdot \nabla_v f_j = 0, \quad x, v \in \mathbb{R}^d, t > 0, \\ -\Delta V = \rho^{\mathrm{cl}}(t,x) - n, \end{cases} \tag{1.4}$$

where

$$\rho^{\mathrm{cl}}(t,x) := \sum_{j=1}^N \int_{\mathbb{R}^d} f_j(t,x,v) \, \mathrm{d}v, \tag{1.5}$$

denotes the classical the charge density. Note that the classical VP system can be obtained from the WP system in the limit $\hbar \to 0$, corresponding to the high-frequency regime. This VP system is very well studied in the mathematical literature, cf. [8, 11] for a broad overview. A particularly interesting feature is a phenomena called *Landau damping* [6], where one considers the propagation of small amplitude waves in an spatially uniform plasma. It is assumed that due to the influence of the self-consistent potential, these small amplitude perturbations are damped out during the course of time, cf. [6, 9, 10] where an exponential time-decay is found for the most simple form of such a perturbation, namely a single plain-wave.

From a rigorous mathematical point of view, the investigation of this damping phenomena turns out to be quite difficult due to the non-linear nature of the problem. One particular setting, where the stability and time-decay properties have been successfully analyzed, is given in [4] and [5]. There, the authors linearize (1.4) around a spatially homogeneous equilibrium state $f_\infty(v) \in L^1(\mathbb{R}^n)$, such that $\int f \mathrm{d}v = n$, and consequently study the long-time behavior of the resulting linearized model in whole space. For that particular setting the authors prove that, for admissible perturbations which are not simply plane-waves, that the corresponding time-decay of the linearized VP system is no longer exponential, but may vary from logarithmic to algebraic, depending on the choice of the equilibrium distribution (which in all cases is assumed to be radial, monotonically decreasing). Most results are given for $d = 1$ only, where the obtained rates are also shown to be essentially sharp [5].

Motivated by the this study in the classical kinetic description of collisionless plasmas, we aim to show that a similar phenomena is also true in a quantum mechanical setting. The main difference to the classical kinetic VP system lies in the non-local action of the potential-operator $\Theta[V]$, which is of strong dispersive nature. In particular, due to the presence of $\Theta[V]$, we are at the moment not able to derive certain a-priori estimates of moments, which are, in fact, heavily used in [4, 5] in order to obtain the time-decay rates.

For a planar flow corresponding to a slab geometry, where the plasma model reduces to its one-dimensional form in phase space, we succeed in analyzing the corresponding problem for the Wigner-Poisson system for a specific choice of the equilibrium function, namely a *Lorentzian distribution*, see (1.7) below, provided the initial data is k-times differentiable with compact support. Lorentzian distributions, which have finite mass and infinite variance (kinetic energy), can be considered among the most important example of equilibrium distributions in plasma physics [7]. Moreover, they allow for certain explicit calculations, which we shall need to circumvent the above mentioned mathematical obstruction. Also the assumption of the regularity and behavior of the initial data is crucial for the time decay rate of the perturbation of the stationary Lorentzian distribution. However, we strongly believe that an analogous result to the one stated below should still hold true for a much larger class of in equilibrium distributions.

We also mention that in the physics literature the phenomena of Landau damping for the the WP system of quantum mechanics has already been investigated, cf. [1] for a non-rigorous study along this lines.

1.2. **Linearization and main result.** In the rest of the manuscript we take $d = 1$, corresponding to a the plasma flow of a one dimensional slab geometry. We first linearize (1.2) around a spatially homogeneous steady state $w_\infty(v)$, given by a Lorentzian distribution centered around a given $v_0 \in \mathbb{R}$, i.e. we consider

$$W(t, x, v) = w_\infty(v) + \varepsilon w(t, x, v), \tag{1.6}$$

where $\varepsilon \ll 1$ is a small (dimensionless) parameter and w_∞ is

$$w_\infty(v) = \frac{n}{\pi} \frac{1}{1 + (v - v_0)^2}. \tag{1.7}$$

This equilibrium state is normalized, such that

$$\int_{\mathbb{R}} w_\infty(v)\,\mathrm{d}v = n,$$

models a charge neutral, spatially homogeneous (in other words infinitely extended and hence classical) steady state, that is radial and monotonically decreasing as a function of its radius.

Formally plugging the ansatz (1.6) into (1.2) and neglecting terms of the order $\mathcal{O}(\varepsilon^2)$ yields the linearized model for the perturbation $w(t,x,v)$:

$$\begin{cases} \partial_t w + v\partial_x w + \Theta[V]w_\infty = 0, \quad x,v\in\mathbb{R}, t>0,\\ -\partial_{xx}V = \int_{\mathbb{R}} w(t,x,v)\mathrm{d}v, \end{cases} \tag{1.8}$$

subject to an initial data $w\big|_{t=0} = w_0(x,v)\in L^2\cap L^1(\mathbb{R}\times\mathbb{R})$, such that

$$\iint_{\mathbb{R}^2} w_0(x,v)\,\mathrm{d}x\,\mathrm{d}v = 0. \tag{1.9}$$

Theorem 1. *For $w_0\in C_0^\alpha(\mathbb{R}\times\mathbb{R})$, with $\alpha\geq 2$, the solution of the linearized equation (1.8) decays as $t\to+\infty$ in the sense that:*

$$\lim_{t\to+\infty}\|\rho(t,\cdot)\|_{L^p(\mathbb{R})} = \mathcal{O}\left(\frac{1}{t^{1-1/p}}\right), \quad \textit{for } p\in[2,\infty]. \tag{1.10}$$

Remark: Note that the estimate (1.10) is *uniform* with respect to the (small) parameter $\hbar$. Indeed the obtained time decay rate is exactly the same as in the classical case, cf. [4, 5]. This indicates that the damping mechanism towards an equilibrium state is essentially a frequency-independent phenomena. Even if at the present moment we are able to prove it only for a particular equilibrium function (Lorentzian distribution), we consequently believe that our result can be extended to a larger class of equilibrium probability distributions.

2. Proof of Theorem 1.

The proof of our theorem will be done in two steps. First we shall derive an appropriate representation for solutions to (1.8) via the method of characteristics the use of the Laplace transform. This representation will allow us, in a second step, to obtain a sufficiently precise control of the time-decay of the particle density $\rho(t,x)$.

2.1. The Fourier transformed setting.

We first integrate (1.8) along characteristics to obtain

$$\begin{aligned} w(t,x,v) &= w_0(x-vt,v) + \int_0^t \Theta[V(x-v(t-s))]w_\infty(v)\,\mathrm{d}s\\ &= w_0(x-vt,v) + \frac{1}{\hbar}\int_0^t\int_{\mathbb{R}} \delta V(x-v(t-s),y)\widehat{w}_\infty(\eta)e^{iv\eta}\,\mathrm{d}\eta\,\mathrm{d}s. \end{aligned}$$

Since the problem is linear, we make use of the Fourier transform as follows. First noticing that w_∞ is given by the Lorentzian distribution (1.7), its Fourier transform is given by

$$\widehat{w}_\infty(\eta) = \int_{\mathbb{R}} w_\infty(v)e^{-i\eta v}\mathrm{d}v = n\,e^{-(|\eta|+iv_0\eta)}.$$

Next, using the fact that for the Poisson equation the symbol $\delta V_\hbar(x,\eta)$ can be written as in [2]

$$\delta V_\hbar(x,\eta) = \frac{1}{4\pi}\int_{\mathbb{R}} K_\hbar(y,\eta)\rho(x-y)\mathrm{d}y, \tag{2.11}$$

with the dipole-kernel (adapted to the one-dimensional case)

$$K_\hbar(y,\eta) := |y+\frac{\hbar\eta}{2}| - |y-\frac{\hbar\eta}{2}|\ , \tag{2.12}$$

we arrive at the following representation formula for the solutions of the linearized system (1.8) :

$$\begin{aligned} w(t,x,v) &= w_0(x-vt,v) \\ &\quad + \frac{1}{\hbar}\int_0^t \iint_{\mathbb{R}\times\mathbb{R}} K_\hbar(y,\eta)\widehat{w}_\infty(\eta)\rho(x-v(t-s)-y)e^{iv\eta}\,\mathrm{d}y\,\mathrm{d}\eta\,\mathrm{d}s. \end{aligned}$$

Since ρ along the characteristic is independent of η, the Fourier transform variable in phase space, the integration with respect to $\eta\in\mathbb{R}$ can be performed separately, and so the particle density $\rho(t,x)$ associated to the solution is given by the following representation formula

$$\rho(t,x) = \rho_0(t,x) + \frac{1}{\hbar}\int_0^t \iint_{\mathbb{R}\times\mathbb{R}} G_\hbar(y,v)\rho(x-v(t-s)-y)\,\mathrm{d}y\,\mathrm{d}v\,\mathrm{d}s, \tag{2.13}$$

where we denote

$$G_\hbar(y,v) := \int_{\mathbb{R}} K_\hbar(y,\eta)\widehat{w}_\infty(\eta)e^{iv\eta}\,\mathrm{d}\eta. \tag{2.14}$$

Moreover, the density corresponding to the initial perturbation $w_0(x,v)$ integrated along the characteristics, is given by

$$\rho_0(t,x) := \int_{\mathbb{R}} w_0(x-vt,v)\mathrm{d}v. \tag{2.15}$$

Next, following the the approach in [4], we investigate the representation formula (2.13) in $x-$Fourier space, where $x\mapsto\xi\in\mathbb{R}$. Thus, denoting by

$$\widehat{\rho}(t,\xi) = \int_{\mathbb{R}} \rho(t,x)e^{-ix\xi}\mathrm{d}x,$$

the corresponding $x-$Fourier transformed particle density (and analogously for $\rho_0(t,x)$), satisfies

$$\begin{aligned} \widehat{\rho}(t,\xi) &= \widehat{\rho}_0(t,\xi) + \frac{i}{\hbar}\int_0^t \iint_{\mathbb{R}\times\mathbb{R}} G_\hbar(y,v)\widehat{\rho}(s,\xi)e^{-i\xi(v(t-s)+y)}\mathrm{d}y\,\mathrm{d}v\,\mathrm{d}s \\ &= \widehat{\rho}_0(t,\xi) + \frac{i}{\hbar}\int_0^t \widehat{\rho}(s,\xi)\left(\int_{\mathbb{R}} \widehat{G}_\hbar(y,\xi(t-s))e^{-i\xi y}\mathrm{d}y\right)\mathrm{d}s, \end{aligned}$$

where now the new integration kernel is computed as follows: Using the representation (2.14) in frequency space and the particular structure (2.11) and (2.12) for the self-consistent potential V, we obtain

$$\begin{aligned} \int_{\mathbb{R}} \widehat{G}_\hbar(y,\xi(t-s))e^{-i\xi y}\mathrm{d}y &= \int_{\mathbb{R}} \widehat{w}_\infty(\eta)\delta_{\eta=\xi(t-s)}\left(\int_{\mathbb{R}} f(y,\eta)e^{-i\xi y}\mathrm{d}y\right)\mathrm{d}\eta \\ &= \frac{\widehat{w}_\infty(\xi(t-s))}{|\xi|^2}\left(e^{i\hbar|\xi|^2(t-s)/2} - e^{-i\hbar|\xi|^2(t-s)/2}\right), \end{aligned}$$

since

$$\int_{\mathbb{R}} |y \pm \frac{\hbar\eta}{2}| \; e^{-i\hbar\xi y} \mathrm{d}y = \frac{e^{\pm \frac{i\hbar}{2}\xi\eta}}{|\xi|^2}.$$

In particular, we arrive at the integral equation

$$\widehat{\rho}(t,\xi) = \widehat{\rho}_0(t,\xi) - \frac{2}{\hbar|\xi|^2} \int_0^t \widehat{w}_\infty(\xi(t-s)) \sin\left(\frac{\hbar|\xi|^2(t-s)}{2}\right) \widehat{\rho}(s,\xi)\,\mathrm{d}s. \tag{2.16}$$

Denoting by

$$F_\hbar(t,\xi) = \frac{2}{\hbar|\xi|^2}\,\widehat{w}_\infty(t\xi)\sin(\hbar|\xi|^2 t/2),$$

equation (2.16) becomes

$$\widehat{\rho}(t,\xi) = \widehat{\rho}_0(t,\xi) - (F_\hbar * \widehat{\rho})(t,\xi), \tag{2.17}$$

where "$*$" denotes the convolution in time.

Remark 2.1. It is worth noticing that this representation (2.16) formula converges (at least formally) in the limit $\hbar \to 0$ to the corresponding classical one for the linearized VP system, cf. [4].

The quantity $\widehat{\rho}(t,\xi)$ which we need to estimate satisfies an integral equation. A Laplace transformation w.r.t. time, i.e.

$$L[f](s) = \int_0^\infty f(t)e^{-st}\,dt.$$

consequently transforms (2.17) into the following algebraic equation for $L[\widehat{\rho}](s,\xi)$:

$$L[\widehat{\rho}](s,\xi) = L[\widehat{\rho}_0](s,\xi) - L[\widehat{\rho}_0](s,\xi)\frac{L[F_\hbar](s,\xi)}{1+L[F_\hbar](s,\xi)}.$$

Following the ideas developed in [4], we define a new kernel $R(t,\xi)$, via

$$L[R_\hbar](s,\xi) := \frac{L[F_\hbar](s,\xi)}{1+L[F_\hbar](s,\xi)}, \tag{2.18}$$

and we consequently rewrite (2.16) as

$$\begin{aligned} \widehat{\rho}(t,\xi) &= \widehat{\rho}_0(t,\xi) - (R_\hbar * \widehat{\rho}_0)(t,s) \\ &= \widehat{\rho}_0(t,\xi) - \int_0^t R_\hbar(s,\xi)\widehat{\rho}_0(t-s,\xi)\,\mathrm{d}s. \end{aligned} \tag{2.19}$$

This finally yields a representation equation for $\widehat{\rho}(t,\xi)$ in terms of the initial state $\widehat{\rho}_0(\xi)$. The estimate for $\widehat{\rho}(t,\xi)$ w.r.t. t will be achieved by obtaining sufficient control of the kernel $R_\hbar(t,\xi)$.

Remark 2.2. We note that an equation analogous to (2.19) indeed holds in any spatial dimension d, since all of the above given calculations can be easily generalized to the d-dimensional case.

2.2. **Time decay estimates for the particle density.** In the particular case under consideration, where $w_\infty(v)$ is a Lorentzian distribution, the kernel $R_\hbar$ can be computed explicitly, since $L[F_\hbar]$ allows for an explicit inversion of $L(R_\hbar)$. Indeed, $L[F_\hbar]$ is given by

$$L[F_\hbar](s,\xi) = \frac{\kappa}{(s+iv_0\xi+|\xi|)^2+\hbar^2|\xi|^4/4},$$

where from now on, we write $\kappa = n/\pi$ for simplicity. Therefore, we obtain from (2.18) the Laplace transformed kernel

$$L[R_\hbar](s,\xi) = \frac{\kappa}{(s+iv_0\xi+|\xi|)^2+\hbar^2|\xi|^4/4+\kappa},$$

and an inverse Laplace transform yields the following expression

$$R_\hbar(t,\xi) = \frac{\kappa}{\sqrt{\kappa+\hbar|\xi|^2/4}}\sin\left(t\sqrt{\kappa+\hbar|\xi|^2/4}\right)\,e^{-t(|\xi|+iv_0\xi)}. \tag{2.20}$$

Note that $R_\hbar(t,\xi)$ converges, as $\hbar\to 0$, to its classical analog, given in [4].

Finally, we can estimate the time evolution for $\widehat{\rho}(t,\xi)$ using the representation formula (2.19) by obtaining control in-time from the kernel $R(t,\xi)$ given in (2.20). More specifically, with this explicit form of $R_\hbar(t,\xi)$ in hand, we can write

$$\begin{aligned}
&\widehat{\rho}(t,\xi)\\
&=\widehat{\rho}_0(t,\xi) - \int_0^t R_\hbar(s,\xi)\widehat{\rho}_0(t-s,\xi)\,\mathrm{d}s\\
&=\widehat{\rho}_0(t,\xi) + \frac{\kappa}{\kappa+\hbar|\xi|^2/4}\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\left(\cos\left(s\sqrt{\kappa+\hbar|\xi|^2/4}\right)\right)\,e^{-s(|\xi|+iv_0\xi)}\widehat{\rho}_0(t-s,\xi)\,\mathrm{d}s.
\end{aligned}$$

Integrating by parts and using the fact that

$$\partial_t\rho_0(t,x)+\partial_x j_0(t,x)=0,$$

where

$$j_0(t,x)=\int_{\mathbb{R}} v w_0(x-vt,v)\mathrm{d}v,$$

we obtain

$$\begin{aligned}
\widehat{\rho}(t,\xi) = &\left(1-\frac{\kappa}{\kappa+\hbar|\xi|^2/4}\right)\widehat{\rho}_0(t,\xi)\\
&+\frac{\kappa}{\kappa+\hbar|\xi|^2/4}\cos\left(t\sqrt{\kappa+\hbar|\xi|^2/4}\right)\,e^{-t(|\xi|+iv_0\xi)}\widehat{\rho}_0(0,\xi)\\
&+\frac{\kappa(|\xi|+iv_0\xi)}{\kappa+\hbar|\xi|^2/4}\int_0^t\cos\left(s\sqrt{\kappa+\hbar|\xi|^2/4}\right)\,e^{-s(|\xi|+iv_0\xi)}\widehat{\rho}_0(t-s,\xi)\,\mathrm{d}s\\
&-\frac{i\kappa\xi}{\kappa+\hbar|\xi|^2/4}\int_0^t\cos\left(s\sqrt{\kappa+\hbar|\xi|^2/4}\right)\,e^{-s(|\xi|+iv_0\xi)}\widehat{j}_0(t-s,\xi)\,\mathrm{d}s.
\end{aligned}$$

From (2.15), we obtain

$$\widehat{\rho}_0(t,\xi)=(2\pi)^{1/2}(\mathcal{F}_2 w_0)(\xi,t\xi),$$

where $\mathcal{F}_2$ denotes the two-dimensional Fourier transform. For any $w_0\in C_0^\alpha(\mathbb{R}\times\mathbb{R})$ this yields

$$|\widehat{\rho}_0(t,\xi)|\le C(1+|\xi|+|t\xi|)^{-\alpha}. \tag{2.21}$$

In addition we can estimate $\widehat{j}_0(t,\xi)$, using the conservation of mass equation

$$\frac{\partial}{\partial t}\widehat{\rho}_0(t,\xi) + i\xi\widehat{j}_0(t,\xi) = 0.$$

Now having in mind that the initial state w_0 is compactly supported, we also get $vw_0 \in C_0^\alpha(\mathbb{R}\times\mathbb{R})$, and so

$$\widehat{j}_0(t,\xi) = (2\pi)^{1/2}(\mathcal{F}_2(vw_0))(\xi, t\xi) \leq C(1+|\xi|+|t\xi|)^{-\alpha}.$$

In particular, we can then estimate $\widehat{\rho}(t,\xi)$ as follows:

$$|\widehat{\rho}(t,\xi)| \leq 2|\widehat{\rho}_0(t,\xi)| + e^{-t|\xi|}|\widehat{\rho}_0(0,\xi)| + \frac{2\kappa|\xi| + |v_0\xi|}{\kappa + \hbar|\xi|^2/4}\int_0^t \frac{e^{-s|\xi|}}{(1+(t-s)|\xi|)^\alpha}\,\mathrm{d}s.$$

Proceeding as in [4], we evaluate the time integral in the subintervals $[0,t/2]$ and $(t/2,t]$ separately. Using once again (2.21) we obtain, following the same estimate as in [4, Equation (42)], that

$$|\widehat{\rho}(t,\xi)| \leq C(1+|t\xi|)^{-\alpha},$$

where $C = C(\hbar)$. Thus, for $\alpha \geq 2$, it holds

$$\int_{\mathbb{R}} |\widehat{\rho}(t,\xi)|\,\mathrm{d}\xi \leq C\int_{\mathbb{R}}(1+|t\xi|)^{-\alpha}\,\mathrm{d}\xi \leq \frac{C}{t}. \tag{2.22}$$

Similarly we get the following L^2-estimate

$$\int_{\mathbb{R}} |\widehat{\rho}(t,\xi)|^2\,\mathrm{d}\xi \leq \frac{\tilde{C}}{t}, \tag{2.23}$$

The general estimate (1.10) then follows by interpolation between (2.22) and (2.23) and a Fourier inversion completes the proof.

□

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