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Asymptotics for a symmetric equation in price formation

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Abstract

We study the existence and asymptotics for large time of the solutions to a one dimensional evolution equation with non-standard right hand side. The right hand side involves the derivative of the solution computed at a given point. Existence is proven through a fixed point argument. When the problem is considered in a bounded interval, it is shown that the solution decays exponentially to the stationary state. This problem is a particular case of a mean-field free boundary model proposed by Lasry-Lions on price formation and dynamic equilibria.

1 Introduction

We consider an idealized population of players consisting of two groups, namely one group of buyers of a certain good and one group of vendors of the same good, that are described by two non-negative densities f_B, f_V depending on $(x,t) \in \mathbb{R} \times \mathbb{R}_+$. In the model, x denotes a possible value of the price. We denote by $p(t)$ the price resulting from a dynamical equilibrium and we assume that there is some friction measured by a positive parameter a. Also, σ is the parameter that measures the randomness. This situation can be described by the following system of free-boundary equations:

$$\begin{cases} \frac{\partial f_B}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 f_B}{\partial x^2} = -\lambda(t)\delta_{x=p(t)-a} \quad \text{if } x \leq p(t),\ t>0, \\ f_B(x,t)>0 \text{ if } x<p(t), \quad f_B(x,t)=0 \text{ if } x \geq p(t), \end{cases} \tag{1.1}$$

together with

$$\begin{cases} \frac{\partial f_V}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 f_V}{\partial x^2} = \lambda(t)\delta_{x=p(t)+a} \quad \text{if } x > p(t),\ t>0, \\ f_V(x,t)>0 \quad \text{if } x>p(t), \quad f_V(x,t)=0 \text{ if } x \leq p(t), \end{cases} \tag{1.2}$$

where

$$\lambda(t) = -\frac{\sigma^2}{2}\frac{\partial f_B}{\partial x}\left(p(t),t\right) = \frac{\sigma^2}{2}\frac{\partial f_V}{\partial x}\left(p(t),t\right). \tag{1.3}$$

The symbol δ denotes the Dirac delta at the indicated point. The multiplier λ represents the number of transactions at time t, so (1.3) means that the flux of buyers which must be equal to the flux of vendors. The initial conditions

$$f_B(x,0) = f_B^0(x) \text{ and } f_V(x,0) = f_V^0(x)$$

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are such that, for some p_0 in $\mathbb{R}$,

$$f_B^0(x) > 0 \text{ if } x < p_0, \quad f_B^0(x) = 0 \text{ if } x \geq p_0$$
$$f_V^0(x) > 0 \text{ if } x > p_0, \quad f_B^0(x) = 0 \text{ if } x \leq p_0.$$

The equation satisfy the property of conservation of mass. Indeed, both

$$\int_{-\infty}^{p(t)} f_V \, dx \quad \text{and} \quad \int_{p(t)}^{+\infty} f_B \, dx$$

remain constant for all $t \geq 0$.

Equations (1.1)-(1.3) describe a mean-field model for the dynamical formation of the price of some good that has been very recently introduced in [6]. Other related works by Lasry-Lions are [7], [8], [9]. They proved that, under suitable assumptions on smoothness and integrability on the initial data, there exists a unique smooth solution (f, p) of (1.1)-(1.3). The most important question in this setting is the following: will the good reach a stable price or will the price oscillate in time? The answer to this question can be found through the study of the large time behavior of the system. However the asymptotics when $t \to +\infty$ of such model is not known.

In this paper we will establish the asymptotic behavior of (1.1)-(1.3) under certain assumptions on the initial data. More precisely, we consider initial data that are symmetric with respect to a general given point p_0, i.e. $f_B^0(p_0 - x) = f_V^0(p_0 + x)$ for all $x \geq 0$. In this case the price $p(t)$ resulting from the dynamical equilibria of the system is a constant function $p(t) = p_0$ for all $t > 0$ and the system (1.1), (1.2) reduces to the single equation

$$\begin{aligned} \frac{\partial f}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} &= \lambda(t)\left[\delta_{x=-a} - \delta_{x=a}\right], \\ f(x,0) &= f_I(x), \end{aligned} \tag{1.4}$$

where

$$\lambda(t) := -\frac{\sigma^2}{2}\frac{\partial f}{\partial x}(0,t),$$

and

$$f := f_B - f_V, \text{and } f_I = f_B^0 - f_V^0.$$

The purpose of this paper is, first to give a direct proof of existence, and second, to show that the solution of this symmetric problem decays exponentially to the stationary state when the problem is considered in a bounded interval.

There are two reasons that justify the interest on the symmetric version (1.4) of (1.1)-(1.3). The first one is that considering symmetric solutions eliminates the free boundary $p(t)$ and makes the problem more accessible while still giving an idea of the general picture. The second, and most important one, is that the linearized problem of (1.1)-(1.3) has, surprisingly, no free boundary $p(t)$; indeed, it is similar to (1.4). The understanding of the asymptotics of the linearized operator is the first step in the study of the general asymptotics. This long time behavior of the general problem is being studied by the authors in [5]. See [1] for some background on free boundary problems.

From a modeling point of view, it is reasonable to assume that the buyers and the vendors trade when the price of the good takes "realistic" values (avoiding the non-realistic situation of infinitely high or low prices). This assumption can be translated in the model (1.4) with the presence of a convection term with confining effects. Therefore, in the whole space, it may be of high interest to study also the following equation

$$\begin{aligned} &\frac{\partial f}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} - \frac{\partial}{\partial x}(xf) = \lambda(t)\left[\delta_{x=-a} - \delta_{x=a}\right], \\ &f(x,0) = f_I(x), \quad \lambda(t) = -\frac{\sigma^2}{2}\frac{\partial f}{\partial x}(0,t). \end{aligned} \tag{1.5}$$

Problem (1.5) has a non trivial stationary solution due to the presence of the confinement. Although we cannot prove the asymptotics for it, we can give a simple argument for existence of solutions. We remark to the reader that (1.5) and (1.4) are related by some suitable self-similar rescaling as in Carrillo-Toscani [3].

The paper is organized as follows: in Section 2 we give the proof if existence for the problem (1.4), first in $\mathbb{R}$ and then in a bounded interval. In section 3 we show the exponential decay of solutions in a bounded interval towards the stationary state. In the last section we consider the modified problem (1.5).

2 Proof of existence

Assume, without loss of generality, that $\frac{\sigma^2}{2} = 1$. In order to motivate the proof of theorems 2.2 and 4.1, we first present an easy direct argument for the existence of solutions of (2.1) in the whole real line $\mathbb{R}$.

2.1 Motivation in $\mathbb{R}$

We seek a function $f(x,t)$, $x \in \mathbb{R}$, $t > 0$, $f \in L^\infty\left(0, T, L^1_x(\mathbb{R})\right)$, $\forall\, T > 0$, odd in the x axis, that solves

$$\begin{aligned} &\frac{\partial f}{\partial t} - \frac{\partial^2 f}{\partial x^2} = \lambda(t)\left[\delta_{x=-a} - \delta_{x=a}\right], \\ &f(x,0) = f_I(x), \end{aligned} \tag{2.1}$$

where the initial condition satisfies $f_I(x) = -f_I(-x)$ and such that $f_I(x) \geq 0$ for $x \leq 0$. The condition $\lambda(t) = -\frac{\partial f}{\partial x}(0,t)$ implies conservation of mass for the equation. Indeed integrating (2.1) with respect to x, we get

$$\int_{-\infty}^{0} f(x,t)dx = \int_{-\infty}^{0} f_I(x)dx \quad \text{for all} \quad t > 0. \tag{2.2}$$

Theorem 2.1. *Let $f_I \in L^1(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$ be an odd function in $\mathbb{R}$, positive for $x < 0$, $f_I(0) = 0$, such that $f_I \in \mathcal{C}^{0,1}$ for $x = 0$. Problem (2.1) has an unique odd, positive for $x < 0$, solution $f(x,t) \in L^\infty\left(0, T, \mathcal{C}^{0,1}(\mathbb{R}) \cap L^1(\mathbb{R})\right)$ for any $T > 0$. In particular,*

$$f(x,t) \in \mathcal{C}\left(0, T, \mathcal{C}^\infty(\mathbb{R}\backslash\{-a, a\})\right).$$

Moreover, $\lambda(t) > 0$ for all $t \in [0,T]$.

Proof. Denote the right hand side of (2.1) as

$$R(x,t) := \lambda(t)\left[\delta_{x=-a} - \delta_{x=a}\right].$$

Using Duhamel's principle, the solution of (2.1) can be written as

$$f(x,t) = \int_{\mathbb{R}} \Gamma(x-y,t) f_I(y)dy + \int_0^t \int_{\mathbb{R}} \Gamma(x-y,t-s)R(y,s)dyds, \tag{2.3}$$

where Γ is the fundamental solution for the heat equation in $\mathbb{R}$, given by

$$\Gamma(x,t) := \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{|x|^2}{4t}}.$$

Note that

$$H(f_I)(x,t) := \int_{\mathbb{R}} \Gamma(x-y,t) f_I(y)dy$$

is the solution of the homogeneous heat equation, and that (2.3) reduces to

$$f(x,t) = H(f_I)(x,t) + \int_0^t \lambda(s)\frac{1}{\sqrt{4\pi(t-s)}}\left(-e^{-\frac{|x-a|^2}{4(t-s)}} + e^{-\frac{|x+a|^2}{4(t-s)}}\right)ds.$$

Differentiating the above function $f(x,t)$ with respect to x, evaluating at $x=0$, we obtain

$$\lambda(t) = \lambda_0(t) + \int_0^t \lambda(s)K(t-s)ds, \tag{2.4}$$

where

$$\lambda_0(t) := -\frac{\partial}{\partial x}H(f_I)(0,t), \tag{2.5}$$

and the kernel K is given by

$$K(t) := \frac{a}{\sqrt{4\pi}t^{3/2}}\, e^{-\frac{a^2}{4t}}.$$

We claim that the integral equation (2.4) has a solution in $[0,T]$ for any $T>0$. It holds

$$K(t) \geq 0 \quad \text{for all} \quad t>0, \qquad \int_0^\infty K(t)dt = 1,$$

so that the operator $S : L^\infty(0,T) \to L^\infty(0,T)$, defined as

$$S(\lambda_n) := \lambda_0 + \int_0^t \lambda_{n-1}(s)K(t-s)ds, \tag{2.6}$$

is contractive in $L^\infty([0,T])$ for any fixed $T<+\infty$.

Let us remark that the condition $f_I \in \mathcal{C}^{0,1}$ at the point $x=0$ is used to obtain a $L^\infty([0,T])$ bound for $\lambda_0(t)$. Indeed, from (2.5) we have

$$\lambda_0(t) = -\frac{1}{\sqrt{\pi}t^{1/2}}\int_{\mathbb{R}} z f_I(2zt^{1/2})e^{-z^2}dz.$$

The existence of $\lambda(t) \in L^\infty([0,T])$ is proven by a fixed point argument for the operator S defined as in (2.6). Since λ_0 is a positive function for all time (this is so because of the Hopf's Lemma), and the kernel K is positive as well, from the iteration scheme $\lambda_n = S(\lambda_{n-1})$ we have that $\lambda_n(t)$ is a positive function. Hopf's Lemma again gives that $\lambda_n(t)$ is a positive function for all $t \in [0,T]$. The regularity for f comes from standard parabolic results, taking into account that $\frac{\partial f}{\partial x} \in L^\infty(\mathbb{R})$ for all $t \geq 0$.

Thus we can construct f as a solution of the heat equation from (2.3) with known right hand side $R(x,t)$. □

2.2 Existence in a bounded interval

In this section we will replace the real line $\mathbb{R}$ with a bounded interval $[-\pi/2, \pi/2]$, and consider the following Neumann problem analogous to (1.4): given $a \in (0, \pi/2)$, we look for an odd function f, positive for $x < 0$, solution of

$$\begin{aligned} &\frac{\partial f}{\partial t} - \frac{\partial^2 f}{\partial x^2} = \lambda(t)\left[\delta_{x=-a} - \delta_{x=a}\right], \quad x \in (-\pi/2, \pi/2),\ t > 0, \\ &\frac{\partial f}{\partial x}(-\pi/2, t) = \frac{\partial f}{\partial x}(\pi/2, t) = 0, \quad f(x, 0) = f_I(x), \\ &\lambda(t) = -\frac{\partial f}{\partial x}(0, t). \end{aligned} \tag{2.7}$$

The initial datum f_I is odd with respect to the point $x = 0$ and satisfies the compatibility conditions at the boundary with $f_I(x) \geq 0$ for $x \in [-\pi/2, 0]$.

The above problem can be reformulated by reflection around the point $x = \pi/2$ as a boundary value problem with zero-Dirichlet boundary conditions on the interval $[0, \pi]$: we seek an even function f with respect to $x = \pi/2$, positive in $(0, \pi)$ with $f(0) = f(\pi) = 0$, that solves

$$\begin{aligned} &\frac{\partial f}{\partial t} - \frac{\partial^2 f}{\partial x^2} = \lambda(t)[\delta_{x=a} + \delta_{x=\pi-a}], \quad x \in (0, \pi),\ t > 0, \\ &f(0, t) = f(\pi, t) = 0, \quad f(x, 0) = f_I(x), \\ &\lambda(t) = \frac{\partial f}{\partial x}(0, t). \end{aligned} \tag{2.8}$$

for an initial datum f_I that satisfy the compatibility conditions. Note that the Neumann boundary condition in (2.7) translates into $f'(\pi/2, t) = 0$ for (2.8). We have that

Theorem 2.2. *Let $\Omega = (0, \pi)$, $0 < a < \pi/2$, and let $f_I \in \mathcal{C}(\Omega)$ be a symmetric function in Ω with respect to $x = \pi/2$, positive, such that the Fourier series for $f_I'(x)$ converges at $x = 0$, and $f_I(0) = f_I(\pi) = 0$. Then problem (2.8) has an unique solution solution $f(x, t) \in L^\infty\left(0, T, \mathcal{C}^{0,1}(\Omega)\right)$ for all $T > 0$ that is positive and symmetric with respect to $x = \pi/2$. In particular,*

$$f(x, t) \in \mathcal{C}\left(0, T, \mathcal{C}^\infty(\Omega \backslash \{a, \pi - a\})\right).$$

Moreover, $\lambda(t) > 0$ for all $t \in [0, T]$.

Proof. The function $\psi(x, y, t)$, defined as

$$\psi(x, y, t) = \frac{2}{\pi} \sum_{n=1}^{+\infty} \sin(nx) \sin(ny) e^{-n^2 t},$$

satisfies the problem

$$\begin{gathered} \psi_t - \psi_{xx} = 0, \\ \psi(0, t) = \psi(\pi, t) = 0, \quad \psi(\cdot, 0) = \delta_{x=y}, \end{gathered}$$

and it is well defined for all $t > 0$. Using Duhamel's formula, the solution to (2.8) can be rewritten as

$$f(x, t) = \int_0^\pi \psi(x, y, t) f_I(y)\, dy + \int_0^\pi \int_0^t \psi(x, y, t - s) R(y, s)\, ds\, dy, \tag{2.9}$$

where $R(x,t) := -\frac{\partial f}{\partial x}(0,t)[\delta_{x=a} + \delta_{x=\pi-a}]$. As before, we obtain an integral equation for $\lambda(t)$,

$$\lambda(t) = \lambda_0(t) + \frac{2}{\pi}\int_0^t \lambda(s)\left[\sum_{n=1}^{+\infty} n(\sin(na) + \sin(n(\pi-a)))e^{-n^2(t-s)}\right] ds,$$

where

$$\lambda_0(t) := \frac{2}{\pi}\int_0^\pi f_I(y)\left[\sum_{n=1}^{+\infty} n\sin(ny)e^{-n^2 t}\right] dy.$$

Note that $\lambda_0 \in L^\infty(0,\pi)$ because on the condition on $f_I'(0)$ and moreover $\lambda_0(t) > 0$ for all $t > 0$ since $\sum_{n=1}^{+\infty} n\sin(ny)e^{-n^2 t} > 0$. Let

$$K(t) := \frac{2}{\pi}\sum_{n=1}^{+\infty} n(\sin(na) + \sin(n(\pi-a)))e^{-n^2 t}.$$

It holds:

$$K(t) \geq 0, \qquad \int_0^{+\infty} K(t)\, dt = 1.$$

The function $\lambda(t)$ will be the fixed-point of the following operator

$$S(\lambda(t)) = \lambda_0(t) + \int_0^t \lambda(s)K(t-s)\, ds.$$

and once $\lambda(t)$ is known, we can easily construct f from (2.9). $\square$

A comparison principle can be shown for this equation.

Lemma 2.3. *Let $f_{I,1}$ and $f_{I,2}$ be two admissible initial conditions for problem* (2.8) *such that $f_{I,1} \leq f_{I,2}$. It holds $\lambda_1 \leq \lambda_2$. As a consequence, $f_1 \leq f_2$.*

Proof. Let us denote with $\xi(y,t) := \sum_{n=1}^{+\infty} n\sin(ny)e^{-n^2 t}$. Since $\xi(x,t) \geq 0$, it holds $\lambda_{0,1} \leq \lambda_{0,2}$, where

$$\lambda_{0,i}(t) = \int_0^\pi f_{I,i}(y)\xi(y,t)\, dy, \quad i = 1,2.$$

The thesis follows from the iteration process

$$\lambda_{n,1}(t) - \lambda_{n,2}(t) = \underbrace{(\lambda_{0,1} - \lambda_{0,2})(t)}_{\leq 0} + \underbrace{\sum_{j=1}^{n} K^j * \underbrace{(\lambda_{0,1} - \lambda_{0,2})}_{\leq 0}}_{\leq 0} \leq 0.$$

A comment on notation: given a function w, the convolution $K^j * w$ for $j > 1$ is trivially defined as

$$K^j * w := K * (K^{j-1} * w).$$

The rest of the proof is standard. The difference of the equations for f_1 and f_2 reads as follows

$$\frac{\partial}{\partial t}(f_1 - f_2) = \frac{\partial^2}{\partial x^2}(f_1 - f_2) + (\lambda_1 - \lambda_2)(\delta_{x=a} + \delta_{x=\pi-a}).$$

we take $\psi = \text{sign}^+_\omega(f_1 - f_2)$, where the positive sign function is approximated by

$$\text{sign}^+_\omega(s) = \begin{cases} 1 & \text{if } s \geq \omega \\ 0 & \text{if } s \leq 0 \\ \frac{e+1}{e-1}\left(\frac{2e^{s/\omega}}{e^{s/\omega}+1} - 1\right) & \text{if } 0 < s < \omega, \end{cases}$$

where $\omega > 0$. Then, for any $s \in \mathbb{R}$, $\text{sign}^+_\omega(s) \to \text{sign}^+(s)$ as $\omega \to 0$. Moreover, $(\text{sign}^+_\omega)'(s) = 2e^{s/\omega}/(\omega(e^{s/\omega}+1)^2)$ for all $0 < s < \omega$ and $(\text{sign}^+_\omega)'(s) = 0$ for all $s < 0$ and $s > \omega$ and thus, it holds $s(\text{sign}^+_\omega)'(s) \to 0$ as $\omega \to 0$ for all $s \in \mathbb{R}$. It holds

$$\int_0^\pi \text{sign}^+_\omega(f_1 - f_2)(f_1 - f_2)_t \, dx \leq \underbrace{\int_0^\pi \text{sign}^+_\omega(f_1 - f_2)(f_1 - f_2)_{xx} \, dx}_{\leq 0}$$
$$+ \int_0^\pi \text{sign}^+_\omega(f_1 - f_2) \underbrace{(\lambda_1 - \lambda_2)}_{\leq 0}(\delta_{x=a} + \delta_{x=\pi-a}) \, dx \leq 0.$$

Passing to the limit $\omega \to 0$ in the above inequality, we obtain

$$\frac{\partial}{\partial t} \int_0^\pi (f_1 - f_2)^+ \, dx \leq 0,$$

and the thesis is proven. □

3 Asymptotic decay

In this section the exponential decay of solutions to (2.8) towards the steady state is given. For simplicity in the computations, we translate the problem from the bounded interval $(0, \pi)$ to $\Omega := (-\pi/2, \pi/2)$. The main result of this section is the following

Theorem 3.1. *Consider the following problem*

$$\begin{aligned} &\frac{\partial f}{\partial t} - \frac{\partial^2 f}{\partial x^2} = \lambda(t)[\delta_{x=-\pi/2+a} + \delta_{x=\pi/2-a}], \quad x \in (-\pi/2, \pi/2), \ t > 0, \\ &f(-\pi/2, t) = f(\pi/2, t) = 0, \quad f(x, 0) = f_I(x), \\ &\lambda(t) = -\frac{\partial f}{\partial x}(\pi/2, t). \end{aligned} \tag{3.1}$$

Under the hypothesis of Theorem 2.2 for f_I, after translation, the unique solution f of (3.1) *decays exponentially to the unique stationary state f_∞, given by*

$$f_\infty(x) := \begin{cases} \beta\,(x - \pi/2) & \text{if } \pi/2 - a \leq x \leq \pi/2, \\ \beta & \text{if } 0 \leq x \leq \pi/2 - a, \end{cases} \tag{3.2}$$

and extended evenly to the negative axis. Here β is the only constant that preserves mass, i.e. $\int_{-\pi/2}^0 f_\infty(x)\, dx = \int_{-\pi/2}^0 f(x)\, dx$ for all $t > 0$. Moreover, the following limit exists,

$$\lim_{t \to +\infty} \lambda(t) = \beta. \tag{3.3}$$

Proof. First of all, it is trivial to derive formula (3.2) for the steady state solution f_∞. Now, the function f is constructed piecewise using the separation of variables method. We divide the interval $(-\pi/2, 0)$ in two parts: $\Omega_1 := (-\pi/2, -\pi/2 + a)$ and $\Omega_2 := (-\pi/2 + a, 0)$. We look for solutions f_1 and f_2 respectively in Ω_1 and Ω_2, where f_1 and f_2 solve the problems

$$\begin{aligned} &\frac{\partial f_1}{\partial t} - \frac{\partial^2 f_1}{\partial x^2} = 0 \quad \text{in } \Omega_1, \\ &f_1(-\pi/2, t) = 0, \quad f_1(x, 0) = f_I(x), \end{aligned}$$

and

$$\begin{aligned} &\frac{\partial f_2}{\partial t} - \frac{\partial^2 f_2}{\partial x^2} = 0 \quad \text{in } \Omega_2, \\ &\frac{\partial f_2}{\partial x}(0, t) = 0, \quad f_2(x, 0) = f_I(x). \end{aligned}$$

We assume that $f_i(x,t) = h_i(x) g_i(t)$ for $i = 1, 2$, where

$$\begin{aligned} g_1(t) &= e^{-\alpha_n^2 t}, \quad h_1(x) = c_1 \sin(\alpha_n x) + c_2 \cos(\alpha_n x), \\ g_2(t) &= e^{-\beta_n^2 t}, \quad h_2(x) = d_1 \sin(\beta_n x) + d_2 \cos(\beta_n x). \end{aligned}$$

Dirichlet boundary condition at $x = -\pi/2$ for f_1 implies that

$$\tan(\alpha_n \pi/2) = \frac{c_2}{c_1}.$$

The evenness condition implies $d_1 = 0$, and therefore

$$f_2(x,t) = d_2 e^{-\beta_n^2 t} \cos(\beta_n x).$$

The function obtained by gluing f_1 and f_2 must be continuous and have a determined jump in the derivative in order to satisfy the equation. From here we derive the *matching conditions* at the point $x = -\pi/2 + a$:

$$\begin{cases} f_1(-\pi/2 + a, t) = f_2(-\pi/2 + a, t), \\ \frac{\partial f_1}{\partial x}(-\pi/2 + a, t) = \frac{\partial f_2}{\partial x}(-\pi/2 + a, t) + \frac{\partial f_1}{\partial x}(-\pi/2, t), \end{cases} \tag{3.4}$$

from which it follows

$$\begin{aligned} &\alpha_n = \beta_n, \\ &c_1 \sin(\alpha_n(-\pi/2 + a)) + c_2 \cos(\alpha_n(-\pi/2 + a)) \\ &\quad = d_2 \cos(\beta_n(-\pi/2 + a)), \\ &c_1 \cos(\alpha_n(-\pi/2 + a)) - c_2 \sin(\alpha_n(-\pi/2 + a)) \\ &\quad = -d_2 \sin(\beta_n(-\pi/2 + a)) + c_1 \cos(\alpha_n \pi/2) + c_2 \sin(\alpha_n \pi/2). \end{aligned}$$

After some computations we get the values for the eigenvalues

$$\alpha_n = \frac{2\pi n}{\pi - a}, \quad n \in \mathbb{N},$$

and

$$d_2 = 2c_1 \frac{\sin(\alpha_n(a - \pi/2))}{\cos(\alpha_n \pi/2)}.$$

By superposition, for $t > 0$, a solution $f_0(x,t)$ has the form

$$f_0(x,t) = \begin{cases} \sum_{n=1}^{+\infty} c_n \sin(\alpha_n(x + \pi/2)) e^{-\alpha_n^2 t}, & -\pi/2 \le x \le -\pi/2 + a, \\ \sum_{n=1}^{+\infty} -2c_n \sin(\alpha_n(\pi/2 - a)) \cos(\alpha_n x) e^{-\alpha_n^2 t}, & -\pi/2 + a \le x \le 0, \end{cases}$$

extended evenly to $[0, \pi/2]$. Note that

$$\int_{-\pi/2}^{0} f_0(x,t)dx = \int_{0}^{\pi/2} f_0(x,t)dx = 0. \tag{3.5}$$

The solution of (3.1) is $f = f_\infty + f_0$, and in particular, $\int_{-\pi/2}^{\pi/2} f(x,t)\,dx = \int_{-\pi/2}^{\pi/2} f_I(x)\,dx$ because of (3.5).

In order to make these formal computations rigorous, we need to understand the convergence of the previous series. Assume that the initial datum f_I is even, positive, $f_I(-\pi/2) = f_I(\pi/2) = 0$, smooth at $\Omega \backslash \{-\pi/2 + a, \pi/2 - a\}$, and that satisfies the matching conditions analogous to (3.4), which means f_I is continuous at $x = -\pi/2 + a$ and

$$\lim_{x \to (-\frac{\pi}{2}+a)^-} f_I'(x) = \lim_{x \to (-\frac{\pi}{2}+a)^+} f_I'(x) \quad + \quad f_I'(-\pi/2). \tag{3.6}$$

Let us construct the Fourier expansion of f_I in two intervals I_1 and I_2 of length $2L$, where $L = \frac{\pi - a}{2}$, such that $\Omega_1 \subseteq I_1$ and $\Omega_2 \subseteq I_2$. Let $\bar{f}_1$ be any periodic function of period $2L$ in I_1 that equals $f_I - f_\infty$ in Ω_1, and consider its Fourier expansion. Let $\bar{f}_2$ be any periodic function of period $2L$ in I_2 that equals $f_I - f_\infty$ in Ω_2, and consider also its Fourier expansion. Both expansions for $\bar{f}_1$ and $\bar{f}_2$ will converge uniformly since $\bar{f}_1$ and $\bar{f}_2$ are piecewise $\mathcal{C}^1$. Moreover, since both f_I and f_∞ satisfy the compatibility conditions (3.4), the Fourier expansion will have the following form

$$f_I(x) = f_\infty + \begin{cases} \sum_{n=1}^{+\infty} c_n \sin(\alpha_n(x + \pi/2)), & -\pi/2 \le x \le -\pi/2 + a, \\ \sum_{n=1}^{+\infty} -2c_n \sin(\alpha_n(\pi/2 - a)) \cos(\alpha_n x), & -\pi/2 + a \le x \le 0, \end{cases}$$

Thus we conclude that the solution of the problem (3.1) must be of the form

$$f(x,t) = f_\infty + \begin{cases} \sum_{n=1}^{+\infty} c_n \sin(\alpha_n(x + \pi/2)) e^{-\alpha_n^2 t}, & -\pi/2 \le x \le -\pi/2 + a, \\ \sum_{n=1}^{+\infty} -2c_n \sin(\alpha_n(\pi/2 - a)) \cos(\alpha_n x) e^{-\alpha_n^2 t}, & -\pi/2 + a \le x \le 0, \end{cases} \tag{3.7}$$

and the convergence of the series is uniform.

Now, a general f_I might not satisfy the matching conditions (3.6). However, any solution $f(x,t)$ to problem (3.1) does so for anytime $t_0 > 0$ (see Theorem 2.2). Consider $f(x, t_0)$ for some $t_0 > 0$ as the new initial condition, note that the solution must be unique. In this case the coefficient c_n in the Fourier expansion of the solution (3.7) will correspond to the Fourier coefficients of $f(x, t_0)$. This is the same phenomenon as in the regular heat equation: we cannot recover the initial conditions from the solution for $t > 0$.

To finish the proof, just note that the exponential decay pointwise in x of f towards the stationary state f_∞ is given by the first non-zero eigenvalue, and it is of order $O\left(e^{-\left(\frac{2\pi}{\pi - a}\right)^2 t}\right)$. □

Remark. It would have been possible to write the exponential convergence results in a more general setting of semigroup theory for the heat equation with non-linear right hand side

$$\frac{\partial u}{\partial t} - A(u) = N(u)$$

Although it would have provided a more general framework for our $N(u)$, the aim of this paper was not to construct a general theory, but to concentrate on this type of non-standard equations.

4 The Fokker-Planck version

Now we turn our attention to the problem with the extra convection term. We seek solutions $f(x,t)$, $x \in \mathbb{R}$, $t > 0$, $f \in L^\infty\left(0,T,L^1(\mathbb{R})\right)$, $\forall T > 0$, f odd in x, and with finite first momentum, $\int_{-\infty}^0 |x|\, f(x,t)dx < +\infty$, of the problem

$$\begin{gathered}\frac{\partial f}{\partial t} - \frac{\partial^2 f}{\partial x^2} - \frac{\partial}{\partial x}(xf) = \lambda(t)\left[\delta_{x=-a} - \delta_{x=a}\right], \\ f(x,0) = f_I(x), \quad \lambda(t) = -\frac{\partial f}{\partial x}(0,t).\end{gathered} \tag{4.1}$$

This a Fokker-Planck equation with an harmonic confining potential. We remind the reader that also in this case, the conservation of mass (2.2) holds. The reader is referred to Frank [4], Risken [10] for the necessary background on the Fokker-Planck equation.

Theorem 4.1. *Let $f_I \in L^1(\mathbb{R})$ be an odd function, positive for $x < 0$, $f_I(0) = 0$, such that $f_I \in \mathcal{C}^{0,1}$ at $x = 0$, and with finite first momentum $\int_{-\infty}^0 |x| f_I(x)dx < \infty$. The problem (4.1) has an unique odd, positive for $x < 0$, solution $f(x,t) \in L^\infty\left(0,T,\mathcal{C}^{0,1}(\mathbb{R}) \cap L^1(\mathbb{R})\right)$ for all $T > 0$, and with finite first momentum. In particular,*

$$f(x,t) \in \mathcal{C}\left(0,T,\mathcal{C}^\infty(\mathbb{R}\backslash\{-a,a\})\right).$$

Moreover, $\lambda(t) > 0$ for all $t \in [0,T]$.

Proof. As before, denote the right hand side of the equation as

$$R(x,t) := \lambda(t)\left[\delta_{x=-a} - \delta_{x=a}\right],$$

and write the solution as

$$f(x,t) = \int_{\mathbb{R}} G(x,y,t) f_I(y)dy + \int_0^t \int_{\mathbb{R}} G(x,y,t-s)R(y,s)dyds, \tag{4.2}$$

where G is the fundamental solution of the Fokker-Planck equation in $\mathbb{R}$, (see Risken [10] for the general background, or Carrillo-Toscani [2], for the notation used in the present paper), and given by the formula

$$G(x,y,t) = \frac{e^t}{\sqrt{2\pi(e^{2t}-1)}}\,\exp\left\{-\frac{\left(e^t x - y\right)^2}{2(e^{2t}-1)}\right\}.$$

Differentiating (4.2) with respect to x and evaluating at $x = 0$, similarly as in the previous theorem, we obtain a similar equation for λ

$$\lambda(t) = \lambda_0(t) + \int_0^t \lambda(s)\tilde{K}(t-s)ds,$$

where

$$\lambda_0(t) := -\frac{\partial}{\partial x} H(f_I)(0,t),$$

for

$$H(f_I)(x,t) := \int_{\mathbb{R}} G(x,y,t) f_I(y)dy,$$

and

$$\tilde{K}(t) := \frac{2a}{\sqrt{2\pi}} \frac{e^{2t}}{\left(e^{2t}-1\right)^{3/2}}\, e^{-\frac{a^2}{2(e^{2t}-1)}}.$$

We can prove that

$$\tilde{K} \geq 0 \quad \text{for all} \quad t, \qquad \int_0^\infty \tilde{K}(t)dt = 1,$$

so that the operator

$$\tilde{S}(\lambda) := \lambda_0 + \int_0^t \lambda(s)\tilde{K}(t-s)ds$$

is contractive in $L^\infty([0,T])$, and as in Theorem 2.1. We remark that $\lambda_0(t)$ is defined as

$$\lambda_0(t) = \sqrt{\frac{2}{\pi}} \int_{\mathbb{R}} \frac{e^{2t}}{(e^{2t}-1)^{1/2}} \, z e^{-z^2} f_I\left(z\sqrt{2(e^{2t}-1)}\right) dz.$$

We use the additional finite first moment condition on the initial data f_I in order to achieve an $L^\infty[0,T]$ bound for λ_0. The rest of the proof can be done following the one of Theorem 2.1. □

Let us write now the stationary state solution f_∞ for (4.1). In the interval $(-\infty, 0]$ it is given by

$$f_\infty(x) = \begin{cases} e^{-x^2/2} & \text{if} \quad x \leq -a, \\ e^{-x^2/2}g(x) & \text{if} \quad -a \leq x \leq 0, \end{cases}$$

for the function

$$g(x) = 1 - \left(\int_{-a}^0 e^{s^2/2}ds\right)^{-1} \int_{-a}^x e^{s^2/2}ds.$$

The solution in the whole line $\mathbb{R}$ is obtained by odd reflection. Note that f_∞ is continuous in $\mathbb{R}$, $f_\infty(0) = 0$, and that f_∞ has a discontinuity in the first derivative at $x = -a, a$.

Remark. Note that equation (4.1) can be obtained from (2.1) by suitable self-similar rescaling as in Carrillo-Toscani [3]. More precisely, if v is a solution of (2.1), then

$$u(x,t) = \alpha(t)u(\alpha(t)x, \beta(t))$$

for

$$\alpha(t) = e^t, \quad \beta(t) = \tfrac{1}{2}\left(e^{2t}-1\right),$$

is a solution of (4.1). Because the Fokker-Planck equation in $\mathbb{R}$ has negative eigenvalues in $\mathbb{R}$, we expect to have some exponential decay results for (4.1). The above rescaling might help in understanding the asymptotic decay of (2.1). This is being studied in [5].

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