THE WEIL REPRESENTATION OVER THE COMPLEX NUMBERS AND LOCALIZATION OF SCHWARTZ SPACES

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Abstract. In this paper a strong form of the Stone-von Neumann property of the Heisenberg representation is stated and proved. Several results in harmonic analysis are obtained as a consequence.

0. Introduction

One of the most fundamental spaces in harmonic analysis is the space $S(\mathbb{R}^N)$ of Schwartz functions. As everybody knows this space consists of infinitely differentiable functions with all their derivatives rapidly decreasing. The Schwartz space is intimately related with many basic transforms in pure and applied mathematics. By far, the most important one is the Fourier transform F. In fact, one of the only reasonable ways to describe F is to define it on $S\left(\mathbb{R}^N\right)$ using an explicit formula

$$F(f)(y) = \int_{\mathbb{R}^N} e^{2\pi i y\cdot x} f(x)dx,$$

and then show that F preserve the standard Hermitian product on $S\left(\mathbb{R}^N\right)$ thus it extends to a unitary operator on $L^2\left(\mathbb{R}^N\right)$. This kind of argument of-course depends on the crucial fact, although not at all a trivial one, that $S\left(\mathbb{R}^N\right)$ is preserved by F.

One of our main goals in this paper is to make a formal sense of the following statement

(*) **The Schwartz space $S\left(\mathbb{R}^N\right)$ is algebraic.**

On the phenomenological level, this means that the Schwartz space enjoys many good properties and exhibit a certain degree of rigidity much like holomorphic functions. In fact, it appears that these two spaces are more alike than what might be expected[1]. As a consequence of (*) we will obtain a novel algebraic formula for the standard Hermitian product on $S\left(\mathbb{R}^N\right)$.

As we mentioned before, $S\left(\mathbb{R}^N\right)$ is related with the operator F of Fourier transform. Surprisingly, it is less well known that the Fourier transform is a particular operator in a family of operators acting on $S\left(\mathbb{R}^N\right)$ and preserving the Hermitian product. As a first approximation, the statement is that there exists a unitary representation of the real symplectic group $Sp = Sp\left(2N, \mathbb{R}\right)$

$$\rho : Sp \longrightarrow U\left(S\left(\mathbb{R}^N\right)\right),$$

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[1]The Schwartz space can be realized (known as the Fock realization) as the space of holomorphic functions satisfying certain growth condition. This is not the type of similarity we mean here. The main difference is that in our realization the growth condition disappear.

so that the operator F appears as $\rho(w)$ for a a particular element $w \in Sp$ called the Weyl element. The representation ρ is called the *Weil representation.*

The correct statement is that ρ is, in fact, a representation of a double cover

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow Mp \longrightarrow Sp \longrightarrow 1,$$

called by Weil [W] the *metaplectic cover*. This fact has many important implications to various fundamental phenomenons in mathematics and physics, including the theory of theta functions and automorphic forms [W], harmonic analysis [F, H1] and last (but probably not least) quantum mechanics [H2, Se2, Sh]. The metaplectic sign was studied by several people [W, LV, V] albeit its precise origin remains to some extent still mysterious. Another main goal of this paper is to make a formal sense of the following statement

The metaplectic sign is of algebraic origin.

0.1. **The Heisenberg representation.** As it turns out, the Schwartz space and the Weil representation are intimately related and both appear as a consequence of a more fundamental structure i.e., the *Heisenberg representation*. We will now proceed to describe this fundamental representation.

The initial data is a $2N$-dimensional real symplectic vector space (V, ω), the reader should think of V as $\mathbb{R}^N \times \mathbb{R}^N$ with the standard symplectic form

$$\omega = \sum_{i=1}^{N} dy_i \wedge dx_i.$$

The vector space V considered as an abelian group admits a non-trivial central extension

$$0 \rightarrow \mathbb{R} \rightarrow H \rightarrow V \rightarrow 0,$$

called the *Heisenberg group*. Concretely, the group H can be presented as $H = V \times \mathbb{R}$ with the multiplication given by

$$(v, z) \cdot (v', z') = (v + v', z + z' + \tfrac{1}{2}\omega(v, v')).$$

The center of H is $Z_H = \{(0, z) : z \in \mathbb{R}\}$. The symplectic group $Sp = Sp(V, \omega)$ acts on H as a group of automorphism via its standard action on the V-coordinate. One of the most important attributes of H is that it admits principally a unique irreducible unitary representation. The formal statement is the content of the celebrated Stone-von Neumann theorem

Theorem 1 (Stone-von Neumann)**.** *Let $\psi_\kappa = e^{\kappa z}$, $\kappa \in i\mathbb{R}$ be a character of the center Z_H. There exists a unique (up to an isomorphism) unitary irreducible representation $(\pi^\kappa, H, \mathcal{H}^\kappa)$ with the center acting by $\pi^\kappa_{|Z_H} = \psi_\kappa \cdot Id_{\mathcal{H}}$.*

The representation π^κ will be referred to as the *Heisenberg representation.*

0.1.1. *The Schwartz space.* The modern point of view suggests that the Schwartz space is naturally identified with the space $\mathcal{H}^\kappa_\infty$ of smooth vectors[2] in the Heisenberg representation. The precise statement is that there exists a particular realization of the Heisenberg representation for which the Hilbert space is the space $L^2(\mathbb{R}^N)$ of square integrable functions and the subspace of smooth vectors is the Schwartz space $S\left(\mathbb{R}^N\right)$.

[2]We remind the reader that given a representation $(\pi, G, \mathcal{H})$ of a Lie group G, a vector $v \in \mathcal{H}$ is called smooth if the map $\pi_v : G \longrightarrow \mathcal{H}$ defined by $\pi_v(g) = \pi(g)v$ is infinitely differentiable at $1 \in G$.

Convention: As a rule, the Heisenberg representation will be considered as a representation of H on the Hilbertian space of smooth vectors. For simplicity, the subscript $(\cdot)_\infty$ will be everywhere omitted, for example from now on instead of writing $\mathcal{H}^\kappa_\infty$ we will write simply $\mathcal{H}^\kappa$.

0.1.2. *The Weil representation.* The nature of the Weil representation is more intricate. A direct consequence of Theorem 1 is the existence of a projective representation

$$\widetilde{\rho} : Sp \to PU(\mathcal{H}^\kappa).$$

The construction of $\widetilde{\rho}$ out of the Heisenberg representation π^κ is standard. The group Sp acts on the category of (unitary) representations of H by

$$\pi \mapsto \pi^g,$$

for every representation π, where π^g acts on the same Hilbert space as π but the action is given by $\pi^g(h) = \pi(g(h))$. Clearly π^κ and $\pi^{\kappa,g}$ have central character ψ_κ hence by Theorem 1 they are isomorphic. Since the space $\mathsf{Hom}_H(\pi^\kappa, \pi^{\kappa,g})$ is one dimensional, choosing for every $g \in Sp$ a non-zero representative $\widetilde{\rho}(g) \in \mathsf{Hom}_H(\pi^\kappa, \pi^{\kappa,g})$ gives the required projective representation. From this point of view however, the fact that the projective representation $\widetilde{\rho}$ can be linearized up to a sign is not transparent.

0.2. **Main results.** The main result of this paper is a formulation of a stronger form of the Stone-von Neumann property of the Heisenberg representation (S-vN property for short). More precisely, It will be shown that the strong S-vN property is governed by an algebraic structure. The following applications of this result will be demonstrated

(1) An algebraic characterization of the Schwartz space $S\left(\mathbb{R}^N\right)$ will be obtained. In addition, a novel algebraic formula for the Hermitian structure of the Heisenberg representation will be established. This will put the Heisenberg-Weil representation on algebraic grounds eliminating all measure theoretic attributes.
(2) A solution to a question of Deligne [DE] will be obtained. This question concerns the existence of a possible natural pairing between various interesting function spaces which are associated with the Heisenberg representation. These pairings generalize the natural pairing between the Heisenberg representations which are associated to opposite central weights κ.
(3) An algebraic origin of the metaplectic sign will be revealed and the obstruction to having an analytic Weil representation of the complexified group $Sp_{\mathbb{C}}$ will be specified. Both of these results are greatly inspired by Deligne's paper [DE] which also address these two issues in the two dimensional setting. The main difference of our approach from that of [DE] is that our constructions are algebraic, hence are strictly of a finite nature, while the approach in [DE] is analytic and requires to manipulate with infinite dimensional objects.

Remark 1. *Result 1 is related to an old question in physics raised by David Bohm and his school concerning the origin of the Hermitian structure in quantum mechanics. In the language of physicists this is the quest for a conceptual explanation of Dirac's Bra/Ket operations.*

We will spend the rest of the introduction to explaining the main ideas which underlay the strong S-vN property, taking result 1 as the main leading theme of our discussion. It will be beneficial to start with a toy example which will turn out to be very suggestive.

0.3. **The space of holomorphic functions (toy example).** The space $\mathcal{O}$ of holomorphic functions in one variable consists of infinitely differentiate functions on $\mathbb{R}^2$ satisfying the Cauchy-Riemann equation

$$\partial_{\overline{z}} f = \left(\tfrac{\partial_x + i\partial_y}{2}\right) f = 0. \tag{0.1}$$

The space $\mathcal{O}$ enjoys many good properties and it exhibit strong rigidity. One of main manifestations of this rigidity is the Cauchy property.

Theorem 2 (Cauchy's Theorem)**.** *Given* $f \in \mathcal{O}$

$$f(w) = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{f(z)}{z-w} dz,$$

for any $w \in \mathbb{R}^2$ *and* $\gamma : S^1 \rightarrow \mathbb{R}^2 \smallsetminus w$ *is any curve homologous to the standard curve* $\gamma_0\left(t\right) = w + e^{2\pi i t}$.

In plain language, the Cauchy property says that the delta functional δ_w when restricted to $\mathcal{O}$ can be calculated using an integral formula, symbolically it can be written as follows

$$\delta_w\left(f\right) = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{f(z)}{z-w} dz \tag{0.2}$$

for every $f \in \mathcal{O}$. Our goal is to track the conceptual mechanism which makes the space of holomorphic functions so rigid. As a result we will obtain a conceptual formulation of the Cauchy property. Then in analogy we will proceed to show that the Schwartz space obeys the same mechanism, which is principally the content of the strong S-vN property. As an application, a Schwartz analogues of the Cauchy property will be demonstrated. Namely, the standard Hermitian product in $S\left(\mathbb{R}^N\right)$ which is given in terms of integration with respect to the Lebegues measure

$$\langle f, g \rangle = \int\limits_{x \in \mathbb{R}^N} f(x)\overline{g(x)} dx,$$

for every $f, g \in S\left(\mathbb{R}^N\right)$ will be replaced by an equivalent algebraic formula of the form

$$\langle f, g \rangle = \int\limits_{\gamma} G(f,g) \tag{0.3}$$

where $G(f,g)$ is a closed N-form on the Lagrangian Grassmanian[3] $Lag = Lag\left(V\right)$ and $\gamma \in H_N(Lag, \mathbb{C})$ is a non-trivial homology class.

As it turns out, the language of algebraic D-modules supplies a convenient formalism to achieving this goal.

[3]The Lagrangian Grassmanian of a symplectic vector space V is the classifying space of all maximal isotropic (Lagrangian) subspaces in V.

Remark 2. *On the philosophical level, the space Lag which appears in our formulas should hold the same importance to Harmonic analysis as the standard N-dimensional Euclidian space $\mathbb{R}^N$, yet it is of a completely different nature i.e., it is compact and contrary to $\mathbb{R}^N$ it is homologically non-trivial. It is tempting to refer to Lag as the "forgotten parameters of harmonic analysis".*

0.3.1. *The Cauchy-Riemann D-module.* Let $D_{\mathbb{R}^2}$ be the algebra of linear differential operators on $\mathbb{R}^2$, also referred to as the two dimensional *Weyl-algebra.* More concretely, $D_{\mathbb{R}^2}$ is a $\mathbb{C}$-algebra generated by

$$x_1, x_2; \partial_{x_1}, \partial_{x_2},$$

subject to the relations

$$\begin{aligned} [x_1, x_2] &= [\partial_{x_1}, \partial_{x_2}] = 0, \\ [\partial_{x_i}, x_j] &= \delta_{ij}. \end{aligned}$$

We shall construct a module M_{CR} over the algebra $D_{\mathbb{R}^2}$ which will encode the Cauchy-Riemann equation. The construction is completely straight forward. Let $I_{CR} \subset D_{\mathbb{R}^2}$ be the left ideal generated by the Cauchy-Riemann differential operator, namely

$$I_{CR} = D_{\mathbb{R}^2} \cdot \partial_{\overline{z}}.$$

We define M_{CR} to be the left $D_{\mathbb{R}^2}$-module given by

$$M_{CR} = D_{\mathbb{R}^2} / I_{CR}.$$

The space of holomorphic functions can be naturally identified with the space of morphisms

$$\mathsf{Hom}_{D_{\mathbb{R}^2}}(M_{CR}, C^\infty_{\mathbb{R}^2}). \tag{0.4}$$

Here, $C^\infty_{\mathbb{R}^2}$ is considered with its natural $D_{\mathbb{R}^2}$-action. The identification is very simple, given a morphism $\varphi : M_{CR} \to C^\infty_{\mathbb{R}^2}$, one associate to it the function $f_\varphi = \varphi(1)$. It is easy to verify that indeed f_φ satisfies the Cauchy-Riemann equation

$$\partial_{\overline{z}} f_\varphi = \partial_{\overline{z}} \varphi(1) = \varphi(\partial_{\overline{z}}) = 0,$$

where the second equality is due to φ being a morphism of $D_{\mathbb{R}^2}$-modules and the third equality is because $\partial_{\overline{z}} \in I_{CR}$. On the other direction, starting from a function f satisfying (0.1) one associate to it a morphism of $D_{\mathbb{R}^2}$-modules

$$\varphi_f : M_{CR} \longrightarrow C^\infty_{\mathbb{R}^2},$$

defined by $\varphi_f(d) = d(f)$ for every $d \in D_{\mathbb{R}^2}$. Again, it is easy to verify that indeed φ_f factors through the quotient M_{CR} since $\partial_{\overline{z}} f = 0$. The space (0.4) is called the space of *solutions of* M_{CR} in the *target* $D_{\mathbb{R}^2}$-module $C^\infty_{\mathbb{R}^2}$.

Summary: the space $\mathcal{O}$ of holomorphic functions is characterized as the space of solutions of an algebraic D-module. This innocent looking observation has far reaching implications. In particular it will allow us to obtain a conceptual explanation of the Cauchy property.

0.3.2. *Algebraic functionals.* The Cauchy property (0.2) will turn out to be the following statement

(**) **The functional δ_w is algebraic.**

Our next goal is to make a formal sense of the above statement. Apriory the functional δ_w is sitting in the "stupid" dual $\mathcal{O}^* = Hom_{\mathbb{C}}(\mathcal{O},\mathbb{C})^*$. The interesting claim is that inside $\mathcal{O}^*$ there exists various much smaller subspaces of *algebraic functionals*. The formal content of (**) is that δ_w lies in one of these algebraic subspaces.

Our plan is to associate to every point $w \in \mathbb{R}^2$ an algebraic subspace $\mathcal{O}_w^\vee \subset \mathcal{O}^*$. The idea behind the construction of $\mathcal{O}_w^\vee$ can be summarized as follows

Main idea: Instead of dualizing $\mathcal{O}$ as a plain vector space, dualize the $D_{\mathbb{R}^2}$-module M_{CR}.

In more details, dualizing the $D_{\mathbb{R}^2}$-module M_{CR} yields another $D_{\mathbb{R}^2}$-module which we denote by $M_{CR}^\vee$. The space $\mathcal{O}_w^\vee$ is taken to be

$$\mathcal{O}_w^\vee \triangleq \mathsf{Hom}_{D_{\mathbb{R}^2}}(M_{CR}^\vee, C^\infty_{\mathbb{R}^2 \smallsetminus w}). \tag{0.5}$$

It remains to explain why $\mathcal{O}_w^\vee$ is naturally sitting inside $\mathcal{O}^*$. This is a consequence of the following fundamental theorem

Theorem 3. *Given a non-trivial class $\gamma \in H_1\left(\mathbb{R}^2 \smallsetminus w, \mathbb{C}\right)$, there exists a natural non-degenerate pairing*

$$B_\gamma : \mathcal{O} \times \mathcal{O}_w^\vee \to \mathbb{C}.$$

From the D-module theoretic point of view Cauchy theorem becomes simply the statement

Theorem 4. *The functional δ_w is algebraic, namely there exists a vector $\Delta_w \in \mathcal{O}_w^\vee$ such that*

$$\delta_w(f) = B_\gamma(f, \Delta_w).$$

for every $f \in \mathcal{O}$.

It should be noted that the integral form of the Cauchy formula follows form the general form of the pairing B_γ.

Remark 3. *The stream of ideas that was demonstrated above agrees with the general methodology of algebraic analysis. This methodology suggests that many analytic phenomena are in fact governed by algebraic structures and can be recast completely in algebraic terms.*

Theorem 3 is a particular case of a general statement concerning the existence of a natural pairing

$$B : \mathsf{Sol}(M) \times \mathsf{Sol}(M^\vee) \longrightarrow \mathbb{C}, \tag{0.6}$$

between the solution space of a D-module M and the solution space of the dual D-module $M^\vee$ [GO]. The main technical ingredient in the construction of such pairing is the Green class associated with the D-module M. The Green class of a D-module appears to be a far reaching generalization of the classical notion of green form of a differential operator. In order to give the reader some intuitive appreciation, we will proceed to give an informal explanation of the construction of the pairing (0.6).

0.4. **Canonical pairings.** Let X be a smooth algebraic manifold. A general enough example for us is the manifold $X = \mathbb{P}_{\mathbb{R}}^N$. We denote by D_X the algebra of linear differential operators on X. For example in the case $X = \mathbb{A}^n$ the algebra D_X is the Weyl algebra with $2n$ generators. The category of finitely generated (left) D_X-module will be denoted by $\mathsf{Coh}(D_X)$. A useful interpretation (however not the only one) of an object M in $\mathsf{Coh}(D_X)$ is viewing it as a generalization of system of linear differential equations. The dictionary is quite simple and follows the same lines as in the Cauchy-Riemann example. Given a differential operator $P \in D_X$ we can associate to it the D_X-module

$$M_P = D_X / I_P,$$

where I_P is the left ideal generated by P, namely

$$I_P = D_X \cdot P.$$

More generally, if $P = (P_1, .., P_l)$ is a system of linear differential operators then we associate to it the D_X-module

$$M_P = \bigoplus_{i=1}^{l} D_X / I_{P_i}.$$

Assumption: For the sake of the introduction, we will only consider modules associated with a single differential operator.

The algebraic D_X-module M_P accounts for the algebraic content of the differential equation

$$Pf = 0,$$

if we want to account also for the solutions of such equation we need to choose a target D_X-module F , which is usually taken to be of an analytic nature. The rule is very simple, for example if we want to consider C^∞-solutions then we take F to be C_X^∞, alternatively if we want to consider generalized solutions then we take F to be the sheaf of generalized functions D'_X. Given a choice of a target module F, the space of F-solutions of the system P is naturally identified with the vector space

$$\mathsf{Sol}(M_P, F) \triangleq \mathsf{Hom}_{D_X}(M_P, F)$$

where the identification sends a morphism $\varphi \in \mathsf{Hom}_{D_X}(M_P, F)$ to the function $\varphi(1) \in F$.

Summary: The D-module theoretic formulation demonstrate a splitting between algebraic structures and analytic ones, where M_P accounts for the algebraic content and the target module F accounts for the analytic content.

Higher solution spaces. The D-module point of view suggest an interesting generalization to the classical notion of solution of a differential equation, namely the notion of *higher solutions*. More precisely, if the target module F is not specified then $\mathsf{Hom}_{D_X}(M_P, \cdot)$ establish a functor

$$\mathsf{Hom}_{D_X}(M_P, \cdot) : D_X\text{-}\mathsf{Mod} \longrightarrow Vect.$$

Following the general yoga of homological algebra, higher solutions appear as the derived functors

$$R^i\mathsf{Hom}_{D_X}(M_P, \cdot) : D_X\text{-}\mathsf{Mod} \longrightarrow Vect,$$

That is, the space of level i, F-solutions of M_p is

$$R^i \mathsf{Hom}_{D_X}(M_P, F)$$

Remark 4. *It is quite curious that although from the homological view point these solution spaces of higher level are as legitimate as the solution space of level zero, they seem to play no essential part in the classical theory of linear partial differential equations.*

0.4.1. *Algebraic functionals.* As we did for the Cauchy-Riemann equation, our goal is to define the notion of an algebraic functional on the solution space $\mathsf{Sol}(M_P, C_X^\infty)$. The main step is to dualize the module M_P. The technical problem is that a rigorous definition of duality in the D-module setting requires us to work in the derived category of coherent D_X-modules. This higher level of sophistication is the price (a small one to the opinion of this author) we have to pay if we want to replace analytic manipulations with algebraic ones. For the sake of the introduction it will be sufficient for us to know that an object in this derived category can be thought of as a complex of D_X-modules

$$M^\bullet : \quad .. \longrightarrow M^{-1} \xrightarrow{d} M^0 \xrightarrow{d} M^1 \longrightarrow ..$$

The category $\mathsf{Coh}(D_X)$ of usual coherent D_X-modules is naturally sitting inside the derived category as complexes supported only at degree 0. The derived category supports an operation of duality, called Verdier duality functor. The Verdier duality functor is an anti-equivalence

$$\mathbb{D} : \mathsf{DCoh}(D_X) \to \mathsf{DCoh}(D_X),$$

Cohen-Macaulay modules. It is quite often that starting from an honest D_X-module M after applying duality, $\mathbb{D}(M)$ is no longer a module but it is a complex supported in various degrees. However, for a D_X-module of the form M_P, the dual is again an honest D_X-module. More precisely $\mathbb{D}(M_P)$ is a complex which is supported at a single cohomological degree, namely it is of the form

$$\mathbb{D}(M_P) : \quad ... \longrightarrow 0 \longrightarrow M_P^\vee \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow ...$$

where $M_P^\vee$ is an honest D_X-module sitting at degree $d = -(\dim X - 1)$. Equivalently we can write

$$\mathbb{D}(M_P) \simeq M_P^\vee[d]$$

where the notation $[d]$ is the standard cohomological shift which means that we consider $M_P^\vee$ as sitting at degree $-d$.

This kind of situation, for which the dual $\mathbb{D}(M)$ is an honest module sitting at a single cohomological degree is exceptional. Such modules are called *Cohen-Macaulay.* In general, the number d varies between 0 and $\dim X$ and it serves as a measure for the size of the module M. The general rule is that the fewer relations M contains the larger d is. Since in our example M_P contains a single relation coming from the differential operator P hence d is almost maximal, that is $d = \dim X - 1$. The other extreme situation is when $d = 0$, such modules are the smallest possible and are called *holonomic*, or in the classical language are sometimes referred to as *maximally over-determined systems.* Holonomic modules appear to play a central role in many diverse areas in mathematics and physics.

We consider the following space of algebraic functionals

$$\mathsf{Sol}(M_P, C_X^\infty)^\vee \triangleq \mathsf{Sol}\left(M_P^\vee, C_X^\infty\right).$$

Similarly as for the toy example of the Cauchy-Riemann equation, also in general there exists a pairing

$$B_{\gamma} : \mathsf{Sol}\left(M_P, C_X^{\infty}\right) \times \mathsf{Sol}(M_P^{\vee}, C_X^{\infty}) \to \mathbb{C}. \tag{0.7}$$

where as suggested by the notation, this pairing depends on a choice of an homology class $\gamma \in H_d(X, \mathbb{C})$.

The main technical tool for constructing the pairing (0.7) is the notion of Green class of a D-module.

The green class (intuitive explanation). The intuitive idea behind the construction of the Green class is straight forward. It will be suggestive to consider for a brief moment the simpler category $Vect$ of vector spaces. Given a vector space V there exits a natural adjunction isomorphism

$$\mathsf{Adj}_V : \mathsf{Hom}_{\mathbb{C}}\left(V, V\right) \xrightarrow{\simeq} \mathsf{Hom}_{\mathbb{C}}\left(\mathbb{C}, V^* \bigotimes\nolimits_{\mathbb{C}} V\right), \tag{0.8}$$

Under this isomorphism, the identity element $Id \in \mathsf{Hom}_{\mathbb{C}}\left(V, V\right)$ is sent to a distinguished element $G_V = \mathsf{Adj}_V\left(Id\right) \in \mathsf{Hom}_{\mathbb{C}}\left(\mathbb{C}, V^* \bigotimes_{\mathbb{C}} V\right)$. The element G_V can described explicitly as

$$G_V = \sum_i e_i^* \otimes e_i,$$

where e_i is an arbitrary basis in V and e_i^* is the corresponding dual basis of V^*.

In the D-module setting there exists an analogue of (0.8), that is for given a D_X-module M there exists a natural adjunction isomorphism

$$\mathsf{Adj}_M\colon \mathsf{Hom}_{D_X}\left(M, M\right) \xrightarrow{\simeq} \mathsf{Hom}_{D_X}\left(\mathcal{O}_X, \mathbb{D}\left(M\right) \bigotimes\nolimits_{\mathcal{O}} M\right),$$

The Green class of M is defined to be the distinguished vector

$$G_M = \mathsf{Adj}_M\left(Id\right) \in \mathsf{Hom}_{D_X}\left(\mathcal{O}_X, \mathbb{D}\left(M\right) \bigotimes\nolimits_{\mathcal{O}} M\right).$$

The de-Rham construction. The Green class G_M can be realized in concrete terms using the de-Rham complex, which, in turn, is probably one of the most basic constructions in the theory of algebraic D-modules. The de-Rham complex $DR^{\bullet}\left(M\right)$ is defined as follows

$$DR^{\bullet}(M): \quad M \xrightarrow{d} M \bigotimes\nolimits_{\mathcal{O}} \Omega^1 \xrightarrow{d} \ldots \xrightarrow{d} M \bigotimes\nolimits_{\mathcal{O}} \Omega^{\dim X},$$

where M is sitting at degree 0 and d is the usual de-Rham differential given in local coordinates by[4]

$$d(m \otimes \omega) = m \otimes d\omega + \sum_{i=1}^{N} \partial_{x_i} m \otimes d_{x_i} \wedge \omega.$$

The Green class. The Green class of M appears as a distinguished class

$$G_M \in H^0(DR^{\bullet}\left(\mathbb{D}\left(M\right) \bigotimes\nolimits_{\mathcal{O}} M\right). \tag{0.9}$$

For our specific example, since $\mathbb{D}\left(M_P\right) \simeq M_P^{\vee}[\dim X - 1]$ it is easy to show that (0.9) takes a simpler form

$$G_{M_P} \in H^{\dim X - 1}\left(DR^{\bullet}\left(M_P^{\vee} \bigotimes\nolimits_{\mathcal{O}} M_P\right)\right).$$

More concretely, the Green class G_{M_P} can be presented by a (non-unique) chain

$$G_{M_P} = \sum_i m_i^{\vee} \otimes m_i \otimes \omega_i,$$

[4]It is a standard argument to show that in fact d does not depend on the choice of the local coordinates.

where $m_i \in M_P$, $m_i^\vee \in M_P^\vee$ and $\omega_i \in \Omega_X^{\dim X-1}$ and which is closed with respect to the de-Rham differential

$$d\left(G_{M_P}\right)=0.$$

Construction of the pairing. Now we are finally ready to describe the pairing (0.7). The construction depends on a choice of an homology class $\gamma \in H_{\dim X-1}\left(X,\mathbb{C}\right)$. Given a pair of solutions

$$\begin{aligned} \nu &\in \mathsf{Sol}(M_P, C_X^\infty),\\ \varphi &\in \mathsf{Sol}(M_P^\vee, C_X^\infty),\end{aligned}$$

the tensor product $\varphi\otimes\nu$ defines a morphism of complexes

$$DR^\bullet\left(M_P^\vee \bigotimes_{\mathcal{O}} M_P\right) \overset{\varphi\otimes\nu}{\longrightarrow} DR^\bullet\left(C_X^\infty\right),$$

which in particular induces a map on the level of cohomologies, in particular it yields a map

$$H^{\dim X-1}(DR^\bullet\left(M_P^\vee \bigotimes_{\mathcal{O}} M_P\right) \overset{\varphi\otimes\nu}{\longrightarrow} H^{\dim X-1}(DR^\bullet\left(C_X^\infty\right))$$

applying the last map to the Green class G_{M_P} one obtains a class in

$$H^{\dim X-1}(DR^\bullet\left(C_X^\infty\right))=H^{\dim X-1}\left(X,\mathbb{C}\right),$$

which we denote by $G(\nu,\varphi)$. We define the pairing between ν and φ by

$$B_\gamma\left(\nu,\varphi\right)=\int\limits_\gamma G(\nu,\varphi).$$

0.4.2. *The Cauchy theorem revisited.* Reconsidering the Cauchy-Riemann D-module M_{CR}. A direct computation reveals that M_{CR} is principally self dual, that is

$$M_{CR}^\vee \simeq M_{CR},$$

therefore, the Green class $G_{_{CR}}=G_{M_{CR}}$ lies in $H^1\left(DR^\bullet\left(M_{CR}\bigotimes_{\mathcal{O}} M_{CR}\right)\right)$. A direct computation reveals that G_{CR} is represented by the following chain

$$G_{CR}=e\otimes e\otimes dz, \tag{0.10}$$

where $e\in M_{CR}$ is the standard generator. Let us verify that (0.10) is a closed chain. Applying the de-Rham differential we obtain

$$\begin{aligned} d\left(G_{CR}\right) &= \partial_{\overline{z}}e\otimes e\otimes d\overline{z}\wedge dz+e\otimes\partial_{\overline{z}}e\otimes d\overline{z}\wedge dz\\ &\quad+\partial_z e\otimes e\otimes dz\wedge dz+e\otimes\partial_z e\otimes dz\wedge dz\\ &= 0+0+0+0=0,\end{aligned}$$

noting that $\partial_{\overline{z}}e=0$ is the relation defining the module M_{CR}. Consider the homology class $\gamma\in H_1(\mathbb{R}^2\backslash w,\mathbb{C})$ which is represented by the closed curve $\gamma\left(t\right)=e^{2\pi it}+w$. Given solutions

$$\begin{aligned} \nu &\in \mathsf{Hom}_{D_{\mathbb{R}^2}}\left(M_{CR}, C_{\mathbb{R}^2}^\infty\right),\\ \varphi &\in \mathsf{Hom}_{D_{\mathbb{R}^2}}\left(M_{CR}^\vee, C_{\mathbb{R}^2\backslash w}^\infty\right),\end{aligned}$$

the pairing $B_\gamma\left(\nu,\varphi\right)$ is given by

$$B_\gamma\left(\nu,\varphi\right)=\int\limits_\gamma G(\nu,\varphi)=\int\limits_\gamma \varphi\left(e\right)\nu\left(e\right)dz. \tag{0.11}$$

If we identify ν and φ with the holomorphic functions $f=\nu(1)$ and $g=\varphi(1)$ respectively then $B_\gamma(f,g)$ takes the form

$$B_\gamma(f,g)=\int\limits_\gamma f(z)\,g(z)\,dz$$

In order to prove the Cauchy Theorem (Theorem 4) we have to exhibit a vector $\Delta_w \in \mathsf{Hom}_{D_{\mathbb{R}^2}}\left(M_{CR}^\vee, C^\infty_{\mathbb{R}^2\setminus w}\right)$ such that

$$\delta_w(f)=B_\gamma(f,\Delta_w),$$

for every $f\in\mathcal{O}$. The answer in this case is very simple, take Δ_w to be the unique solution satisfying

$$\Delta_w(e)=\tfrac{1}{2\pi i(z-w)}.$$

Remark 5. *The reader might think that the pairing B_γ always admits a simple form like (0.11). This is not the case! already for a general enough differential operator P the green class G_{M_P} might be quite complicated and as a result B_γ takes a complicated form as well.*

0.5. **The strong Stone-von Neumann property.** Let us summarize what we learned so far from our toy example. The space $\mathcal{O}$ of holomorphic functions in one variable can be characterized as the space

$$Hom_{D_{\mathbb{R}^2}}(M_{CR}, C^\infty_{\mathbb{R}^2}),$$

of solutions of an algebraic $D_{\mathbb{R}^2}$-module M_{CR}. In this framework, using duality in the category of D-modules, we were able to exhibit various subspaces $\mathcal{O}_w^\vee\subset\mathcal{O}^*$ of algebraic functionals (0.5), defined as

$$\mathcal{O}_w^\vee = Hom_{D_{\mathbb{R}^2}}\left(M_{CR}^\vee, C^\infty_{\mathbb{R}^2\smallsetminus w}\right),$$

where $M_{CR}^\vee$ is the dual of the D-module M_{CR} (more precisely, the Verdier dual $\mathbb{D}(M_{CR})$ is isomorphic to $M_{CR}^\vee[1]$). The pairing between $\mathcal{O}$ and $\mathcal{O}_w^\vee$ turn out to depend on a choice of a non-trivial homology class $\gamma\in H_1(\mathbb{R}^2\smallsetminus w,\mathbb{C})$. Given such a class, for any pair of solutions $\nu\in\mathcal{O}$ and $\varphi\in\mathcal{O}_w^\vee$ the pairing $B_\gamma(\nu,\varphi)$ is given by

$$B_\gamma(\nu,\varphi)=\int\limits_\gamma G(\nu,\varphi),$$

We concluded that from this point of view, the Cauchy Theorem becomes the assertion that the functional $\delta_w\in\mathcal{O}^*$ is algebraic, namely it can be presented by

$$\delta_w(f)=B_\gamma(f,\Delta_w),$$

where Δ_w is a fixed vector in the algebraic space $\mathcal{O}_w^\vee$ and f runs inside $\mathcal{O}$.

0.5.1. *The Schwartz space revisited.* The strong S-vN property asserts that much like $\mathcal{O}$, also the Schwartz space $S\left(\mathbb{R}^N\right)$ can be characterized as a space of solutions of an algebraic D-module which we denote[5] by M^κ, namely

$$S\left(\mathbb{R}^N\right)=\mathsf{Hom}_{D_X}(M^\kappa, C^\infty_X).$$

[5]The superscript κ is related to the central weight of the Heisenberg repreentation which silently governs the picture.

However, In order to realize this assertion, a radical change of perspective should take place. Unlike the situation for $\mathcal{O}$ where the D-module was defined on the same space of parameters $\mathbb{R}^2$, for Schwartz functions the space X is not simply $\mathbb{R}^N$ as might be expected, but should be taken to be a much larger space of parameters. We will proceed to give an explanation of the last assertion.
Canonical model of the Heisenberg representation. As was mentioned before, the Schwartz space $S\left(\mathbb{R}^N\right)$ is identified with the space $\mathcal{H}^\kappa$ of smooth vectors in the Heisenberg representation. One of the most important attributes of the Heisenberg representation is that it admits a multitude of different models. These models appear in families and a particular family of such models is associated with maximal isotropic (Lagrangian) spaces in V. In more details given a choice of a Lagrangian subspace L in V there exists a model of the Heisenberg representation which we denote by

$$(\pi_L^\kappa, H, \mathcal{H}_L^\kappa),$$

At this point it is sufficient for us to know only that the space $\mathcal{H}_L^\kappa$ is a subspace of functions in $C^\infty(H)$. We will use the notation $Lag = Lag(V)$ to denote the Lagrangian Grassmanian associated to V. All the models $\mathcal{H}_L^\kappa$ are just different realizations of the same object, more precisely all of these models are equivalent as representations of H which is a consequence of Theorem 1. This means that for every pair of Lagrangians $L, M \in Lag$ there exists an isomorphism of H-representations (intertwiner)

$$F_{M,L} \in \mathsf{Hom}_H(\mathcal{H}_L^\kappa, \mathcal{H}_M^\kappa).$$

We are now ready to formulate a naive interpretation of the strong Sv-N property.

Strong S-vN property: For every pair $(M, L) \in Lag \times Lag$ there exists a canonical choice of an intertwiner $F_{M,L}$.

Fixing this system of canonical intertwiners, any smooth vector $v \in \mathcal{H}^\kappa$ in the Heisenberg representation can be presented as system of vectors

$$(v_L \in \mathcal{H}_L^\kappa : L \in Lag),$$

satisfying the compatibility condition

$$v_M = F_{M,L}(v_L), \tag{0.12}$$

for every $M, L \in Lag$.

Now putting this idea in a more formal language, we can say that the vector v can be considered as a single function in $C^\infty(Lag \times H)$ satisfying a system P of differential equations which encodes the compatibility condition (0.12).

The previous discussion suggests that the right choice of the parameter space X is

$$X = Lag \times H.$$

Now we can use our acquired language of D-modules and claim that there exists a D_X-module

$$M^\kappa = M_P,$$

such that the space $\mathcal{H}^\kappa$ of smooth vectors can be identified with the space of solutions

$$\mathsf{Hom}_{D_X}(M^\kappa, C_X^\infty)$$

We are now ready to spell out an "almost" precise formulation of the strong S-vN property.

Theorem 5 (strong S-vN property)**.** *There exists a D_X-module M^κ so that the Schwartz space $S\left(\mathbb{R}^N\right)$ is naturally identified as an H-representation*[6] *with*

$$\mathsf{Hom}_{D_X}\left(M^\kappa, C_X^\infty\right).$$

The D-module M^κ will be called the *Weil D-module.*

Remark 6. *In the case $\dim V = 2$, the module M^κ, under appropriate trivializations, corresponds to the heat differential operator. Therefore, roughly speaking, the Schwartz space $S(\mathbb{R})$ can be identified with solutions of the heat equation on the one dimensional real projective line.*

Why just "almost" precise. We use the term "almost", since the actual statement involves some technicalities. The most prominent one is that M^κ is not an honest D_X-module but is a module over slightly more general differential algebra which is associated to the determinant line bundle on Lag. For this reason we spend a large portion of Section 1 to explain the formalism of such kind of differential algebras.

Intuitively, this technicality is related to the fact that the system $F_{M,L}$ of intertwiners do not form a flat connection but a projective one, namely they do not satisfy the multiplicativity condition

$$F_{N,M} \circ F_{M,L} = F_{N,L},$$

for every triple of Lagrangian subspaces $M, N, L \in Lag$. Instead [LV], there exists a canonical function

$$c : Lag \times Lag \times Lag \longrightarrow \mathbb{C},$$

so that

$$F_{N,M} \circ F_{M,L} = c(L,N,M) F_{N,L}.$$

The function c is intimately related to the Maslov index.

0.5.2. *Algebraic Hilbertian structure.* The Weil D-module M^κ enjoys many desired properties, in particular it is Cohen-Macaulay. In fact its dual can be described explicitly

$$\mathbb{D}\left(M^\kappa\right) \simeq M^{-\kappa}[N].$$

where $M^{-\kappa}$ is the module associated to the Heisenberg representation of the opposite central weight. As a consequence of this last statement, using the general construction (0.7) we can obtain a novel formula for the natural pairing between the corresponding spaces of smooth vectors $\mathcal{H}^\kappa$ and $\mathcal{H}^{-\kappa}$. In more details, if we choose a non-trivial class $\gamma \in H_N\left(Lag, \mathbb{C}\right)$ we can define a natural pairing

$$B_\gamma : \mathcal{H}^\kappa \times \mathcal{H}^{-\kappa} \longrightarrow \mathbb{C}. \tag{0.13}$$

If one assumes in addition that the central weight κ is purely imaginary, that is $\kappa \in i\mathbb{R}$ then it can be shown that complex conjugation yields an anti-linear isomorphism

$$\overline{(\cdot)} : \mathcal{H}^\kappa \longrightarrow \mathcal{H}^{-\kappa},$$

therefore one can define an H-invariant Hermitian product on $\mathcal{H}^\kappa$ by

$$\left\langle \nu, \varphi \right\rangle_\gamma \triangleq B_\gamma(\nu, \overline{\varphi}) = \int_\gamma G(\nu, \overline{\varphi}).$$

[6]The H-action on $\mathsf{Hom}_{D_X}\left(M^\kappa, C_X^\infty\right)$ is explained below.

Since $\mathcal{H}^{\kappa}$ admits a unique (up to scalar multiplication) H-invariant Hermitian product, it implies that $\langle \cdot, \cdot \rangle_{\gamma}$ is proportional to the standard Hermitian product given by integration with respect to the Lebegues measure on $\mathbb{R}^N$, namely

$$\int_{\mathbb{R}^N} \nu(x)\overline{\varphi}(x)dx = \int_{\gamma} G(\nu, \overline{\varphi}). \tag{0.14}$$

The reader should note that while the left side of (0.14) is given by integration of a function on the non-compact domain $\mathbb{R}^N$, the right side is given by integration of a closed N-form on a compact homology class sitting in a completely different space of parameters, namely the Lagrangian Grassmanian Lag.

0.5.3. *The Heisenberg representation revisited.* The Heisenberg representation, at least on the infinitesimal level, is algebraic. The precise meaning of the last assertion is that the action $d\pi^{\kappa}$ of the Heisenberg Lie algebra $\mathfrak{h}$ on the Hilbertian space

$$\mathcal{H}^{\kappa} = \mathsf{Sol}(M^{\kappa}, C_X^{\infty}),$$

is encoded in the algebraic structure of the D_X-module M^{κ}. In more details, there exists a map of algebras

$$\Pi^{\kappa} : \mathcal{U}(\mathfrak{h})^{\circ} \longrightarrow \mathsf{Hom}_{D_X}(M^{\kappa}, M^{\kappa}),$$

where $\mathcal{U}(\mathfrak{h})^{\circ}$ is the universal enveloping algebra of $\mathfrak{h}$ considered with the opposite multiplication. The action $d\pi^{\kappa}$ can be defined in terms of the map Π^{κ} as follows. Given an element $\xi \in \mathfrak{h}$ and a vector $\varphi \in \mathsf{Sol}(M^{\kappa}, C_X^{\infty})$, we let

$$d\pi^{\kappa}(\xi)\varphi \triangleq \varphi \circ \Pi^{\kappa}(\xi).$$

The fact that this action preserves the Hermitian product is principally a tautology. Since the Hermitian product is defined using the canonical class

$$G_{M^{\kappa}} \in H^0\left(DR^{\bullet}\left(\mathbb{D}\left(M^{\kappa}\right) \bigotimes_{\mathcal{O}} M^{\kappa}\right)\right),$$

this implies that any automorphism θ of the module M^{κ} fixes $G_{M^{\kappa}}$, namely

$$\theta(G_{M^{\kappa}}) = G_{M^{\kappa}}. \tag{0.15}$$

Therefore if we consider endomorphisms of the form $\Pi^{\kappa}(\xi)$ as infinitesimal generators of such automorphism, then (0.15) implies that

$$\Pi^{\kappa}(\xi)(G_{M^{\kappa}}) = 0,$$

which in turns yields the result.

0.5.4. *The Weil representation revisited.* A similar argument works for the Weil representation as well. That is, the infinitesimal Weil representation is encoded in the algebraic structure of the D_X-module M^{κ}. The mechanism is the same. There exists a map of algebras

$$\Theta^{\kappa} : \mathcal{U}(\mathfrak{sp})^{\circ} \longrightarrow \mathsf{Hom}_{D_X}(M^{\kappa}, M^{\kappa}),$$

where $\mathcal{U}(\mathfrak{sp})^{\circ}$ is the universal enveloping algebra of $\mathfrak{sp}$ considered with the opposite multiplication. Using the map Θ^{κ}, the action $d\rho$ of $\mathfrak{sp}$ on $\mathcal{H}^{\kappa}$ is defined by

$$d\rho^{\kappa}(\xi)\varphi \triangleq \varphi \circ \Theta^{\kappa}(\xi).$$

for every element $\xi \in \mathfrak{sp}$ and a vector $\varphi \in \mathsf{Sol}(M^{\kappa}, C_X^{\infty})$. Again the fact that this action preserves the Hermitian product follows from similar arguments as in the Heisenberg case.

The difference from the previous situation of the Heisenberg action is that in the case of the symplectic group, the action of the Lie algebra can be "exponentiated" to an action of the Lie group Sp. Formally, this is the statement that M^{κ} admits a (weak) Sp-equivariant structure, which roughly means that there exists an homomorphism of groups

$$\theta^{\kappa} : Sp \to Aut_{\mathbb{C}}(M^{\kappa}). \tag{0.16}$$

such that

$$d\theta^{\kappa}\left(\xi\right) - \xi^{\#} = \Theta^{\kappa}\left(\xi\right),$$

for every $\xi \in \mathfrak{sp}$. Here $\xi^{\#}$ is the vector field on Lag associated to ξ.

Remark 7. *Looking at (0.16) the reader should ask (and rightly so) where do the metaplectic sign comes from. The answer is hidden in the small technicality issue that we mentioned after stating the strong S-vN property. Roughly, the explanation is that the module M^{κ} is equipped with a canonical projective connection. In order to consider solutions, this projector must be linearized first (make it flat). This principally can be done in two ways. The upshot is that non of these two linearization is preserved by the group Sp. In fact, Sp switches between the two.*

0.6. **Structure of the paper.** Apart from the introduction, this paper is logically divided into four sections.

- In Section 1, we recall some facts and notations from the algebraic theory of D-modules. Specifically, we introduce the notion of a differential algebra and its associated category of coherent modules. We mainly concentrate our attention on differential algebras associated with infinitesimal symmetries of principal vector bundles. This particular type of differential algebras is used in order to give a rigorous definition to the classical notion of projective connection. We discuss the notion of a target module and the associated spaces of solutions. We discuss the Verdier duality functor both in the context of usual D-modules and also in the context of more general differential algebras. We define the Green class of a D-module, which generalizes the classical notion of Green form of a partial differential operator. Using the Green class we are able to define canonical pairings between solution spaces of a D-module and its Verdier dual. Finally we discuss (weak) equivariance structures and associated representations on the space of solutions.
- In Section 2, the basic constructions are introduced and the main result of this paper is formulated - the strong Stone-von Neumann property of the Heisenberg representation. We begin by introducing various Differential algebras. Then an infinite dimensional holomorphic vector bundle $\mathcal{W}^{\kappa}$ is constructed, a-lá Deligne, on the Lagrangian Grassmanian Lag. The Strong Sv-N property is equivalent to the existence of a canonical projective connection on $\mathcal{W}^{\kappa}$, which formally is given by an action of a differential algebra which is associated to the determinant line bundle. This form of the strong Sv-N property is not yet satisfactory. Next step we take is to recast the construction in the language of algebraic D-modules. In this step we introduce an (algebraic) D-module $\mathcal{M}^{\kappa}$ on $Lag \times H$, which finally puts the Sv-N property on algebraic grounds. A technical advantage of the algebraic setting is the ability to use powerful techniques from homological algebra. We study various solution spaces associated to $\mathcal{M}^{\kappa}$. The main theorem of this

section concerns the explicit description of the Verdier dual of $\mathcal{M}^{\kappa}$. Finally we show that $\mathcal{M}^{\kappa}$ is equipped with a natural Sp-equivariance structure.
- In Section 3, several applications of the strong Sv-N property in its D-module theoretic formulation are established. First application concerns the existence of a canonical pairing between various function spaces, this establish, in particular, an affirmative answer to a question of Deligne [DE]. Second application concerns the construction of the Weil representation of the real symplectic group, here we show that the strong Sv-N property directly implies the metaplectic sign.
- In Appendix A, proofs of all the statements that appear in the paper are given.

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1. Preliminaries from the theory of D-modules

We need to recall some facts and notations about (algebraic) D-modules and also about slightly more general differential algebras and their categories of modules. Let X be a smooth algebraic variety over $\mathbb{C}$. Throughout the paper we will make use of the following convention. Calligraphic letters are used to denote sheaves of vector spaces/algebras/modules (for example $\mathcal{D}_X$ denotes the sheaf of algebras of differential operators on X). Usual uppercase letters are used to denote vector spaces/algebras/modules (for example $D_X = \Gamma(X, \mathcal{D}_X)$ is the algebra of global linear differential operators). Most of the material presented in this section about usual D-modules appears in one form or another in [BO], [KA] and [BE]. The material about general differential algebras is partly taken from [BB]. The notion of Green class of a usual D-module is discussed thoroughly in [GO].

1.1. Differential algebras.

1.1.1. *Standard setting.* Let $\mathcal{D}_X$ denote the sheaf of algebras of linear differential operators on X or equivalently saying $\mathcal{D}_X = \mathsf{Diff}(\mathcal{O}_X)$ is the sheaf of all differential endomorphisms of the structure sheaf.

Example 1. *Assume $X = \mathbb{A}^N$. In this case the algebra $D_{\mathbb{A}^N} = \Gamma(X, \mathcal{D}_{\mathbb{A}^N})$ is generated by*

$$x_1, .., x_N, \partial_1, .., \partial_N,$$

subject to the relations

$$\begin{aligned} [x_i, x_j] &= 0,\ [\partial_i, \partial_j] = 0, \\ [\partial_i, x_j] &= \delta_{ij.} \end{aligned}$$

The algebra $D_{\mathbb{A}^N}$ is called the Weyl algebra.

An important example is the D-module associated to a differential equation or more generally to a system of differential equations

Example 2. *Let* $P = \sum_{\alpha} a_\alpha(x)\partial^\alpha \in D_{\mathbb{A}^N}$ *be a differential operator. We use the standard convention where the indices* α *are* N *tuples* $\alpha = (\alpha_1, .., \alpha_N)$ *and* ∂^α *stands for* $\partial_{x_1}^{\alpha_1} \cdot .. \cdot \partial_{x_n}^{\alpha n}$. *Define*

$$\mathcal{M}_P = \mathcal{D}_{\mathbb{A}^N}/\mathcal{D}_{\mathbb{A}^N}P.$$

More generally, given a matrix of differential operators $(P_{ij})_{1\leq i\leq m; 1\leq j\leq n}$, *define*

$$\mathcal{M}_{(P_{ij})} = \mathsf{coker}\left((P_{ij}) : \bigoplus_{i=1}^{m} \mathcal{D}_{\mathbb{A}^N} \longrightarrow \bigoplus_{i=1}^{n} \mathcal{D}_{\mathbb{A}^N}\right).$$

We denote by $\mathsf{Coh}(\mathcal{D}_X)$ $({}^R\mathsf{Coh}(\mathcal{D}_X)$) the category of coherent sheaves of left (right) $\mathcal{D}_X$-modules and by $\mathsf{Dcoh}(\mathcal{D}_X)$ $({}^R\mathsf{Dcoh}(\mathcal{D}_X)$) the associated derived category.

Remark 8. *In case* $\mathcal{M}$ *is a vector bundle then an action of* $\mathcal{D}_X$ *is equivalent to the classical notion of a flat connection on* $\mathcal{M}$.

Solution spaces. Let $\mathcal{M} \in \mathsf{Coh}(\mathcal{D}_X)$ or more generally an object in the derived category $\mathsf{DCoh}(\mathcal{D}_X)$. The reader should note that $\mathcal{M}$ can be thought of as a generalization of the classical notion of system of (algebraic) linear partial differential equations (In fact a D-module with a system of generators stands as the precise generalization but we will not be so pedantic here). Given another $\mathcal{D}_X$-module $\mathcal{F}$, not necessarily quasicoherent (usually $\mathcal{F}$ is taken to be a module of an analytic nature) one can consider the space of *solutions* of $\mathcal{M}$ in $\mathcal{F}$

$$\mathsf{Sol}(\mathcal{M}, \mathcal{F}) \triangleq \mathsf{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{F}),$$

which generalizes the classical notion of solution of a linear partial differential equation. The module $\mathcal{F}$ is sometimes called the *target module*. For example, if we consider a module of the form $\mathcal{M} = \mathcal{M}_P$ where $P \in D_X$ is a linear algebraic differential operators (see Example 2) and take $\mathcal{F} = \mathcal{O}_X^{an}$ then $\mathsf{Sol}(\mathcal{M}_P, \mathcal{F})$ can be identified with the space $\mathsf{Sol}(\mathsf{P}, \mathcal{O}_X^{an})$ of analytic solutions of the differential equation $Pf = 0$ via

$$\varphi \in \mathsf{Sol}(\mathcal{M}, \mathcal{F}) \mapsto \varphi(1),$$

where $1 \in M_P = \Gamma(X, \mathcal{M}_P)$ is the standard generator of $\mathcal{M}_P$. The D-module point of view immediately leads to an interesting generalization of *higher solution* spaces

$$\mathsf{Sol}^i(\mathcal{M}, \mathcal{F}) \triangleq R^i\mathsf{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{F}).$$

In this paper we consider two types of target modules.

- Holomorphic type. Let $\mathcal{V}$ be an analytic vector bundle on X^{an} equipped with a flat connection, take

  $$\mathcal{F}^{an} = \mathcal{V},$$

  considered as a sheaf on the Zarisky site of X.
- C^∞ type. This type is associated with a choice of a real structure on X. Equivalently, assume X is a smooth scheme over $\mathbb{R}$. Let us assume in addition that $X(\mathbb{R})$, the space of real points of X, is a smooth manifold of dimension $\dim X(\mathbb{R}) = \dim X$. We denote by ex the morphism of schemes

  $$\mathsf{ex} : X(\mathbb{C}) \longrightarrow X,$$

obtained by extension of scalars. Let $\mathcal{V}$ be a complex vector bundle on $X(\mathbb{R})$ equipped with a flat connection, $\mathcal{V}$ can be considered as a sheaf on the Zarisky site of X. Define

$$\mathcal{F}^{\infty}=\mathsf{ex}^{*}\mathcal{V}=\mathcal{O}_{X(\mathbb{C})}\bigotimes_{\mathsf{ex}^{\cdot}\mathcal{O}_{X}}\mathsf{ex}^{\cdot}\mathcal{V}.$$

The sheaf $\mathcal{F}^{\infty}$ is naturally equipped with a $\mathcal{D}_{X(\mathbb{C})}$-action.

Example 3. *Consider the real scheme $X=\mathbb{A}^2_{\mathbb{R}}$. In this case $X(\mathbb{C})=\mathbb{C}^2$ and $X(\mathbb{R})=\mathbb{R}^2$. Let $\mathcal{M}_{CR}\in\mathsf{Coh}(\mathcal{D}_{X(\mathbb{C})})$ be the D-module associated with the Cauchy-Riemann equation, that is*

$$\mathcal{M}_{CR}=\mathcal{D}_{\mathbb{C}^2}/\mathcal{D}_{\mathbb{C}^2}\left(\frac{\partial_x+i\partial_y}{2}\right).$$

Let $\mathcal{F}^{\infty}=\mathsf{ex}^{*}C^{\infty}_{X(\mathbb{R})}$ then $\mathsf{Sol}(\mathcal{M}_{CR},\mathcal{F}^{\infty})$ is naturally identified with the space of holomorphic functions on $\mathbb{R}^2$.

1.1.2. *General setting.* We begin with some general definitions (taken from [BB]).

Definition 1 (Differential bimodule)**.** *A differential $\mathcal{O}_X$-bimodule on X is a quasicoherent sheaf on $X\times X$ supported on the diagonal $\Delta\subset X\times X$.*

Definition 2 (Differential algebra)**.** *An $\mathcal{O}_X$-differential algebra, or simply a D-algebra on X is a sheaf $\mathcal{D}$ of associative algebras on X equipped with a morphism of algebras $i:\mathcal{O}_X\longrightarrow\mathcal{D}$ such that $\mathcal{D}$ is a differential $\mathcal{O}_X$-bimodule.*

It is sometimes convenient to describe a differential algebra as the universal enveloping algebra of a Lie algebroid.

Definition 3. *A Lie algebroid $\mathcal{T}$ on X is a quasicoherent $\mathcal{O}_X$-module equipped with a morphism of $\mathcal{O}_X$-modules $\sigma:\mathcal{T}\longrightarrow Tan_X$ (where Tan_X is the tangent sheaf on X) and a $\mathbb{C}$-linear pairing $[\cdot,\cdot]:\mathcal{T}\bigotimes_{\mathbb{C}}\mathcal{T}\longrightarrow\mathcal{T}$ such that*

- *$[\cdot,\cdot]$ is a Lie algebra bracket and σ commutes with the brackets.*
- *for $\xi_1,\xi_2\in\mathcal{T}$, and $f\in\mathcal{O}_X$ one has $[\xi_1,f\xi_2]=f[\xi_1,\xi_2]+\sigma(\xi_1)(f)\xi_2$*

Given a Lie algebroid $\mathcal{T}$, its *universal enveloping* D-algebra $\mathcal{U}(\mathcal{T})$ is a sheaf of algebras equipped with the morphisms of sheaves $i:\mathcal{O}_X\longrightarrow\mathcal{U}(\mathcal{T})$, $i_{\mathcal{T}}:\mathcal{T}\longrightarrow\mathcal{U}(\mathcal{T})$ and it is generated as an algebra by the images of these morphisms subject to the following relations

- i is a morphism of algebras.
- $i_{\mathcal{T}}$ is a morphism of Lie algebras.
- For $f\in\mathcal{O}_X$, $\xi\in\mathcal{T}$ one has $i_{\mathcal{T}}(f\xi)=i(f)i_{\mathcal{T}}(\xi)$ and $[i_{\mathcal{T}}(\xi),i(f)]=i(\sigma(\xi)f)$.

The relation between a Lie algebroid and its universal enveloping D-algebra is similar to the relation between a Lie algebra and its associated universal enveloping algebra.

Example 4. *The most standard example of a Lie algebroid is the tangent sheaf Tan_X, where $[\cdot,\cdot]$ is given by the commutator operation between vector fields and σ is the identity. The sheaf of linear differential operators $\mathcal{D}_X$ is the universal enveloping algebra of Tan_X, namely we have*

$$\mathcal{D}_X=\mathcal{U}(Tan_X).$$

Example 5. *Assume X is equipped with an action $\alpha : G\times X \longrightarrow X$ of an algebraic group G. The action α yields an anti homomorphism of Lie algebras*

$$d\alpha : \mathfrak{g} \longrightarrow \Gamma\left(X, Tan_X\right),$$

sending an element $\xi \in \mathfrak{g}$ to the vector field $\xi^{\#} \triangleq d\alpha\left(\xi\right)$.The sheaf $\mathfrak{g}_X = \mathfrak{g}\otimes\mathcal{O}_X$ is equipped with a natural structure of a Lie algebroid as follows

- *The commutator is given by*

$$[\xi \otimes f, \eta \otimes g] = [\xi, \eta] \otimes f \cdot g - \eta \otimes f \cdot \xi^{\#}\left(g\right) + \xi \otimes \eta^{\#}\left(f\right) \cdot g.$$

- *The morphism $\sigma : \mathfrak{g}_X \to Tan_X$ is given by*

$$\sigma\left(\xi \otimes f\right) = -f \cdot \xi^{\#}.$$

Given a differential algebra $\mathcal{D}$, we denote by $\mathsf{Coh}(\mathcal{D})$ $\ ({}^{R}\mathsf{Coh}(\mathcal{D}))$ the category of left (right) coherent $\mathcal{D}$-modules and by $\mathsf{DCoh}(\mathcal{D})$ $\ ({}^{R}\mathsf{DCoh}(\mathcal{D}))$ the associated derived category.

D-algebras associated with principal bundles. In this paper we will be interested with a particular type of differential algebras which are associated with infinitesimal symmetries of principal bundles. More precisely, let G be a connected algebraic group. Let $P \xrightarrow{\pi} X$ be a principal right G-bundle. Let $\mathcal{T}_P$ be the Lie algebroid of infinitesimal symmetries of P. Formally $\mathcal{T}_P$ is given by

$$\mathcal{T}_P = \pi_*(Tan_P)^G,$$

where Tan_P is the tangent sheaf of P. the operation of push forward of vector fields supplies the morphism $\sigma : \mathcal{T}_P \to Tan_X$. The kernel $\mathsf{ker}(\sigma)$ is denoted by $\mathcal{T}_P^{\mathsf{v}}$ and consists of G-invariant vertical vector fields on P. The center $Z(\mathcal{T}_P)$ can be canonically identified with the constant sheaf $Z(\mathfrak{g})$. The universal enveloping algebra $\mathcal{U}(\mathcal{T}_P)$ is denoted by $\mathcal{D}_P$. Its center $Z(\mathcal{D}_P)$ is canonically identified with the constant sheaf associated with the symmetric algebra $S^{\cdot}(Z(\mathfrak{g}))$.

Monodromic algebras. The algebra $\mathcal{D}_P$ can be specialized with respect to characters of its center. More precisely, given a character $\lambda \in Z(\mathfrak{g})^*$, we can define the quotient algebra

$$\mathcal{D}_P^{\lambda} = \mathcal{D}_P \bigotimes_{S^{\cdot}(Z(\mathfrak{g}))} \mathbb{C}_{\lambda}.$$

Algebras of the form $\mathcal{D}_P^{\lambda}$ are called *monodromic* differential algebras. An object $\mathcal{M} \in \mathsf{Coh}(\mathcal{D}_P^{\lambda})$ is called λ-*monodromic.*

Commutative example. A particularly important example for us is when the group G is commutative, namely $G \simeq \mathbb{G}_m^r \times \mathbb{G}_a^s$. In this case $Z(\mathcal{T}_P) = \mathfrak{g}$ and therefore $Z(\mathcal{D}_P) = S^{\cdot}(\mathfrak{g})$. Fixing a character $\lambda \in \mathfrak{g}^*$, the monodromic algebra is given by

$$\mathcal{D}_P^{\lambda} = \mathcal{D}_P \otimes_{S^{\cdot}(g)} \mathbb{C}_{\lambda}.$$

A vector bundle with a projective connection is by definition a vector bundle $\mathcal{M}$ equipped with a λ-monodromic structure, that is a $\mathcal{D}_P^{\lambda}$-action.

Solution spaces. Formally, the notion of solution space of a $\mathcal{D}_P$-module is similar to the corresponding notion in the standard setting, however, a classical analogy in terms of solutions of differential equations is less naive. Let $M \in \mathsf{Coh}(D_P)$ or more generally an object in the derived category $\mathsf{DCoh}(\mathcal{D}_P)$ and $\mathcal{F}$ be another $\mathcal{D}_P$-module called the *target module*. The space of *solutions* of $\mathcal{M}$ in $\mathcal{F}$ is

$$\mathsf{Sol}(\mathcal{M}, \mathcal{F}) \triangleq \mathsf{Hom}_{\mathcal{D}_P}(\mathcal{M}, \mathcal{F}),$$

and more generally, the space of *higher solutions* is

$$\mathsf{Sol}^i(\mathcal{M},\mathcal{F}) \triangleq R^i\mathsf{Hom}_{\mathcal{D}_P}(\mathcal{M},\mathcal{F}).$$

As before, we consider two types of target modules.

- Holomorphic type. Let (ρ, G, V) be an algebraic representation of the local symmetry group of P, and let $\mathcal{V} = P^{an} \times_{G^{an}} V$ be the associated analytic vector bundle. We take
$$\mathcal{F}^{an} = \mathcal{V},$$
considered as a sheaf on the Zarisky site of X. Clearly, $\mathcal{F}$ is a $\mathcal{D}_P$-module.
- C^∞ type. Assume X, P and G are defined over $\mathbb{R}$ and moreover the corresponding spaces of real points are smooth manifolds of dimensions equal to the krull dimension of the corresponding schemes. In addition, assume the projection map $\pi : P \longrightarrow X$ is defined over $\mathbb{R}$. Let $(\rho, G(\mathbb{R}), V)$ be a finite dimensional (complex) representation of the Lie group $G(\mathbb{R})$ and let $\mathcal{V} = P(\mathbb{R}) \times_{G(\mathbb{R})} V$ be the associated (complex) vector bundle. The vector bundle $\mathcal{V}$ is naturally equipped with a $\mathcal{D}_{P(\mathbb{R})}$-action where $\mathcal{D}_{P(\mathbb{R})}$ is the universal enveloping algebra of the Lie algebroid
$$\mathcal{T}_{P(\mathbb{R})} = \mathbb{C} \bigotimes_{\mathbb{R}} \pi_*(Tan_{P(\mathbb{R})})^{G(\mathbb{R})}.$$
Considering $\mathcal{V}$ as a sheaf on the Zarisky site of X, define
$$\mathcal{F}^\infty = \mathsf{ex}^*\mathcal{V} = \mathcal{O}_{X(\mathbb{C})} \bigotimes_{\mathsf{ex}^{\cdot}\mathcal{O}_X} \mathsf{ex}^{\cdot}\mathcal{V}.$$
The sheaf $\mathcal{F}^\infty$ is naturally equipped with a $\mathcal{D}_{P(\mathbb{C})}$-action.

1.2. **Verdier Duality.** Existence of a duality functor for D-modules is a most interesting and a non-trivial phenomenon. Let us first recall its construction in the standard setting of usual D-modules.

1.2.1. *Standard setting.* The category $\mathsf{DCoh}(\mathcal{D}_X)$ admits an anti-equivalence (called *Verdier duality)*

$$\mathbb{D} : \mathsf{DCoh}(\mathcal{D}_X) \longrightarrow \mathsf{DCoh}(\mathcal{D}_X).$$

defined by $\mathbb{D}(\mathcal{M}) = R\mathcal{H}om(\mathcal{M}, \mathcal{D}_X^{\Omega^{-\mathsf{top}}})[\dim X]$, where

$$\mathcal{D}_X^{\Omega^{-\mathsf{top}}} = \mathcal{D}_X \bigotimes_{\mathcal{O}_X} \Omega_X^{-\mathsf{top}} = \mathcal{D}_X \bigotimes_{\mathcal{O}_X} \bigwedge^{\dim X} Tan_X,$$

is the dualizing module. In more details, $\mathcal{D}_X^{\Omega^{-\mathsf{top}}}$ is a bimodule admitting two commuting left actions of $\mathcal{D}_X$. The first action is given by the standard left action of $\mathcal{D}_X$ on itself and the second is defined as follows. Given an element $d \otimes \alpha \in \mathcal{D}_X^{\Omega^{-\mathsf{top}}}$ we have

- Action of functions. for every $f \in \mathcal{O}_X$

(1.1) $$f \rhd d \otimes \alpha = df \otimes \alpha = d \otimes f\alpha\ .$$

- Action of vector fields. for every $\xi \in \mathcal{T}_X$

(1.2) $$\xi \rhd d \otimes \alpha = d \otimes [\xi, \alpha] - d\xi \otimes \alpha.$$

The verification that the above formulas extend to a left action of $\mathcal{D}_X$ is left to the reader. Another way to think about $\mathbb{D}$ is as the composition of two functors

$$\mathbb{D} = \left(\cdot \bigotimes_{\mathcal{O}} \bigwedge^{\dim X} Tan_X\right) \circ \widetilde{\mathbb{D}},$$

where

$$\widetilde{\mathbb{D}} : \mathsf{DCoh}(\mathcal{D}_X) \longrightarrow {}^R\mathsf{DCoh}(\mathcal{D}_X)$$

is the an anti-equivalence sending a left $\mathcal{D}_X$-module $\mathcal{M} \in \mathsf{DCoh}(\mathcal{D}_X)$ to the right $\mathcal{D}_X$-module $\widetilde{\mathbb{D}}(\mathcal{M}) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)[\dim X]$ and

$$\left(\cdot \bigotimes_{\mathcal{O}} \bigwedge^{\dim X} Tan_X\right) : \ {}^R\mathsf{DCoh}(\mathcal{D}_X) \longrightarrow \mathsf{DCoh}(\mathcal{D}_X),$$

is the standard equivalence between the categories of right and left $\mathcal{D}_X$-modules, i.e., given a right module $\mathcal{M} \in {}^R\mathsf{Coh}(\mathcal{D}_X)$ the left action on $\mathcal{M} \bigotimes_{\mathcal{O}} \bigwedge^{\dim X} Tan_X$ is given by (1.1) and (1.2).

Cohen-Macaulay D-modules. It is most common that starting from a coherent module, its dual is no longer an honest module but is a complex with non-trivial cohomologies in various degrees. Yet, for particular type of modules the dual is itself a module (up-to a cohomological shift). This phenomenon is non-trivial and quite rare and therefore considered interesting. This motivates the following definition.

Definition 4. *A module $\mathcal{M} \in \mathsf{Coh}(\mathcal{D}_X)$ is called Cohen-Macaulay if $\mathbb{D}(\mathcal{M}) \simeq \mathcal{M}^\vee[d]$ for some $\mathcal{M}^\vee \in \mathsf{Coh}(\mathcal{D}_X)$.*

Given a Cohen-Macaulay module $\mathcal{M} \in \mathsf{Coh}(\mathcal{D}_X)$ we call $\mathcal{M}^\vee = \mathbb{D}(\mathcal{M})[-d]$ the naive dual of $\mathcal{M}$.

Explicit examples. Let us compute the Verdier dual for some explicit modules.

Example 6. *Let $X = \mathbb{A}^1$. Consider $\mathcal{M} = \mathcal{O}_{\mathbb{A}^1}$. We will show that*

$$\mathbb{D}(\mathcal{O}_{\mathbb{A}^1}) \simeq \mathcal{O}_{\mathbb{A}^1}.$$

In order to compute $\mathbb{D}(\mathcal{M})$ we need to consider a free resolution $\mathcal{P}^{\cdot} \xrightarrow{q.i} \mathcal{M}$. Such a resolution is[7]

$$P^{\cdot} : \ \underset{-1}{\mathcal{D}_{\mathbb{A}^1}} \xrightarrow{\partial_x} \underset{0}{\mathcal{D}_{\mathbb{A}^1}},$$

where ∂_x acts by multiplication from the right. We have

$$\begin{aligned} \mathbb{D}(\mathcal{M}) &\simeq R\mathcal{H}om_{\mathcal{D}_{\mathbb{A}^1}}(\mathcal{P}^{\cdot}, \mathcal{D}_{\mathbb{A}^1}^{\Omega^{-1}})[1] \\ &\simeq \left(\underset{-1}{\mathcal{D}_{\mathbb{A}^1}^{\Omega^{-1}}} \xrightarrow{\partial_x} \underset{0}{\mathcal{D}_{\mathbb{A}^1}^{\Omega^{-1}}} \right), \end{aligned}$$

where the map in the last expression is multiplication by ∂_x from the left. Finally we use the identification $\mathcal{D}_{\mathbb{A}^1} \xrightarrow{\simeq} \mathcal{D}_{\mathbb{A}^1}^{\Omega^{-1}}$ sending d to $d \otimes dx^{-1}$ and we obtain

$$\left(\underset{-1}{\mathcal{D}_{\mathbb{A}^1}^{\Omega^{-1}}} \xrightarrow{\partial_x} \underset{0}{\mathcal{D}_{\mathbb{A}^1}^{\Omega^{-1}}} \right) \simeq \left(\underset{-1}{\mathcal{D}_{\mathbb{A}^1}} \xrightarrow{-\partial_x} \underset{0}{\mathcal{D}_{\mathbb{A}^1}} \right)$$

The right side of the last isomorphism is quasi-isomorphic to $\mathcal{O}_{\mathbb{A}^1}$, which yields the result.

Example 7. *This example is a generalization of the previous example. We take X to be an arbitrary smooth algebraic variety. We will show that $\mathbb{D}(\mathcal{O}_X) \simeq \mathcal{O}_X$. We use the Koszul resolution*

$$Koz^{\bullet}(\mathcal{D}_X) \xrightarrow{q.i} \mathcal{O}_X,$$

where $Koz^{-i}(\mathcal{D}_X) = \mathcal{D}_X \bigotimes_{\mathcal{O}} \bigwedge^i \mathcal{T}_X$ and the differential is given by Cartan formulas, for example

- *$d_{-1} : Koz^{-1}(\mathcal{D}_X) \to Koz^0(\mathcal{D}_X)$ is given by $d \otimes \partial \longmapsto d\partial$.*

[7] The small integer numbers which appear below the complex denote the cohomological degree.

- $d_{-2} : Koz^{-2}(\mathcal{D}_X) \to Koz^{-1}(\mathcal{D}_X)$ *is given by* $d \otimes \partial_0 \wedge \partial_1 \longmapsto d\partial_1 \otimes \partial_0 - d\partial_0 \otimes \partial_1 - d \otimes [\partial_0, \partial_1]$.

We have

$$\mathbb{D}(\mathcal{O}_X) \simeq R\mathcal{H}om_{\mathcal{D}_X}(Koz^{\bullet}(\mathcal{D}_X), \mathcal{D}_X^{\Omega^{-\mathrm{top}}})[\dim X].$$

The left side can be computed in two stages. First one shows that $R\mathcal{H}om_{\mathcal{D}_X}(Koz^{\bullet}(\mathcal{D}_X), \mathcal{D}_X)$ *is quasi-isomorphic to*

$$\mathcal{D}R^{\bullet}(\mathcal{D}_X) : \quad \underset{0}{\mathcal{D}_X} \xrightarrow{d} \underset{1}{\mathcal{D}_X \bigotimes_{\mathcal{O}} \Omega^1} \xrightarrow{d} \ldots \xrightarrow{d} \underset{\dim X}{\mathcal{D}_X \bigotimes_{\mathcal{O}} \Omega^{\dim X}},$$

second one shows that $\mathcal{D}R^{\bullet}(\mathcal{D}_X) \bigotimes_{\mathcal{O}} \mathcal{D}_X^{\Omega^{-\mathrm{top}}}$ *is quasi-isomorphic to* $Koz^{\bullet}(\mathcal{D}_X)[-\dim X]$. *These two arguments conclude the computation.*

Example 8. *In this example we study the D-module associated with a linear differential operator on* $X = \mathbb{A}^N$. *Let* $P \in D_{\mathbb{A}^N}$, *namely* $P = \sum_{\alpha} a_{\alpha}(x)\partial^{\alpha}$. *Let* $\mathcal{M}_P = \mathcal{D}_{\mathbb{A}^N}/\mathcal{D}_{\mathbb{A}^N}P$. *We will show that*

$$\mathbb{D}(\mathcal{M}_P) \simeq \mathcal{M}_{P^t}[N-1],$$

where $P^t = \sum_{\alpha} (-\partial)^{\alpha} a_{\alpha}(x)$ *is the transpose of* P, *with* $(-\partial)^{\alpha} = (-\partial_{x_1})^{\alpha_1} \cdot .. \cdot (-\partial_{x_n})^{\alpha_N}$. *We consider the resolution* $\mathcal{P}^{\bullet} \xrightarrow{q.i} \mathcal{M}_P$ *given by*

$$\underset{-1}{\mathcal{D}_{\mathbb{A}^N}} \xrightarrow{P} \underset{0}{\mathcal{D}_{\mathbb{A}^N}},$$

where P *is acting by multiplication from the right. We have*

$$\begin{aligned} \mathbb{D}(\mathcal{M}_P) &\simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{P}^{\bullet}, \mathcal{D}_{\mathbb{A}^N}^{\Omega^{-N}})[N] \\ &\simeq \left(\underset{-N}{\mathcal{D}_{\mathbb{A}^N}^{\Omega^{-N}}} \xrightarrow{P} \underset{-N+1}{\mathcal{D}_{\mathbb{A}^N}^{\Omega^{-N}}} \right), \end{aligned}$$

where in the last expression P *is acting by multiplication from the left. Using the identification* $\mathcal{D}_{\mathbb{A}^N} \xrightarrow{\simeq} \mathcal{D}_{\mathbb{A}^N}^{\Omega^{-N}}$ *given by* $d \longmapsto d \otimes dx^{-1}$ *we can write*

$$\left(\underset{-N}{\mathcal{D}_{\mathbb{A}^N}^{\Omega^{-N}}} \xrightarrow{P} \underset{-N+1}{\mathcal{D}_{\mathbb{A}^N}^{\Omega^{-N}}} \right) \simeq \left(\underset{-N}{\mathcal{D}_{\mathbb{A}^N}} \xrightarrow{P^t} \underset{-N+1}{\mathcal{D}_{\mathbb{A}^N}} \right),$$

yielding the result.

1.2.2. *General setting.* The functor of Verdier duality exists also in the setting of general differential algebras. For the sake of concreteness, we will restrict ourselves to the case of differential algebras of the form $\mathcal{D}_P$ for $P \xrightarrow{\pi} X$ a principal G-bundle. The category $\mathsf{DCoh}(\mathcal{D}_P)$ admits an anti-equivalence $\mathbb{D} : \mathsf{DCoh}(\mathcal{D}_P) \longrightarrow \mathsf{DCoh}(\mathcal{D}_P)$,defined by

$$\mathbb{D}(\mathcal{M}) = R\mathcal{H}om_{\mathcal{D}_P}(M, \mathcal{D}_P^{\Omega^{-\mathrm{top}}})[\dim P],$$

where

$$\mathcal{D}_P^{\Omega^{-\mathrm{top}}} = \mathcal{D}_P \bigotimes_{\mathcal{O}_X} \pi_*(\Omega_P^{-\mathsf{top}})^G = \mathcal{D}_P \bigotimes_{\mathcal{O}_X} \bigwedge^{\dim P} \mathcal{T}_P,$$

is the dualizing bimodule. The two commuting left $\mathcal{D}_P$-actions are given by similar formulas as in the standard setting. The Verdier functor restricts to give a duality functor on the monodromic categories. More precisely, for any character $\lambda \in Z(\mathfrak{g})^*$ we have an induced functor

$$\mathbb{D} : \mathsf{DCoh}(\mathcal{D}_P^{\lambda}) \longrightarrow \mathsf{DCoh}(\mathcal{D}_P^{-\lambda}),$$

given by

$$\mathbb{D}(\mathcal{M}){=}R\mathcal{H}om_{\mathcal{D}_P}(\mathcal{M},\mathcal{D}_P^{\lambda,\Omega^{-\mathrm{top}}})[\dim P-\dim Z(G)]. \tag{1.3}$$

where $\mathcal{D}_P^{\lambda,\Omega^{-1}}=\mathcal{D}_P^{\lambda}\bigotimes_{\mathcal{O}_X}\bigwedge^{\dim P}\mathcal{T}_P$. In the case when G is commutative, since $\dim Z(G)=\dim G$ then (1.3) is given by

$$\mathbb{D}(\mathcal{M}){=}R\mathcal{H}om_{\mathcal{D}_P}(\mathcal{M},\mathcal{D}_P^{\lambda,\Omega^{-\mathrm{top}}})[\dim X].$$

As before we distinguish a class of modules which are well behaved with respect to $\mathbb{D}$.

Definition 5. *A module* $\mathcal{M}\in$*Coh*$(\mathcal{D}_P)$ *(Coh*$(\mathcal{D}_P^{\lambda})$*) is called Cohen-Macaulay if* $\mathbb{D}(\mathcal{M})\simeq\mathcal{M}^{\vee}[d]$ *for some* $\mathcal{M}^{\vee}\in\mathsf{Coh}(\mathcal{D}_P)$ $(\mathsf{Coh}(\mathcal{D}_P^{-\lambda}))$*.*

1.3. **The Green class of a D-module.**

> *"... we can say that there is only one formula (which we shall call "fundamental formula") in the whole theory of partial differential equations, no matter to which type they belong. "*
>
> J. Hadamard, Lectures on the Cauchy problem.

The notion of Green class of a D-module generalizes the classical notion of green form of a partial differential operator. It is satisfying to note that using the homological language of derived categories this notion can be defined in a straight forward manner and apply to much more general situations. We begin by explaining this notion in the standard setting of usual D-modules and then introduce its generalization.

1.3.1. *Standard setting.* The construction is based on the following fundamental adjunction property of the Verdier duality functor.

Theorem 6 (Adjunction property)**.** *Let* $\mathcal{M},\mathcal{N},\mathcal{L}\in\mathsf{DCoh}(\mathcal{D}_X)$*. There exists a canonical isomorphism*

$$\mathsf{RHom}_{\mathcal{D}_X}(\mathcal{N}\bigotimes_{\mathcal{O}}\mathcal{M},\mathcal{L})\simeq\mathsf{RHom}_{\mathcal{D}_X}(\mathcal{N},\mathbb{D}(\mathcal{M})\bigotimes_{\mathcal{O}}\mathcal{L}).$$

Applying Theorem 6 to $\mathcal{N}=\mathcal{O}_X$ and $\mathcal{M}=\mathcal{L}$ we obtain

$$\mathsf{RHom}_{\mathcal{D}_X}(\mathcal{M},\mathcal{M})\simeq\mathsf{RHom}_{\mathcal{D}_X}(\mathcal{O}_X,\mathbb{D}(\mathcal{M})\bigotimes_{\mathcal{O}}\mathcal{M}).$$

In particular we have

$$\mathsf{R}^0\mathsf{Hom}_{\mathcal{D}_X}(\mathcal{M},\mathcal{M})\simeq\mathsf{R}^0\mathsf{Hom}_{\mathcal{D}_X}(\mathcal{O}_X,\mathbb{D}(\mathcal{M})\bigotimes_{\mathcal{O}}\mathcal{M}). \tag{1.4}$$

Let us interpret both sides of (1.4).

- The left hand side of (1.4) is simply $\mathsf{Hom}_{\mathcal{D}_X}(\mathcal{M},\mathcal{M})$.
- The right hand side can be interpreted as follows. Consider the Koszul resolution $Koz^{\bullet}(\mathcal{D}_X)\xrightarrow{q.i}\mathcal{O}_X$. We have

$$RHom_{\mathcal{D}_X}(\mathcal{O}_X,\mathbb{D}(\mathcal{M})\bigotimes_{\mathcal{O}}\mathcal{M})\simeq\mathsf{Hom}_{\mathcal{D}_X}(Koz^{\bullet}(\mathcal{D}_X),\mathbb{D}(\mathcal{M})\bigotimes_{\mathcal{O}}\mathcal{M})$$

The right hand side of the above equation can be written as

$$\mathsf{Hom}_{\mathcal{D}_X}(Koz^{\bullet}(\mathcal{D}_X),\mathbb{D}(\mathcal{M})\bigotimes_{\mathcal{O}}\mathcal{M})=DR^{\bullet}(\mathbb{D}(\mathcal{M})\bigotimes_{\mathcal{O}}\mathcal{M})),$$

where $DR^{\bullet}$ stands for the standard de-Rham construction, that is for a coherent module $\mathcal{N} \in \mathsf{Coh}(\mathcal{D}_X)$

$$\begin{aligned} DR^{\bullet}(\mathcal{N}) &= R\Gamma(\mathcal{D}R^{\bullet}(\mathcal{N})) \\ &= R\Gamma(\underset{0}{\mathcal{N}} \xrightarrow{d} \underset{1}{\mathcal{N}} \bigotimes_{\mathcal{O}} \Omega^1 \xrightarrow{d} \dots \xrightarrow{d} \underset{\dim X}{\mathcal{N}} \bigotimes_{\mathcal{O}} \Omega^{\dim X}), \end{aligned}$$

where d is the standard de-Rham differential. In particular,

$$R^0\mathsf{Hom}_{\mathcal{D}_X}(\mathcal{O}_X, \mathbb{D}(\mathcal{M}) \bigotimes_{\mathcal{O}} \mathcal{M}) \simeq H^0(DR^{\bullet}(\mathbb{D}(\mathcal{M}) \bigotimes_{\mathcal{O}} \mathcal{M})).$$

The left hand side of (1.4) consists of a canonical element, namely the identity morphism $Id \in \mathsf{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{M})$. The adjunction isomorphism sends it to a distinguished class $G_{\mathcal{M}} \in H^0(DR^{\bullet}(\mathbb{D}(\mathcal{M}) \bigotimes_{\mathcal{O}} \mathcal{M}))$. The class $G_{\mathcal{M}}$ is called the *Green class* of the D-module $\mathcal{M}$. Let us explain the construction in more concrete terms. For simplicity assume X is affine therefore $R\Gamma = \Gamma$ and in addition assume $\mathcal{M}$ is Cohen-Macaulay, namely $\mathcal{M} \in \mathsf{Coh}(\mathcal{D}_X)$ and $\mathbb{D}(\mathcal{M}) \simeq \mathcal{M}^{\vee}[d]$ for some $\mathcal{M}^{\vee} \in \mathsf{Coh}(\mathcal{D}_X)$. In this situation we can write

$$\begin{aligned} \mathsf{Hom}_{\mathcal{D}_X}(\mathcal{O}_X, \mathbb{D}(\mathcal{M}) \bigotimes_{\mathcal{O}} \mathcal{M}) &\simeq \mathsf{Hom}_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M}^{\vee}[d] \bigotimes_{\mathcal{O}} \mathcal{M}) \\ &\simeq R^d\mathsf{Hom}_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M}^{\vee} \bigotimes_{\mathcal{O}} \mathcal{M}) \\ &\simeq H^d(DR^{\bullet}(\mathcal{M}^{\vee} \bigotimes_{\mathcal{O}} \mathcal{M})), \end{aligned}$$

therefore, the class $G_{\mathcal{M}}$ is represented by a degree d cycle

$$G_{\mathcal{M}} = \sum_i m_i^{\vee} \otimes m_i \otimes \omega_i \in DR^d(\mathcal{M}^{\vee} \bigotimes_{\mathcal{O}} \mathcal{M})$$

where $m_i \in \mathcal{M}$, $m_i^{\vee} \in \mathcal{M}^{\vee}$ and $\omega_i \in \Omega^d$. If we drop the assumption that X is affine then representing the Green class $G_{\mathcal{M}}$ is slightly more complicated. It can be done using the Čech resolution. Choose an affine covering $\mathcal{U} = \{U_i\}$ of X . We can write

$$\begin{aligned} DR^{\bullet}(\mathcal{M}^{\vee} \bigotimes_{\mathcal{O}} \mathcal{M}) &= R\Gamma(X, \mathcal{D}R^{\bullet}(\mathcal{M}^{\vee} \bigotimes_{\mathcal{O}} \mathcal{M})) \\ &\simeq \check{C}ech^{\bullet}(\mathcal{U}, \mathcal{D}R^{\bullet}(\mathcal{M}^{\vee} \bigotimes_{\mathcal{O}} \mathcal{M})) \end{aligned}$$

The Green class can be represented by a cycle

$$G_{\mathcal{M}} \in H^d\left(\check{C}ech^{\bullet}(\mathcal{U}, \mathcal{D}R^{\bullet}(\mathcal{M}^{\vee} \bigotimes_{\mathcal{O}} \mathcal{M}))\right).$$

1.3.2. *General setting.* The notion of Green class can be generalized to the setting of differential algebras of the form $\mathcal{D}_P$ and their monodromic specializations $\mathcal{D}_P^{\lambda}$. The construction is based on a generalization of Theorem 6.

Theorem 7. *Let $\mathcal{M}, \mathcal{N}, \mathcal{L} \in \mathsf{DCoh}(\mathcal{D}_P)$. There exists a canonical isomorphism*

(1.5) $$\mathsf{RHom}_{\mathcal{D}_P}(\mathcal{N} \bigotimes_{\mathcal{O}} \mathcal{M}, \mathcal{L}) \simeq \mathsf{RHom}_{\mathcal{D}_P}(\mathcal{N}, \mathbb{D}(\mathcal{M}) \bigotimes_{\mathcal{O}} \mathcal{L}).$$

In the monodromic situation when $\mathcal{M} \in \mathsf{DCoh}(\mathcal{D}_P^{\lambda})$, $\mathcal{N} \in \mathsf{DCoh}(\mathcal{D}_P^{\mu})$ and $\mathcal{L} \in \mathsf{DCoh}(\mathcal{D}_P^{\lambda+\mu})$ the adjunction isomorphism can be written as

(1.6) $$\mathsf{RHom}_{\mathcal{D}_P^{\lambda+\mu}}(\mathcal{N} \bigotimes_{\mathcal{O}} \mathcal{M}, \mathcal{L}) \simeq \mathsf{RHom}_{\mathcal{D}_P^{\lambda}}(\mathcal{N}, \mathbb{D}(\mathcal{M}) \bigotimes_{\mathcal{O}} \mathcal{L}).$$

Note that $\mathcal{N} \bigotimes_{\mathcal{O}} \mathcal{M}, \mathcal{L} \in \mathsf{DCoh}(\mathcal{D}_P^{\lambda+\mu})$ and $\mathcal{N}, \mathbb{D}(\mathcal{M}) \bigotimes_{\mathcal{O}} \mathcal{L} \in \mathsf{DCoh}(\mathcal{D}_P^{\lambda})$hence both sides of (1.6) make sense. Taking $\mathcal{N} = \mathcal{O}_X$ and $\mathcal{M} = \mathcal{L}$, we obtain

$$\mathsf{RHom}_{\mathcal{D}_P}(\mathcal{M}, \mathcal{M}) \simeq \mathsf{RHom}_{\mathcal{D}_P}(\mathcal{O}_X, \mathbb{D}(\mathcal{M}) \bigotimes_{\mathcal{O}} \mathcal{M}),$$

which in particular implies that

(1.7) $$\mathsf{Hom}_{\mathcal{D}_P}(\mathcal{M}, \mathcal{M}) \simeq \mathsf{R}^0\mathsf{Hom}_{\mathcal{D}_P}(\mathcal{O}_X, \mathbb{D}(\mathcal{M}) \bigotimes_{\mathcal{O}} \mathcal{M}),$$

The right hand side of (1.7) can be written as $H^0(DR_P^\bullet(\mathbb{D}(\mathcal{M})\bigotimes_{\mathcal{O}}\mathcal{M}))$. Here $DR_P^\bullet(\mathcal{N})$ is a de-Rham like complex associated with a $\mathcal{D}_P$-module $\mathcal{N}$

$$\begin{aligned} DR_P^\bullet(\mathcal{N}) &= R\Gamma(\mathcal{D}R_P^\bullet(\mathcal{N})) \\ &= R\Gamma(\underset{0}{\mathcal{N}}\xrightarrow{d}\underset{1}{\mathcal{N}}\bigotimes_{\mathcal{O}}\mathcal{T}_P^*\xrightarrow{d}\dots\xrightarrow{d}\underset{\dim X}{\mathcal{N}}\bigotimes_{\mathcal{O}}\textstyle\bigwedge^{\mathsf{top}}\mathcal{T}_P^*), \end{aligned}$$

with the differential given by the same formulas as in the standard setting. The distinguished class $G_{\mathcal{M}}\in H^0(DR_P^\bullet(\mathbb{D}(\mathcal{M})\bigotimes_{\mathcal{O}}\mathcal{M}))$ which corresponds to $Id\in\mathsf{Hom}_{\mathcal{D}_P^\lambda}(\mathcal{M},\mathcal{M})$ is called the *Green class* of $\mathcal{M}$. We are interested in a particular situation when G is commutative and $\mathcal{M}\in\mathsf{DCoh}(\mathcal{D}_P^\lambda)$. In this case using (1.6) we can write

$$\begin{aligned} \mathsf{Hom}_{\mathcal{D}_P^\lambda}(\mathcal{M},\mathcal{M}) &\simeq \mathsf{Hom}_{\mathcal{D}_P^0}(\mathcal{O}_X,\mathbb{D}(\mathcal{M})\bigotimes_{\mathcal{O}}\mathcal{M}) \\ &\simeq \mathsf{Hom}_{\mathcal{D}_X}(\mathcal{O}_X,\mathbb{D}(\mathcal{M})\bigotimes_{\mathcal{O}}\mathcal{M}) \\ &\simeq H^0(DR^\bullet(\mathbb{D}(\mathcal{M})\bigotimes_{\mathcal{O}}\mathcal{M})). \end{aligned}$$

If we assume that X is affine and $\mathcal{M}$ is Caohen-Macaulay, namely $\mathbb{D}(\mathcal{M})\simeq\mathcal{M}^\vee[d]$ for some $\mathcal{M}^\vee\in\mathsf{Coh}(\mathcal{D}_X)$ then

$$G_{\mathcal{M}}\in H^d(DR^\bullet(\mathcal{M}^\vee\bigotimes_{\mathcal{O}}\mathcal{M})),$$

therefore in this case the class $G_{\mathcal{M}}$ is represented by a degree d cycle

$$G_{\mathcal{M}}=\sum_i m_i^\vee\otimes m_i\otimes\omega_i\in DR^d(\mathcal{M}^\vee\bigotimes_{\mathcal{O}}\mathcal{M})$$

where $m_i\in\mathcal{M}$, $m_i^\vee\in\mathcal{M}^\vee$ and $\omega_i\in\Omega^d$. As before, without the assumption that X is affine we only have

$$G_{\mathcal{M}}\in H^d\left(\check{C}ech^\bullet(\mathcal{U},\mathcal{D}R^\bullet(\mathcal{M}^\vee\bigotimes_{\mathcal{O}}\mathcal{M}))\right).$$

1.4. **Natural pairings.** Using the Green class one can construct a natural pairing between solution spaces of a D-module and its Verdier dual. This construction is very basic and extremely useful. As we always do, we first explain it in the setting of usual D-modules and then introduce its generalization.

1.4.1. *Standard setting.*
Canonical pairing between solution spaces of holomorphic type. Before we introduce the construction in its full generality it might be beneficial to start with more particular circumstance. Let us assume that X is affine and $\mathcal{M}$ is Cohen-Macaulay, namely $\mathbb{D}(\mathcal{M})\simeq\mathcal{M}^\vee[d]$ for some $\mathcal{M}^\vee\in\mathsf{Coh}(\mathcal{D}_X)$. Let $\mathcal{V}$ be a holomorphic vector bundle on X^{an} equipped with a flat connection and $\mathcal{F}^{an}$ be the corresponding target D_X-module. Let $\gamma\in H_d(X(\mathbb{R}),\mathbb{C})$ be a non-trivial homology class.

We will define a pairing

$$B_\gamma:\mathsf{Sol}(\mathcal{M},\mathcal{F}^{an})\times\mathsf{Sol}(\mathcal{M}^\vee,\mathcal{F}^{an*})\longrightarrow\mathbb{C},$$

where $\mathcal{F}^{an*}$ is associated with the dual vector bundle $\mathcal{V}^*$. The pairing B_γ is defined as follows. Given solutions

$$\begin{aligned} \nu &\in \mathsf{Sol}(\mathcal{M},\mathcal{F}^{an}), \\ \varphi &\in \mathsf{Sol}(\mathcal{M}^\vee,\mathcal{F}^{an*}). \end{aligned}$$

Applying $\varphi\otimes\nu$ to the Green class $G_{\mathcal{M}}\in H^d(DR^\bullet(\mathcal{M}^\vee\bigotimes_{\mathcal{O}}\mathcal{M}))$ we obtain a class

$$\varphi\otimes\nu(G_{\mathcal{M}})\in H^d(DR^\bullet(\mathcal{F}^{an*}\bigotimes_{\mathcal{O}}\mathcal{F}^{an})),$$

Applying further the canonical morphism of $\mathcal{D}_X$-modules

$$m : \mathcal{F}^{an*} \bigotimes_{\mathcal{O}} \mathcal{F}^{an} \longrightarrow \mathcal{O}_X^{an},$$

yields an honest cohomology class

$$m \circ \varphi \otimes \nu(G_{\mathcal{M}}) \in H^d(DR^{\bullet}(\mathcal{O}_X^{an})) = H^d(X^{an}, \mathbb{C}).$$

Now, define

$$B_\gamma(\nu, \varphi) := \langle m \circ \varphi \otimes \nu(G_{\mathcal{M}}), \gamma \rangle ,$$

where $\langle \cdot, \cdot \rangle$ stands for the canonical pairing between homology and cohomology. The previous construction is a particular case of the following general statement

Theorem 8. *Let* $\mathcal{M} \in \mathsf{DCoh}(\mathcal{D}_X)$ *and* $\gamma \in H_d(X, \mathbb{C})$ *a non-trivial class. There exists a natural pairing*

$$B_\gamma : \mathsf{R}^i\mathsf{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{F}^{an}) \times \mathsf{R}^{2N-i-d}\mathsf{Hom}_{\mathcal{D}_X}(\mathbb{D}\mathcal{M}, \mathcal{F}^{*an}) \longrightarrow \mathbb{C}, \tag{1.8}$$

where $N = \dim X$

Canonical pairing between solution spaces of C^∞ type. Let us assume first that X is affine and $\mathcal{M}$ is Cohen-Macaulay, namely $\mathbb{D}(\mathcal{M}) \simeq \mathcal{M}^\vee[d]$ for some $\mathcal{M}^\vee \in \mathsf{Coh}(\mathcal{D}_X)$. Assume X is defined over $\mathbb{R}$ so that $X(\mathbb{R})$ is a smooth manifold of dimension $\dim X(\mathbb{R}) = \dim X$. Let $\mathcal{V}$ be a complex vector bundle on the manifold $X(\mathbb{R})$ equipped with a flat connection. Let $\mathcal{F}^\infty = \mathsf{ex}^*\mathcal{V}$ be the corresponding target module. Let $\gamma \in H_d(X(\mathbb{R}), \mathbb{C})$ be a non-trivial homology class. In the same manner as before we can define a pairing

$$B_\gamma : \mathsf{Sol}(\mathcal{M}, \mathcal{F}^\infty) \times \mathsf{Sol}(\mathcal{M}^\vee, \mathcal{F}^{\infty *}) \longrightarrow \mathbb{C}.$$

Given solutions

$$\begin{aligned} \nu &\in \mathsf{Sol}(\mathcal{M}, \mathcal{F}^\infty), \\ \varphi &\in \mathsf{Sol}(\mathcal{M}, \mathcal{F}^{\infty *}), \end{aligned}$$

applying $\varphi \otimes \nu$ to the Green class $G_{\mathcal{M}} \in H^d(DR^{\bullet}(\mathcal{M}^\vee \bigotimes_{\mathcal{O}} \mathcal{M}))$ yields a class

$$\varphi \otimes \nu(G_{\mathcal{M}}) \in H^d(DR^{\bullet}(\mathcal{F}^{\infty *} \bigotimes_{\mathcal{O}} \mathcal{F}^\infty)),$$

Applying further the canonical morphism of $\mathcal{D}_X$-modules

$$m : \mathcal{F}^{\infty *} \bigotimes_{\mathcal{O}} \mathcal{F}^\infty \longrightarrow \mathsf{ex}^* C^\infty_{X(\mathbb{R})}$$

We obtain an honest cohomology class

$$m \circ \varphi \otimes \nu(G_{\mathcal{M}}) \in H^d(DR^{\bullet}(\mathsf{ex}^* C^\infty_{X(\mathbb{R})})) = H^d(X(\mathbb{R}), \mathbb{C}).$$

Define

$$B_\gamma(\nu, \varphi) := \langle m \circ \varphi \otimes \nu(G_{\mathcal{M}}), \gamma \rangle ,$$

The general statement is

Theorem 9. *Let* $\mathcal{M} \in \mathsf{DCoh}(\mathcal{D}_X)$ *and* $\gamma \in H_d(X(\mathbb{R}), \mathbb{C})$ *a non-trivial class. There exists a natural pairing*

$$B_\gamma : \mathsf{R}^i\mathsf{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{F}^\infty) \times \mathsf{R}^{2N-i-d}\mathsf{Hom}_{\mathcal{D}_X}(\mathbb{D}\mathcal{M}, \mathcal{F}^{*\infty}) \longrightarrow \mathbb{C}, \tag{1.9}$$

where $N = \dim X$.

1.4.2. *General setting.*

Canonical pairing between solution spaces of holomorphic type. Let $\mathcal{V}$ be an analytic vector bundle equipped with a $\mathcal{D}_P$-action and $\mathcal{F}^{an}$ the associated target $\mathcal{D}_P$-module. The following theorem is a generalization of Theorem 8.

Theorem 10. *Let $\mathcal{M} \in \mathsf{DCoh}(\mathcal{D}_P)$ and $\gamma \in H_d(X, \mathbb{C})$ a non-trivial class. There exists a natural paring*

$$B_\gamma : \mathsf{R}^i\mathsf{Hom}_{\mathcal{D}_P}(\mathcal{M}, \mathcal{F}^{an}) \times \mathsf{R}^{2N-i-d}\mathsf{Hom}_{\mathcal{D}_P}(\mathbb{D}\mathcal{M}, \mathcal{F}^{an*}) \longrightarrow \mathbb{C}, \tag{1.10}$$

where $N = \dim X$.

Canonical pairing between solution spaces of C^∞ type. Assume X, P and G are defined over $\mathbb{R}$ so that $X(\mathbb{R}), P(\mathbb{R})$ and $G(\mathbb{R})$ are smooth manifold of the correct dimensions. Let $\mathcal{V}$ be a complex vector bundle on the manifold $X(\mathbb{R})$ equipped with a $\mathcal{D}_{P(\mathbb{R})}$-action and Let $\mathcal{F}^\infty = \mathsf{ex}^*\mathcal{V}$ be the corresponding target module. The following theorem is a generalization of Theorem 9.

Theorem 11. *Let $\mathcal{M} \in \mathsf{DCoh}(\mathcal{D}_P)$ and $\gamma \in H_d(X, \mathbb{C})$ a non-trivial class. There exists a natural paring*

$$B_\gamma : \mathsf{R}^i\mathsf{Hom}_{\mathcal{D}_P}(\mathcal{M}, \mathcal{F}^{\infty}) \times \mathsf{R}^{2N-i-d}\mathsf{Hom}_{\mathcal{D}_P}(\mathbb{D}\mathcal{M}, \mathcal{F}^{\infty*}) \longrightarrow \mathbb{C}, \tag{1.11}$$

where $N = \dim X$.

1.5. **Equivariant structures .** There are two main notions of equivariance in the theory of D-modules. The first kind is weak equivariance (also called weak Harish-Chandra structure) and the second is strong equivariance (strong Harish-Chandra structure). In this paper we will mainly be interested in the first kind which will simply be called equivariance structure.

1.5.1. *Standard setting.* Assume the variety X is equipped with a group action

$$\alpha : G \times X \longrightarrow X,$$

where G is a connected reductive group. Let $\mathcal{M} \in \mathsf{Coh}(\mathcal{D}_X)$.
Naive definition. As a first approximation, a G-equivariant structure on $\mathcal{M}$ is a family of isomorphisms of $\mathcal{D}_X$-modules

$$\theta_g : \mathcal{M} \xrightarrow{\simeq} g^*\mathcal{M} \triangleq \alpha_g^*\mathcal{M},$$

satisfying the following multiplicative condition

$$\theta_{gh} = h^*\theta_g \circ \theta_h. \tag{1.12}$$

Explicitly,

$$\begin{aligned} \theta_{gh} &: \mathcal{M} \longrightarrow (gh)^*\mathcal{M} \simeq h^*g^*\mathcal{M}, \\ h^*\theta_g \circ \theta_h &: \mathcal{M} \xrightarrow{\theta_h} h^*\mathcal{M} \xrightarrow{h^*\theta_g} h^*g^*\mathcal{M}. \end{aligned}$$

Formal definition. The formal definition, of-course, has to take into account the topology of G. Let p_X and p_G denote the projection maps from $G \times X$ on G and X respectively. Let $Tan^{\mathsf{vert}}_{G\times X} \subset Tan_{G\times X}$ be the Lie subalgebroid of the tangent sheaf consisting of vertical vector fields with respect to the projection p_G. Precisely, an element $\xi \in Tan^{\mathsf{vert}}_{G\times X}$ if and only if $p_{G*}\xi = 0$. Let $\mathcal{D}^{\mathsf{vert}}_{G\times X} = \mathcal{U}(Tan^{\mathsf{vert}}_{G\times X}) \subset \mathcal{D}_{G\times X}$ the corresponding universal enveloping algebra.

Definition 6. *A G-equivariant structure on $\mathcal{M}$ is an isomorphism of $\mathcal{D}^{\mathsf{vert}}_{G\times X}$-modules*

$$\theta : p_X^*\mathcal{M} \overset{\simeq}{\longrightarrow} \alpha^*\mathcal{M},$$

satisfying a cocycle condition on $G \times G \times X$.

The cocycle condition in Definition 6 is defined as follows. We have three maps

$$\begin{aligned}(Id,\alpha) &: G\times G\times X \longrightarrow G\times X,\\ (m,Id) &: G\times G\times X \longrightarrow G\times X,\\ p_{23} &: G\times G\times X \longrightarrow G\times X,\end{aligned}$$

where p_{23} is the projection on the right $G \times X$ copy. We have two maps form $(m, Id)^*p_X^*\mathcal{M}$ to $(Id, \alpha)^*\alpha^*\mathcal{M}$ defined by the following two chains of compositions

$$(m,Id)^*p_X^*\mathcal{M} \overset{(m,Id)^*\theta}{\longrightarrow} (m,Id)^*\alpha^*\mathcal{M} \simeq (Id,\alpha)^*\alpha^*\mathcal{M},$$

$$\begin{aligned}(m,Id)^*p_X^*\mathcal{M} &\simeq p_{23}^*p_X^*\mathcal{M} \overset{p_{23}^*\theta}{\longrightarrow} p_{23}^*\alpha^*\mathcal{M} \simeq (Id,\alpha)^*p_X^*\mathcal{M}\\ &\overset{(Id,\alpha)^*\theta}{\longrightarrow} (Id,\alpha)^*\alpha^*\mathcal{M}.\end{aligned}$$

All the morphisms in the above chains are isomorphisms. The cocycle condition is requirement for the equality of this pair of maps. The reader should convince himself that this condition implies the multiplicativity condition (1.12).
Examples of equivariant modules. If $\mathcal{M}$ is of the form $\mathcal{M} = \mathcal{D}_X/\mathcal{I}$, where $\mathcal{I}$ is a sheaf of left ideals then $\mathcal{M}$ is G-equivariant if and only if

$$g(\mathcal{I}) \subset \mathcal{I}, \quad \text{for every } g \in G.$$

For this particular type of situation, $\mathcal{M}$ is strongly G-equivariant if in addition we require

$$\xi^{\#} \in \mathcal{I}, \quad \text{for every } \xi \in \mathfrak{g},$$

where $\xi^{\#} \in Tan_X$ is the vector field associated to the element ξ. We will not develop this notion further.

Let us write some specific modules.

Example 9. *Let $X = G = \mathbb{G}_m$ and $\mathcal{M} = \mathcal{D}_{\mathbb{G}_m}/\mathcal{D}_{\mathbb{G}_m}(x\partial_x - \lambda)$. The differential operator $x\partial_x - \lambda$ is G-invariant, therefore $\mathcal{M}$ is G-equivariant. In the case $\lambda = 0$, $\mathcal{M}$ is strongly G-equivariant. Assume $\lambda = n \in \mathbb{Z}$, the solution space $\mathsf{Sol}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}^{an}_{\mathbb{G}_m})$ is the one dimensional representation of $\mathbb{G}_m$ associated with the character z^n.*

Example 10. *Let $X = G = \mathbb{G}_a$ and $\mathcal{M} = \mathcal{D}_{\mathbb{G}_a}/\mathcal{D}_{\mathbb{G}_a}(\partial_x - \lambda)$. The differential operator $\partial_x - \lambda$ is G-invariant, therefore $\mathcal{M}$ is G-equivariant. In the case $\lambda = 0$, $\mathcal{M}$ is strongly G-equivariant. The solution space $\mathsf{Sol}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}^{an}_{\mathbb{G}_a})$ is the one dimensional representation of $\mathbb{G}_a$ associated with the character $e^{\lambda z}$.*

Action of the group G on solution spaces. If the module $\mathcal{M}$ is G-equivariant and $\mathcal{F}$ is a G-equivariant target $\mathcal{D}_X$-module then the group G acts in a natural manner on the space of solutions

$$\mathsf{Sol}(\mathcal{M},\mathcal{F}).$$

As a first approximation this action is obtained as follows. Define

$$\mathsf{Sol}(\mathcal{M},\mathcal{F}) \xrightarrow{g^*} \mathsf{Sol}(g^*\mathcal{M}, g^*\mathcal{F}) \overset{(\theta_g,\phi_g^{-1})}{\longrightarrow} \mathsf{Sol}(\mathcal{M},\mathcal{F}).$$

where $\phi_g : \mathcal{F} \xrightarrow{\simeq} g^*\mathcal{F}$, for every $g \in G$ is the system of isomorphisms coming form the equivariance property of $\mathcal{F}$. Formally, we have to consider the isomorphisms

$$\begin{aligned} \theta &: p_X^*\mathcal{M} \xrightarrow{\simeq} \alpha^*\mathcal{M}, \\ \phi &: p_X^*\mathcal{F} \xrightarrow{\simeq} \alpha^*\mathcal{F}, \end{aligned}$$

using these we can define

$$\mathsf{Sol}(\mathcal{M},\mathcal{F}) \xrightarrow{\alpha^*} \mathsf{Hom}_{\mathcal{D}^{\mathsf{vert}}_{G\times X}}(\alpha^*\mathcal{M}, \alpha^*\mathcal{F}) \overset{(\theta,\phi^{-1})}{\longrightarrow} \mathsf{Hom}_{\mathcal{D}^{\mathsf{vert}}_{G\times X}}(p_X^*\mathcal{M}, p_X^*\mathcal{F}),$$

finally, we have an injective map

$$\mathsf{Hom}_{\mathcal{D}^{\mathsf{vert}}_{G\times X}}(p_X^*\mathcal{M}, p_X^*\mathcal{F}) \hookrightarrow p_X^{\cdot}\mathsf{Sol}(\mathcal{M},\mathcal{F}),$$

altogether, we obtain a map

$$\mathsf{Sol}(\alpha) : \mathsf{Sol}(\mathcal{M},\mathcal{F}) \longrightarrow p_X^{\cdot}\mathsf{Sol}(\mathcal{M},\mathcal{F}).$$

The map $\mathsf{Sol}(\alpha)$ gives the required action. Given an element $g \in G$. The restriction from $G \times X$ to the fiber $\{g\} \times X$ yields a map

$$Res_g : p_X^{\cdot}\mathsf{Sol}(\mathcal{M},\mathcal{F}) \longrightarrow \mathsf{Sol}(\mathcal{M},\mathcal{F}).$$

The composition $Res_g \circ \mathsf{Sol}(\alpha)$ gives the action of the element g on $\mathsf{Sol}(\mathcal{M},\mathcal{F})$.

1.5.2. *General setting.* We will discuss G-equivariant structures only for coherent modules over differential algebras of the form $\mathcal{D}_P$. We shall denote the group of local symmetries of P by K in order to distinguish it from the group G of "global" symmetries. Assume P is equipped with a G-action

$$\alpha : G \times P \longrightarrow P,$$

which commutes with the K action, namely $\alpha_g(pk) = \alpha_g(p)k$. We denote the induced action of G on X by $\overline{\alpha}$. Let $\mathcal{M} \in \mathsf{Coh}(\mathcal{D}_P)$.

Naive definition. We repeat the same exposition as before. As a first approximation a G-equivariant structure on $\mathcal{M}$ is a family of isomorphisms of $\mathcal{D}_P$-modules

$$\theta_g : \mathcal{M} \xrightarrow{\simeq} \overline{\alpha}_g^*\mathcal{M} = \mathcal{O}_X \bigotimes_{\overline{\alpha}_g^{\cdot}\mathcal{O}_X} \overline{\alpha}_g^{\cdot}\mathcal{M},$$

satisfying the multiplicative condition

$$\theta_{gh} = \overline{\alpha}_h^*\theta_g \circ \theta_h.$$

Note that $\overline{\alpha}_g^*\mathcal{M}$ is equipped with a $\mathcal{D}_P$-action which is generated by

- Action of functions. For every $f \in \mathcal{O}_X$, $f \triangleright (g \otimes m) = fg \otimes m$.
- Action of vector fields. For every $\beta \in \mathcal{T}_P$, $\beta \triangleright (g \otimes m) = \sigma_\beta(g) \otimes m + g \otimes \alpha_{g*}(\beta)m$.

Remark 9. *The above formulas are very similar to the standard ones. We use the map α in order to push forward K-invariant vector fields on P.*

Formal definition. Let p_P and p_G denote the projection maps from $G\times P$ to P and G respectively. Let $\mathcal{T}^{\mathsf{vert}}_{G\times P}\subset\mathcal{T}_{G\times P}$ be the Lie subalgebroid on $G\times X$ of vertical K-invariant vector fields with respect to the projection p_G. More precisely, an element $\beta\in\mathcal{T}^{\mathsf{vert}}_{G\times P}$ if and only if $p_{G*}(\beta)=0$. We denote by $\mathcal{D}^{\mathsf{vert}}_{G\times P}=\mathcal{U}(\mathcal{T}^{\mathsf{vert}}_{G\times P})$ the corresponding universal enveloping algebra. A G-equivariant structure on $\mathcal{M}$ is an isomorphism of $\mathcal{D}^{\mathsf{vert}}_{G\times P}$-modules

$$\theta:p_X^*\mathcal{M}\xrightarrow{\simeq}\overline{\alpha}^*\mathcal{M},$$

where $p_X:G\times X\to X$ is the projector on X. Note that both $p_X^*\mathcal{M}$ and $\overline{\alpha}^*\mathcal{M}$ are equipped with a natural $\mathcal{D}^{\mathsf{vert}}_{G\times P}$-action (we leave the verification of this fact to the reader). The isomorphism θ should satisfy a cocycle condition on $G\times G\times X$, identical to the one in the standard setting thus we omit it.
Examples of equivariant modules. A standard example is when $\mathcal{M}$ is of the form $\mathcal{M}=\mathcal{D}_P/\mathcal{I}$ where $\mathcal{I}\subset\mathcal{D}_P$ is a sheaf of left ideals. In this case $\mathcal{M}$ is G-equivariant if and only if $g(\mathcal{I})\subset\mathcal{I}$ for every $g\in G$. In this particular situation, $\mathcal{M}$ is called strongly G-equivariant if $\xi^{\#}\in\mathcal{I}$ for every $\xi\in\mathfrak{g}$ where $\xi^{\#}\in\Gamma(X,\mathcal{T}_P)$ is the global vector field on P associated with ξ.
Action of the group G on the solution space. If the module $\mathcal{M}$ is equipped with a G-equivariant structure then the group G acts in a natural manner on the solution space $\mathsf{Sol}(\mathcal{M},\mathcal{F})$, yielding a map

$$\mathsf{Sol}\,(\alpha):\mathsf{Sol}(\mathcal{M},\mathcal{F})\longrightarrow p_X^{\cdot}\mathsf{Sol}(\mathcal{M},\mathcal{F})$$

where $\mathcal{F}$ is any G-equivariant target $\mathcal{D}_P$-module.

2. The strong Stone-von Neumann property

In this section, the algebraic formulation of the strong Stone-von Neumann property of the Heisenberg representation is formulated. Throughout this section we use the following terminology. Let (V,ω) be a $2N$-dimensional symplectic vector space over $\mathbb{C}$ and let $Lag=Lag\,(V)$ denotes the Lagrangian Grassmanian associated to the vector space V.

2.1. **Basic differential algebras.**

2.1.1. *Differential algebras on the Lagrangian Grassmanian.*
The total algebra. Let $Fr\to Lag$ be the canonical frame bundle. An element in the fiber $Fr_{|L}$ is an ordered basis $\overrightarrow{e}=(e_1,..,e_N)$ of L. The frame bundle Fr is a right principal GL_N-bundle with the action given by

$$(e_1,..,e_N)\rightarrow\left(\sum_{i=1}^{N}g_{i1}e_i,..,\sum_{i=1}^{N}g_{iN}e_N\right)\text{ for }g=(g_{ij})_{1\leq i,j\leq N}.$$

Let $\mathcal{T}_{Fr}$ be the Lie algebroid of infinitesimal symmetries of Fr. Recall the definition

$$\mathcal{T}_{Fr}=\pi_*(Tan_{Fr})^{GL_N}.$$

The fibers of $\mathcal{T}_{Fr}$ can be described in linear algebraic terms. An element $\beta\in\mathcal{T}_{Fr|L}$ is a linear map

$$\beta:L\to V,$$

satisfying that $\omega_\beta = \omega(\beta(\cdot), \cdot)$ is a symmetric (possibly degenerate) bilinear form on L. The fibers of the vertical subalgebra $\mathcal{T}^{\mathsf{v}}_{Fr}$ take the form

$$\mathcal{T}^{\mathsf{v}}_{Fr|L} = \mathsf{Hom}(L, L).$$

As a consequence there exists a canonical global section $Id \in \Gamma(Lag, \mathcal{T}^{\mathsf{v}}_{Fr})$ which spans the center $Z(\mathcal{T}_{Fr})$ of $\mathcal{D}_{Fr}$.

The determinant algebra. We denote by $C \to Lag$ the canonical vector bundle on Lag with fibers $C_{|L} = L$. The top wedge product $C^{\wedge N}$ is called the *determinant* line bundle and it is denoted by Det. Let $Det^{\times} \to Lag$ be the associated principal $\mathbb{G}_m$-bundle, that is $Det^{\times}$ is the complement to the zero section in Det. Let $\mathcal{T}_{Det^{\times}}$ be the Lie algebroid of infinitesimal symmetries of $Det^{\times}$ and $\mathcal{D}_{Det^{\times}}$ be its universal enveloping algebra. The vertical subalgebra $\mathcal{T}^{\mathsf{v}}_{Det^{\times}}$ can be easily described

$$\mathcal{T}^{\mathsf{v}}_{Det^{\times}|L} = \mathsf{Hom}(\wedge^{top} L, \wedge^{top} L).$$

As a consequence there exists a canonical global section $Id \in \Gamma(Lag, \mathcal{T}^{\mathsf{v}}_{Det^{\times}})$ which spans the center $Z(\mathcal{T}_{Det^{\times}})$.

We have a surjective morphism of Lie algebroids

$$\mathcal{T}_{Fr} \twoheadrightarrow \mathcal{T}_{Det^{\times}},$$

sending the central element Id_{Fr} to $N \cdot Id_{Det^{\times}}$ with kernel $\mathcal{K}$ given by

$$\mathcal{K}_{|L} = \{\beta : L \to L : Tr(\beta) = 0\}.$$

Monodromic algebras. Since both $Z(\mathcal{D}_{Fr})$ and $Z(\mathcal{D}_{Det^{\times}})$ admit a canonical generator hence specifying a complex number $\lambda \in \mathbb{C}$ we can define the monodromic algebras

$$\begin{aligned} \mathcal{D}^{\lambda}_{Fr} &= \mathcal{D}_{Fr}/\mathcal{D}_{Fr}(Id - \lambda), \\ \mathcal{D}^{\lambda}_{Det^{\times}} &= \mathcal{D}_{Det^{\times}}/\mathcal{D}_{Det^{\times}}(Id - \lambda). \end{aligned}$$

Example 11. *We have the following natural identifications*

- *Let $\lambda = -1$. The monodromic algebra $\mathcal{D}^{-1}_{Det^{\times}}$ is canonically identified with* Diff(Det) *the sheaf of differential endomorphisms of the determinant bundle.*
- *Let $\lambda = -1/2$. The D-algebra $\mathcal{D}^{1/2}_{Det^{\times}}$ "is" the sheaf of algebras of differential operators of the virtual square root $Det^{1/2}$ of the determinant line bundle. This statement can be made precise if one restricts to an open set U where $Det_{|U}$ admits a square root $Det^{1/2}$. On such an open set we have*

$$\mathcal{D}^{-1/2}_{Det^{\times}} = \mathsf{Diff}(Det^{1/2}).$$

The case $\lambda = 1/2$ is similar but with respect the square root of the dual line bundle Det^{-1}.

2.1.2. *Differential algebras on the symplectic vector space.* Let $H = H_V = V \times \mathbb{C}$ be the Heisenberg group which is now a complex algebraic group.

The Heisenberg algebra. Considering H as merely a Z_H-principal bundle (Torsor)

$$H \xrightarrow{\pi} V,$$

We can associate to it the Lie algebroid $\mathcal{T}_H = \pi_*(Tan_H)^{Z_H}$ and its universal enveloping algebra $\mathcal{D}_H = \mathcal{U}(\mathcal{T}_H)$. Let us denote by $\hbar$ the standard generator

$(0,1) \in Z_H$ hence we can (and will) identify Z_H with the additive group $\mathbb{G}_a$. We have

$$Z(\mathcal{T}_H) = \mathbb{C} \cdot \hbar,$$
$$Z(\mathcal{D}_H) = \mathbb{C}[\hbar].$$

For any $\kappa \in \mathbb{C}$ the monodromic algebra $\mathcal{D}_H^\kappa$ is naturally identified with the sheaf of differential endomorphisms of the associated line bundle[8] $\mathcal{L}^\kappa = H \times_{Z_H} \mathbb{C}_\kappa$. Since the variety V is affine, the sheaf $\mathcal{D}_H$ is determined by the algebra D_H of its global sections. Precisely $\mathcal{D}_H = \mathcal{O}_V \bigotimes_{\Gamma(\mathcal{O}_V)} D_H$. The same relation holds of-course between $\mathcal{D}_H^\kappa$ and the algebra D_H^κ of its global sections. It will be convenient to describe D_H and D_H^κ in terms of the group structure of H. We have the following simple lemma.

Lemma 1.

$$D_H = \mathcal{U}(\mathfrak{h})^\circ \bigotimes_{\mathbb{C}[\hbar]} \mathcal{U}(\mathfrak{h}),$$
$$D_H^\kappa = D_H \bigotimes_{\mathbb{C}[\hbar]} \mathbb{C}_\kappa = \mathcal{U}^\kappa(\mathfrak{h})^\circ \bigotimes_{\mathbb{C}[\hbar]} \mathcal{U}^\kappa(\mathfrak{h}),$$

where $\mathcal{U}^\kappa(\mathfrak{h}) = \mathcal{U}(\mathfrak{h}) \bigotimes_{\mathbb{C}[\hbar]} \mathbb{C}_\kappa$ *and* $\mathcal{U}(\mathfrak{h})^\circ$ *denotes the opposite algebra to* $\mathcal{U}(\mathfrak{h})$.

2.1.3. *Differential algebras on the total space* $Lag \times V$. We have differential algebras $\mathcal{D}_{Fr\times H}$ and $\mathcal{D}_{Det^\times \times H}$ associated with the $GL_N \times Z_H$-principal bundle $Fr \times H \to Lag \times V$ and the $\mathbb{G}_m \times Z_H$-principal bundle $Det^\times \times H \to Lag \times V$ respectively. We will use the following notations

$$\mathcal{D}_{\mathsf{tot}} = \mathcal{D}_{Fr\times H},$$
$$\mathcal{D}_{\mathsf{det}} = \mathcal{D}_{Det^\times \times H}.$$

These algebras split into an exterior tensor product of the previously discussed algebras

$$\mathcal{D}_{\mathsf{tot}} = \mathcal{D}_{Fr} \boxtimes \mathcal{D}_H,$$
$$\mathcal{D}_{\mathsf{det}} = \mathcal{D}_{Det^\times} \boxtimes \mathcal{D}_H.$$

For any pair of complex numbers $\lambda, \kappa \in \mathbb{C}$ we have the monodromic algebras

$$\mathcal{D}_{\mathsf{tot}}^{\lambda,\kappa} = \mathcal{D}_{Fr}^\lambda \boxtimes \mathcal{D}_H^\kappa$$
$$\mathcal{D}_{\mathsf{det}}^{\lambda,\kappa} = \mathcal{D}_{Det^\times}^\lambda \boxtimes \mathcal{D}_H^\kappa.$$

We will also consider the partial specializations

$$\mathcal{D}_{\mathsf{tot}}^{\kappa} = \mathcal{D}_{Fr} \boxtimes \mathcal{D}_H^\kappa$$
$$\mathcal{D}_{\mathsf{det}}^{\kappa} = \mathcal{D}_{Det^\times} \boxtimes \mathcal{D}_H^\kappa.$$

2.2. **The strong Stone-von Neumann property.** In this subsection, an infinite dimensional vector bundle $\mathcal{W}^\kappa$ on Lag with a canonical projective connection will be constructed. Principally, our construction resembles that of Deligne [DE], except that here the construction is given for any dimension and uses the more elegant formalism of differential algebras what makes the considerations a bit more transparent (we hope). In this subsection and the next all spaces are considered as complex analytic with the usual analytic topology.

[8]The notation $\mathbb{C}_\kappa$ stands for the one dimensional representation of the Z_H associated to the central character $\psi_\kappa(z) = e^{\kappa z}$.

2.2.1. *The vector bundle.* The groups H and its opposite H° act on the space of global sections $\Gamma(V,\mathcal{L}^{\kappa})$ through right and left translations respectively. These actions clearly commute. We define the fiber of the vector bundle $\mathcal{W}^{\kappa}$ at a point $L\in Lag$ to be

$$\mathcal{W}^{\kappa}_{|L}= {}^{L}\Gamma(V,\mathcal{L}^{\kappa}),$$

where ${}^{L}\Gamma(V,\mathcal{L}^{\kappa})$ is the subspace of L-invariant sections when L is considered as a subgroup of H°.

Lemma 2. *The spaces $\mathcal{W}^{\kappa}_{|L}$ glue into an holomorphic (infinite dimensional) vector bundle on Lag.*

2.2.2. *Projective connection.* Our goal is to exhibit a $\mathcal{D}^{1/2}_{Det^{\times}}$-module structure on $\mathcal{W}^{\kappa}$. The strategy is to first exhibit a $\mathcal{D}_{Fr}$-action and then to show that this action factors through $\mathcal{D}^{1/2}_{Det^{\times}}$.

The main step is the construction of a linear map

$$\tau^{\kappa}:\mathcal{T}_{Fr}\longrightarrow\mathcal{U}^{\kappa}(\mathfrak{h})_{Lag}/C\cdot\mathcal{U}^{\kappa}(\mathfrak{h})_{Lag},$$

where $\mathcal{U}^{\kappa}(\mathfrak{h})_{Lag}$ is the sheaf of algebras $\mathcal{U}^{\kappa}(\mathfrak{h})_{Lag}=\mathcal{U}^{\kappa}(\mathfrak{h})\bigotimes_{\mathbb{C}}\mathcal{O}_{Lag}$ and the tautological vector bundle C is naturally considered as a subsheaf of commutative algebras in $\mathcal{U}^{\kappa}(\mathfrak{h})_{Lag}$.

Given $\beta\in\mathcal{T}_{Fr}$ and $s\in\mathcal{W}^{\kappa}$, the action of β on s is defined as follows

$$\nabla_{\beta}(s)=\beta\triangleright s+\tau(\beta)^{\flat}\triangleright s, \tag{2.1}$$

where

- $\beta\triangleright s$ is the derivative of s with respect to the vector field β. Here we consider s as a GL_N-invariant function on $Fr\times H$. Since β is a GL_N-invariant vector field hence $\beta(s)$ remains GL_N-invariant.
- $\tau(\beta)^{\flat}\triangleright s$ is an application of a right H-invariant differential operator to s. In more details, the group H acts on itself by left translations, this yields an anti-homomorphism of sheaves of algebras

  $$(\cdot)^{\flat}:\mathcal{U}(\mathfrak{h})_{Lag}\longrightarrow End(\mathcal{W}^{\kappa}),$$

  sending an element $a\in\mathcal{U}(\mathfrak{h})$ to a right H-invariant differential operator on H. The map $(\cdot)^{\flat}$ clearly factors through $\mathcal{U}^{\kappa}(\mathfrak{h})_{Lag}/C\cdot\mathcal{U}^{\kappa}(\mathfrak{h})_{Lag}$.

Construction of the map τ. it is sufficient to explain the construction on fibers. Fix a point $L\in Lag$. The fiber τ_L is a morphism of vector spaces

$$\tau_L:\mathcal{T}_{Fr|L}\longrightarrow\mathcal{U}^{\kappa}(\mathfrak{h})/L\cdot\mathcal{U}^{\kappa}(\mathfrak{h}),$$

We recall that the fiber $\mathcal{T}_{Fr|L}$ can be naturally identified with $\mathsf{Hom}^{sym}(L,V)$ where the later consists of all linear maps $\beta:L\to V$ so that $\omega_{\beta}=\omega(\beta(\cdot),\cdot)$ is a symmetric bilinear form on L. The following proposition[9] is simultaneously a characterization and a construction of τ_L.

Proposition 1. *There exists a unique map*

$$\tau_L:\mathsf{Hom}^{sym}(L,V)\to\mathcal{U}^{\kappa}(\mathfrak{h})/L\cdot\mathcal{U}^{\kappa}(\mathfrak{h}),$$

which satisfies the following two properties

[9]This is probably the main technical proposition in this paper.

(1) *(Linearity)* $\tau_L(\beta+\beta') = \tau_L(\beta) + \tau_L(\beta')$ *for every* $\beta, \beta' \in \mathsf{Hom}^{sym}(L,V)$.
(2) *(Formula) If* ω_β *is non-degenerate then* $\tau_L(\beta)$ *is given by the following formula*

$$\tau_L(\beta) = \frac{1}{2\kappa}\sum_{i=1}^{N}\beta(e_i)^2, \tag{2.2}$$

for $\overrightarrow{e} = (e_1,..,e_N)$ *an orthonormal basis*[10] *with respect to* ω_β.

Proposition 2. *The action (2.1) defines a* D_{Fr}*-action on the vector bundle* $\mathcal{W}^\kappa$.

Proposition 3. *The* $\mathcal{D}_{Fr}$*-action on* $\mathcal{W}^\kappa$ *induced from (2.1) factors through a* $\mathcal{D}^{1/2}_{Det^\times}$*-action.*

2.3. **Function spaces.** In this subsection we show how to obtain various interesting function spaces from $\mathcal{W}^\kappa$.

2.3.1. *Fundamental gerbe.* We will not discuss to any depth the notion of a gerbe in this paper. It is enough to our purposes to say that a gerbe on a variety X is a sheaf of groupoids satisfying certain local triviality conditions. Roughly speaking, a gerbe might be thought of as a categorical analogue of a line bundle. A particular gerbe which plays a prominent role in our context is the gerbe of square roots [We] of the canonical line bundle Det. This gerbe will be denoted by $\mathfrak{Det}^{1/2}$. It is defined as follows. For every open set $U \subset Lag$, the groupoid $\mathfrak{Det}^{1/2}(U)$ of sections over U is

- An object $\Upsilon \in \mathfrak{Det}^{1/2}(U)$ is a pair $(Det^{1/2}, \alpha)$ where $Det^{1/2}$ is a line bundle on U and α is an isomorphism
$$\alpha : Det^{1/2} \bigotimes Det^{1/2} \xrightarrow{\simeq} Det_{|U}.$$
- A morphism $\theta : \Upsilon \to \Upsilon'$ is an isomorphism of line bundles
$$\theta : Det^{1/2} \xrightarrow{\simeq} Det^{1/2\prime},$$
satisfying $\theta^{\otimes 2} \circ \alpha = \alpha' \circ \theta^{\otimes 2}$.

In the same manner we define the dual gerbe $\mathfrak{Det}^{-1/2}$ of square roots of the dual line bundle Det^{-1}.

2.3.2. *Flat sections.* Given an open set $U \subset Lag$ we would like to consider the space "$\Gamma_{flat}(U, \mathcal{W}^\kappa)$" of flat sections of $\mathcal{W}^\kappa$. In order to do that we need to linearize the connection. This can be formally done as follows. Choose an object $\Upsilon = (Det^{1/2}, \alpha) \in \mathfrak{Det}^{1/2}(U)$. Since the line bundle $Det^{1/2}$ is naturally a $\mathcal{D}^{-1/2}_{Det^\times}$-module, the tensor product $Det^{1/2} \bigotimes_{\mathcal{O}} \mathcal{W}^\kappa$ is a plain $\mathcal{D}_U$-module. The space of flat section is defined by

$$\begin{aligned} \mathcal{H}^\kappa_\Upsilon &= \Gamma_{flat}(U, Det^{1/2} \textstyle\bigotimes_{\mathcal{O}} \mathcal{W}^\kappa_{|U}) \\ &\triangleq \mathsf{Hom}_{\mathcal{D}_U}(\mathcal{O}_U, Det^{1/2} \textstyle\bigotimes_{\mathcal{O}} \mathcal{W}^\kappa_{|U}). \end{aligned}$$

Heisenberg action. The (complex) Heisenberg group H acts on the fibers of $\mathcal{W}^\kappa$ by right translations thus commuting with the $\mathcal{D}^{1/2}_{Det^\times}$-action and hence it induces an action of H on the space $\mathcal{H}^\kappa_\Upsilon$.

[10] As will be proved below, formula (2.2) does not depend on the choice of the basis $\overrightarrow{e}$.

2.4. **The Weil D-module.** The vector bundle $\mathcal{W}^\kappa$ consists of a mixture of algebraic structure (projective connection) and analytic structure (the fibers of the bundle $\mathcal{W}^\kappa$). The main idea is to separate as much as possible between these two structures. This is done as follows. The projective connection will encoded as an (algebraic) $\mathcal{D}_{\mathsf{det}}^{-1/2,\kappa}$-module $\mathcal{M}^\kappa$ such that the analytic space $\mathcal{H}_\Upsilon^\kappa$ of "flat" sections is realized as a solution space of $\mathcal{M}^\kappa$ in an appropriate target module $\mathcal{F}$.

In this section all spaces are algebraic.

2.4.1. *Construction of the module $\mathcal{M}^\kappa$.* The strategy is to construct $\mathcal{M}^\kappa$ as a $\mathcal{D}_{\mathsf{tot}}^\kappa$-module and then to show that the $\mathcal{D}_{\mathsf{tot}}^\kappa$-action factors through a $\mathcal{D}_{\mathsf{det}}^{-1/2,\kappa}$-action. Define

$$\mathcal{M}^\kappa = \mathcal{D}_{\mathsf{tot}}^\kappa / \mathcal{I}^\kappa.$$

Here $\mathcal{I}^\kappa \subset \mathcal{D}_{\mathsf{tot}}^\kappa$ is a sheaf of left ideals defined as follows

$$\mathcal{I}^\kappa = (c^\flat, \beta + \widetilde{\tau}(\beta)^\flat : c \in C, \beta \in \mathcal{T}_{Fr}). \tag{2.3}$$

where

- The element $\widetilde{\tau}(\beta)$ is any lifting of $\tau(\beta)$ to $\mathcal{U}(\mathfrak{h})_{Lag}$.
- The elements $c^\flat \in \mathcal{D}_{\mathsf{tot}}^\kappa$ and $\widetilde{\tau}(\beta)^\flat$ are defined using the anti-linear homomorphism

  $$(\cdot)^\flat : \mathcal{U}(\mathfrak{h})_{Lag} \longrightarrow pr_{Lag*}(\mathcal{D}_{\mathsf{tot}}^\kappa) = \mathcal{D}_{Fr}^\kappa \bigotimes_{\mathbb{C}} \left(\mathcal{U}^\kappa(\mathfrak{h})^\circ \bigotimes_{\mathbb{C}[h]} \mathcal{U}^\kappa(\mathfrak{h}) \right),$$

  sending an element $a \in \mathcal{U}(\mathfrak{h})$ to the right H-invariant differential operator $a^\flat = 1 \otimes (a \otimes 1)$.

Remark 10. *It is important to note here that the ideal $\mathcal{I}^\kappa$ is well defined and does not depend on the choice of the lifting $\widetilde{\tau}$.*

Proposition 4. *The $\mathcal{D}_{\mathsf{tot}}^\kappa$-action on $\mathcal{M}^\kappa$ factors through $\mathcal{D}_{\mathsf{det}}^{-1/2,\kappa}$.*

2.5. **Solution spaces.** We will use the superscripts $(\cdot)^{an}$ and $(\cdot)^\infty$ to denote sheaves of analytic and C^∞ type respectively.

2.5.1. *Solution spaces of holomorphic type.* Specify the following data

- Let $j : U \hookrightarrow Lag^{an}$ be an open set in the analytic topology.
- Let $\Upsilon = (Det^{1/2}, \theta) \in \mathfrak{Det}^{1/2}(U)$ be a square root of the determinant line bundle Det^{an} on U.

Define

$$\mathcal{F}_\Upsilon^{an}(1/2, \kappa) = j_* \left(Det^{1/2} \right) \boxtimes \mathcal{L}^\kappa,$$

The sheaf $\mathcal{F}_\Upsilon^{an}(1/2, \kappa)$ is equipped with a natural action of $\mathcal{D}_{\mathsf{det}}^{-1/2,\kappa}$. In the same manner we define $\mathcal{F}_\Upsilon^{an}(-1/2, \kappa)$ for $\Upsilon \in \mathfrak{Det}^{-1/2}(U)$. There exists a natural pairing map

$$\mathcal{F}_\Upsilon^{an}(1/2, \kappa) \bigotimes_{\mathcal{O}} \mathcal{F}_\Upsilon^{an}(-1/2, -\kappa) \longrightarrow j_* \mathcal{O}_U^{an} \boxtimes \mathcal{O}_V^{an}.$$

We define the solution space

$$\mathcal{H}_\Upsilon^{an}(\kappa) = \mathsf{Sol}(\mathcal{M}^\kappa, \mathcal{F}_\Upsilon^{an}(1/2, \kappa)).$$

When Υ, κ or $\lambda = \pm 1/2$ are clear from the context they will be omitted from the notation.

2.5.2. *Solution spaces of C^{∞} type.* Fix a choice of a real structure on V. Specify the following data

- Let $\Upsilon = (Det^{1/2}, \theta) \in \mathfrak{Det}^{1/2}(Lag(\mathbb{R}))$ be a square root of the (complexified) determinant line bundle on the manifold $Lag(\mathbb{R})$.
- Assume the central weight κ is pure imaginary, $\kappa \in i\mathbb{R}$. Let $\mathcal{L}^{\kappa} = H(\mathbb{R}) \times_{Z_{H(\mathbb{R})}} \mathbb{C}_{\kappa}$ be the associated line bundle to the central character ψ_{κ}.

Define

$$\mathcal{F}^{\infty}_{\Upsilon}(1/2, \kappa) = \mathsf{ex}^{*}\left(Det^{1/2} \boxtimes \mathcal{L}^{\kappa}\right).$$

The sheaf $\mathcal{F}^{\infty}_{\Upsilon}(1/2, \kappa)$ is equipped with an action of $\mathcal{D}_{\mathsf{det}}^{-1/2,\kappa}$. In the same manner we define $\mathcal{F}^{\infty}_{\Upsilon}(-1/2, \kappa)$ for $\Upsilon \in \mathfrak{Det}^{-1/2}(Lag(\mathbb{R}))$. There exists a natural pairing map

$$\mathcal{F}^{\infty}_{\Upsilon}(1/2, \kappa) \bigotimes\nolimits_{\mathcal{O}} \mathcal{F}^{\infty}_{\Upsilon}(-1/2, -\kappa) \longrightarrow \mathsf{ex}^{*} C^{\infty}_{Lag(\mathbb{R})}. \tag{2.4}$$

We denote by $\mathcal{H}^{\infty}_{\Upsilon}(\kappa)$ the solution space

$$\mathcal{H}^{\infty}_{\Upsilon}(\kappa) = \mathsf{Sol}(\mathcal{M}^{\kappa}, \mathcal{F}^{\infty}_{\Upsilon}(1/2, \kappa)).$$

When Υ , κ or $\lambda = \pm 1/2$ are clear from the context they will be omitted from the notation.

2.6. **Explicit formulas.** The constructions given so far are rather abstract. It might hold a certain pedagogical value if we write some explicit formulas. Assume $N = 1$. We will show that under appropriate trivializations, the module $\mathcal{M}^{\kappa}$ is equivalent to the honest D-module associated to the heat equation.

Choose coordinates $V \simeq \mathbb{C}^2 = \{(x, y)\}$ such that ω becomes the standard symplectic form $\omega = dx \wedge dy$. Consider the map

$$f : \mathbb{C} \times \mathbb{C}^2 \longrightarrow Fr \times H,$$

given by $f(t, (x, y)) = (e_t, (x, y, 0))$ where $e_t = (1, t) \in Fr$. A direct computation reveals that

$$f^{*}\mathcal{M}^{\kappa} = \mathcal{D}_{\mathbb{C}\times\mathbb{C}^2}/\mathcal{D}_{\mathbb{C}\times\mathbb{C}^2}(X, H),$$

where

$$\begin{aligned} H &= \partial_t - \frac{1}{2\kappa}\left(\partial_y - \frac{\kappa}{2}x\right)^2, \\ X &= \partial_x - \frac{\kappa t x}{2} + t\partial_y + \frac{\kappa y}{2}. \end{aligned}$$

Consider the map

$$g : \mathbb{C}^2 \to \mathbb{C} \times \mathbb{C}^2,$$

given by $g(t, y) = (t, (0, y))$. A direct computation reveals

$$\mathcal{N}^{\kappa} := g^{*} f^{*} \mathcal{M}^{\kappa} = \mathcal{D}_{\mathbb{C}^2}/\mathcal{D}_{\mathbb{C}^2}(\partial_t - \frac{1}{2\kappa}\partial_y^2),$$

which is the D-module associated with the heat differential operator. Denote by h the composition $f \circ g : \mathbb{C}^2 \longrightarrow Fr \times H$. Let U be the open set $U = \mathrm{Im}\,(pr_{Lag} \circ f)$.

Concretely, U is the the complement to the south pole of the projective line $\mathbb{P}^1_{\mathbb{C}}$. Let $\Upsilon \in \mathfrak{Det}^{1/2}(U)$. On the level of solution spaces we have the following identifications.

$$\begin{aligned} \mathcal{H}^{an} &= \mathsf{Hom}_{\mathcal{D}^{1/2,\kappa}_{\mathsf{det}}}(\mathcal{M}^{\kappa}, \mathcal{F}^{an}) \\ &\xrightarrow{\simeq} \mathsf{Hom}_{\mathcal{D}_{\mathbb{C}^2}}(h^*\mathcal{M}^{\kappa}, h^*\mathcal{F}^{an}) \\ &\xrightarrow{\simeq} \mathsf{Hom}_{\mathcal{D}_{\mathbb{C}^2}}(\mathcal{N}^{\kappa}, \mathcal{O}^{an}_{\mathbb{C}^2}), \end{aligned} \tag{2.5}$$

where the last term in (2.5) is the solution space of the heat equation $\partial_t - \frac{1}{2\kappa}\partial_y^2$.

2.7. **The Heisenberg representation.** The action of the Heisenberg Lie algebra can be reconstructed from the algebraic structure of the module $\mathcal{M}^{\kappa}$. More precisely we have a map of algebras

$$\Pi^{\kappa} : \mathcal{U}(\mathfrak{h})^{\circ} \longrightarrow End_{\mathcal{D}^{\kappa}_{\mathsf{tot}}}(\mathcal{M}^{\kappa}),$$

sending an element $a \in \mathcal{U}(\mathfrak{h})$ to the endomorphism $\Pi^{\kappa}(a) : \mathcal{M}^{\kappa} \longrightarrow \mathcal{M}^{\kappa}$ defined by right multiplication with the element $1 \otimes 1 \otimes a \in \mathcal{D}^{\kappa}_{\mathsf{tot}}$

$$\Pi^{\kappa}(a)(m) = m \cdot (1 \otimes 1 \otimes a). \tag{2.6}$$

Note that since the ideal $\mathcal{I}^{\kappa}$ is defined in terms of elements of the form $1 \otimes a \otimes 1$, formula (2.6) indeed yields an endomorphism. On the level of solution spaces the infinitesimal action of $\mathfrak{h} = Lie(H)$ on $\mathcal{H}^{an/\infty}$ can be written in terms of Π^{κ}. More precisely, given an element $\xi \in \mathfrak{h}$ we have two actions

$$\begin{aligned} \Pi^{\kappa}(\xi) &: \mathcal{H}^{an/\infty} \longrightarrow \mathcal{H}^{an/\infty}, \\ d\pi^{\kappa}(\xi) &: \mathcal{H}^{an/\infty} \longrightarrow \mathcal{H}^{an/\infty}, \end{aligned}$$

which are defined as follows. Let $\varphi \in \mathcal{H}^{an/\infty} = \mathsf{Sol}\left(\mathcal{M}^{\kappa}, \mathcal{F}^{an/\infty}\right)$ be a vector in the solution space. The first action is defined by composition $\Pi^{\kappa}(\xi)\varphi = \varphi \circ \Pi^{\kappa}(\xi)$. In order to define the second action, it is enough to specify $d\pi^{\kappa}(\xi)\varphi$ at the point $1 \in \mathcal{M}^{\kappa}$. We take

$$d\pi^{\kappa}(\xi)\varphi : 1 \longmapsto R_{\xi}(\varphi(1)),$$

where the last expression is taking the derivative of $\varphi(1)$ with respect to the left invariant vector field associated with ξ.

Lemma 3. *for every* $\xi \in \mathfrak{h}$

$$\Pi^{\kappa}(\xi) = d\pi^{\kappa}(\xi).$$

Proof. The proof is obvious. □

2.8. **Duality.** In this subsection we state one of the main results of this paper regarding the duality relation between the Weil D-modules $\mathcal{M}^{\kappa}$ and $\mathcal{M}^{-\kappa}$. This duality is given in terms of the Verdier duality functor. Recall that in general we have a functor

$$\mathbb{D} : \mathsf{DCoh}(\mathcal{D}^{\lambda,\kappa}_{\mathsf{det}}) \longrightarrow \mathsf{DCoh}(\mathcal{D}^{-\lambda,-\kappa}_{\mathsf{det}}).$$

In the case $\lambda = -1/2$, it will be convenient to consider a twisted duality functor

$$\widetilde{\mathbb{D}} : \mathsf{DCoh}(\mathcal{D}^{-1/2,\kappa}_{\mathsf{det}}) \longrightarrow \mathsf{DCoh}(\mathcal{D}^{-1/2,-\kappa}_{\mathsf{det}}),$$

which is defined by

$$\widetilde{\mathbb{D}}(\mathcal{M}) = pr^*_{Lag}Det \bigotimes_{\mathcal{O}} \mathbb{D}(\mathcal{M}).$$

Note that since Det is naturally a $\mathcal{D}^{-1}_{Det^{\times}}$-module, therefore, tensoring with it gives a functor from $\mathsf{DCoh}(\mathcal{D}^{1/2,\kappa}_{\mathsf{det}})$ to $\mathsf{DCoh}(\mathcal{D}^{-1/2,\kappa}_{\mathsf{det}})$.

Theorem 12. *The module $\mathcal{M}^{\kappa}$ is Cohen-Macaulay. Moreover, there exists a canonical isomorphism $\widetilde{\mathbb{D}}(\mathcal{M}^{\kappa}) \xrightarrow{\simeq} \mathcal{M}^{-\kappa}[N]$.*

2.9. **Equivariance structure.** We consider $\mathcal{M}^{\kappa}$ as a coherent module over the algebra $\mathcal{D}^{\kappa}_{\mathsf{tot}}$. Recall that $\mathcal{D}_{\mathsf{tot}}$ is associated with the $G = GL_N \times \mathbb{G}_a$ principle bundle $Fr \times H \rightarrow Lag \times V$. Since the symplectic group Sp acts on the bundle $Fr \times H$ and this action commutes with the action of the local symmetry group G, the notion of Sp-equivariant object in the category $\mathsf{Coh}(\mathcal{D}^{\kappa}_{\mathsf{tot}})$ makes sense.

Proposition 5. *The module $\mathcal{M}^{\kappa}$ is equipped with a natural Sp-equivariant structure.*

3. Applications

In this section, several applications of the strong Stone-von Neumann property are established. First application concerns the existence of a canonical pairing between various solution spaces. As a corollary, an affirmative answer to a question of Deligne is obtained. Second application is to the construction of the Weil representation of the real symplectic group, it will be shown that the strong S-vN property directly implies the metaplectic sign.

In this section, it will be convenient to assume that (V, ω) is defined over $\mathbb{R}$, namely V is a smooth scheme over $\mathbb{R}$ with

$$\omega : V \times V \longrightarrow \mathbb{A}^1_{\mathbb{R}},$$

a skew symmetric morphism of schemes. As a consequence all associated spaces and groups that we consider are smooth schemes over $\mathbb{R}$.

3.1. **Canonical pairings.**

3.1.1. *Canonical pairing between holomorphic solution spaces.* Specify the following data

- An analytic open set $U \subset Lag\,(\mathbb{C})^{an}$ such that $H_N(U, \mathbb{C})$ is non trivial.
- An analytic square root $\Upsilon = (Det^{1/2}, \theta) \in \mathfrak{Det}^{1/2}(U)$.

Theorem 13. *Given a non-trivial class $\gamma \in H_N(U, \mathbb{C})$, there exists a natural non-degenerate $H\,(\mathbb{C})$-invariant pairing*

$$B_{\gamma} : \mathcal{H}^{an}\,(\kappa) \times \mathcal{H}^{an}\,(-\kappa) \longrightarrow \mathbb{C}.$$

Concrete formulas. It might hold a certain pedagogical value to write the paring B_{γ} in concrete coordinates. For doing this, let us assume that $N = 1$. We choose coordinates $V \simeq \mathbb{C}^2$ so that ω becomes the standard symplectic form $\omega = dx \wedge dy$. The variety $Lag\,(\mathbb{C})$ is identified with the projective line $P^1_{\mathbb{C}}$. Consider the maps

$$\begin{aligned} f &: \mathbb{C}^{\times} \times \mathbb{C}^2 \longrightarrow Fr\,(\mathbb{C}) \times H\,(\mathbb{C})\,, \\ g &: \mathbb{C}^{\times} \times \mathbb{C} \longrightarrow \mathbb{C}^{\times} \times \mathbb{C}^2, \\ h &= f \circ g : \mathbb{C}^{\times} \times \mathbb{C} \longrightarrow Fr\,(\mathbb{C}) \times H\,(\mathbb{C})\,, \end{aligned}$$

where f is given by $f(t, x, y) = (e_t, (x, y, 0))$ and g is given by $g(t, y) = (t, (0, y))$. Using the map h we can pull-back the modules $\mathcal{M}^{\pm\kappa}$ to honest D-modules on $\mathbb{C}^{\times} \times \mathbb{C}$. We define

$$\begin{aligned} \mathcal{N}^{\kappa} &= h^{*}\mathcal{M}^{\kappa}, \\ \mathcal{N}^{-\kappa} &= h^{*}\mathcal{M}^{-\kappa}. \end{aligned}$$

A direct calculation reveals that

$$\begin{aligned}\mathcal{N}^{\kappa} &\simeq \mathcal{D}_{\mathbb{C}^{\times}\times\mathbb{C}}/\mathcal{D}_{\mathbb{C}^{\times}\times\mathbb{C}}\left(\partial_t - \tfrac{1}{2\kappa}\partial_y^2\right),\\ \mathcal{N}^{-\kappa} &\simeq \mathcal{D}_{\mathbb{C}^{\times}\times\mathbb{C}}/\mathcal{D}_{\mathbb{C}^{\times}\times\mathbb{C}}\left(\partial_t + \tfrac{1}{2\kappa}\partial_y^2\right).\end{aligned}$$

That is, $\mathcal{N}^{\kappa}$ is the D-module associated to the Heat equation and $\mathcal{N}^{-\kappa}$ is associated to its transposed. On the level of solution spaces, the pull-back functor induces isomorphisms

$$\begin{aligned}h^* &: \mathsf{Sol}\left(\mathcal{M}^{\kappa}, \mathcal{F}^{an}\left(1/2,\kappa\right)\right) \xrightarrow{\simeq} \mathsf{Sol}(\mathcal{N}^{\kappa}, \mathcal{O}^{an}_{\mathbb{C}^{\times}\times\mathbb{C}}),\\ h^* &: \mathsf{Sol}\left(\mathcal{M}^{-\kappa}, \mathcal{F}^{an}(1/2,-\kappa\right) \xrightarrow{\simeq} \mathsf{Sol}(\mathcal{N}^{-\kappa}, \mathcal{O}^{an}_{\mathbb{C}^{\times}\times\mathbb{C}}).\end{aligned}$$

We would like to write the pairing B_{γ} in terms of the concrete spaces

$$\begin{aligned}\mathcal{H}^{\kappa} &\triangleq \mathsf{Sol}(\mathcal{N}^{\kappa}, \mathcal{O}^{an}_{\mathbb{C}^{\times}\times\mathbb{C}}),\\ \mathcal{H}^{-\kappa} &\triangleq \mathsf{Sol}(\mathcal{N}^{-\kappa}, \mathcal{O}^{an}_{\mathbb{C}^{\times}\times\mathbb{C}})\end{aligned}$$

For every pair of solutions

$$\begin{aligned}\nu &\in \mathsf{Sol}\left(\mathcal{M}^{\kappa}, \mathcal{F}^{an}\left(1/2,\kappa\right)\right),\\ \varphi &\in \mathsf{Sol}\left(\mathcal{M}^{-\kappa}, \mathcal{F}^{an}\left(1/2,-\kappa\right)\right),\end{aligned}$$

we have

$$h^*\left(\varphi\otimes\nu(G_{\mathcal{M}^{\kappa}})\right) = h^*\varphi\otimes h^*\nu(h^*G_{\mathcal{M}^{\kappa}}).$$

By functoriality we have $h^*(G_{\mathcal{M}^{\kappa}}) = G_{\mathcal{N}^{\kappa}}$, if we denote by $\widetilde{\nu}$ and $\widetilde{\varphi}$ the pullbacks $h^*\nu \in \mathcal{H}^{\kappa}$ and $h^*\varphi \in \mathcal{H}^{-\kappa}$ respectively we can write

$$h^*\left(\varphi\otimes\nu(G_{\mathcal{M}^{\kappa}})\right) = \widetilde{\varphi}\otimes\widetilde{\nu}(G_{\mathcal{N}^{\kappa}}).$$

Let us compute the Green class $G_{\mathcal{N}^{\kappa}}$. A direct calculation reveals that $\mathbb{D}\mathcal{N}^{\kappa} \simeq \mathcal{N}^{-\kappa}[1]$ therefore we have

$$\begin{aligned}G_{\mathcal{N}^{\kappa}} &\in R^1\mathsf{Hom}_{\mathcal{D}_{\mathbb{C}^{\times}\times\mathbb{C}}}(\mathcal{O}_{\mathbb{C}^{\times}\times\mathbb{C}}, \mathcal{N}^{-\kappa}\bigotimes_{\mathcal{O}}\mathcal{N}^{\kappa})\\ &= H^1(DR^{\bullet}(\mathcal{N}^{-\kappa}\bigotimes_{\mathcal{O}}\mathcal{N}^{\kappa})).\end{aligned}$$

A direct calculation reveals that $G_{\mathcal{N}^{\kappa}}$ is represented by the following explicit chain in $DR^1(\mathcal{N}^{-\kappa}\bigotimes_{\mathcal{O}}\mathcal{N}^{\kappa})$

$$G_{\mathcal{N}^{\kappa}} = dy\otimes 1\otimes 1 - \frac{1}{2\kappa}\left(dt\otimes\partial_y\otimes 1 - dt\otimes 1\otimes\partial_y\right).$$

Let us fix a curve $\gamma : S^1 \longrightarrow \mathbb{C}^{\times}\times\mathbb{C}$ representing a non-trivial element in $H_1(\mathbb{C}^{\times}\times\mathbb{C},\mathbb{C})$, for example we can take $\gamma(t) = (t,0)$. If we denote by $r = \widetilde{\nu}(1)$ and by $s = \widetilde{\varphi}(1)$ the corresponding holomorphic functions on $\mathbb{C}^{\times}\times\mathbb{C}$, we can write

$$\begin{aligned}B_{\gamma}(r,s) &= \int_{\gamma}\widetilde{\varphi}\otimes\widetilde{\nu}(G_{\mathcal{N}^{\kappa}})\\ &= \int_{t\in S^1}\left(\partial_y s(t,0)\cdot r(t,0) - s(t,0)\partial_y r(t,0)\right)dt.\end{aligned}$$

3.1.2. *Canonical pairing between C^{∞} function spaces.* Specify the following data

- A C^{∞}-square root $\Upsilon = (Det^{1/2}, \theta) \in \mathfrak{Det}^{1/2}(Lag\,(\mathbb{R}))$.
- Assume that the central weight κ is pure imaginary, $\kappa \in i\mathbb{R}$.

Theorem 14. *For every non-trivial class $\gamma \in H_N(Lag\,(\mathbb{R})\,, \mathbb{C})$, there exists a natural non-degenerate $H\,(\mathbb{R})$-invariant pairing*

$$B_{\gamma} : \mathcal{H}^{\infty}\,(\kappa) \times \mathcal{H}^{\infty}\,(-\kappa) \longrightarrow \mathbb{C}.$$

3.2. **Deligne's question.** Deligne's question concerns the following situation. Assume $N = 1$. Let $\gamma : S^1 \to Lag\,(\mathbb{C})$ be a curve so that $Lag\,(\mathbb{C}) \setminus \mathrm{Im}(\gamma)$ is a disjoint union of two contractible open sets U_+ and U_-. Let us denote by $Z_{\pm}$ the closed subsets $Z_{\pm} = U_{\pm} \cup \mathrm{Im}\,\gamma$ respectively. Choose objects $\Upsilon^{\pm} \in \mathfrak{Det}^{1/2}(Z_{\pm})$ and an isomorphism

$$\theta : \Upsilon^{+}_{|\,\mathrm{Im}\,\gamma} \overset{\simeq}{\longrightarrow} \Upsilon^{-}_{|\,\mathrm{Im}\,\gamma}.$$

Deligne's question: Does there exists a canonical $H\,(\mathbb{C})$-invariant pairing

$$B_{\theta,\gamma} : \mathcal{H}^{an}_{\Upsilon^+}\,(\kappa) \times \mathcal{H}^{an}_{\Upsilon^-}\,(-\kappa) \to \mathbb{C}. \tag{3.1}$$

We will formulate and prove a generalized variant of (3.1).

Specify the following data

- Open cover $Lag^{an}\,(\mathbb{C}) = U_+ \cup U_-$.
- Objects $\Upsilon^{\pm} \in \mathfrak{Det}^{1/2}(U_{\pm})$
- Isomorphism

$$\theta : \Upsilon^{+}_{|U_+ \cap U_-} \overset{\simeq}{\longrightarrow} \Upsilon^{-}_{|U_+ \cap U_-}.$$

Assume $H_N(U_+ \cap U_-, \mathbb{C}) \neq 0$ and let $\gamma \in H_N(U_+ \cap U_-, \mathbb{C})$ be a non trivial class.

Theorem 15. *There exists a non-trivial $H\,(\mathbb{C})$-invariant paring*

$$B_{\theta,\gamma} : \mathcal{H}^{an}_{\Upsilon^+}\,(\kappa) \times \mathcal{H}^{an}_{\Upsilon^-}\,(-\kappa) \to \mathbb{C}. \tag{3.2}$$

3.2.1. *The Heisenberg representation (analytic models).* An important application of (3.2) appears when the decomposition $Lag = U_+ \cup U_-$ is compatible with the real structure on V. More precisely, assume $\overline{U}_+ = U_-$ and $U_+ \cap U_-$ is an homotopic retract of $Lag\,(\mathbb{R})$, hence

$$H_*\,(U_+ \cap U_-, \mathbb{C}) = H_*\,(Lag\,(\mathbb{R})\,, \mathbb{C})\,.$$

Fix a non-trivial class $\gamma \in H_N\,(Lag\,(\mathbb{R})\,, \mathbb{C})$. In this particular situation (3.2) implies the existence of an $H\,(\mathbb{R})$-invariant Hilbertian structure on $\mathcal{H}^{an}_{\Upsilon^+}\,(\kappa)$, which is obtained as follows. The Galois action defines an anti-complex isomorphism

$$\overline{(\cdot)} : \mathcal{H}^{an}_{\Upsilon^+}\,(\kappa) \to \mathcal{H}^{an}_{\Upsilon^-}\,(-\kappa)\,.$$

We obtain an $H\,(\mathbb{R})$-invariant Hermitian structure

$$\langle \cdot, \cdot \rangle_{\theta,\gamma} : \mathcal{H}^{an}_{\Upsilon^+}\,(\kappa) \times \mathcal{H}^{an}_{\Upsilon^+}\,(\kappa) \to \mathbb{C},$$

given by

$$\langle \nu, \varphi \rangle_{\theta,\gamma} = B_{\gamma,\theta}(\nu, \overline{\varphi}) = \int\limits_{\gamma} G\,(\nu, \overline{\varphi})\,.$$

Summary: This example establishes an Hilbertian model $\left(\pi^{\kappa}_{U_+}, H, \mathcal{H}^{\kappa}_{U_+}\right)$ of the Heisenberg representation consisting of a certain class of H-holomorphic vectors. Interestingly, a large variety of such models appears depending on the choice of the open set U_+. The larger U_+ the smaller the model $\mathcal{H}^{\kappa}_{U_+}$ is.

3.3. **The Weil representation over the reals.** In this subsection we explain how the strong S-vN property implies the existence of the Weil representation of the real symplectic group. Our plan is to construct a unitary representation of a double cover Mp of the real symplectic group $Sp(\mathbb{R})$. First, we construct the Hilbertian space $\mathcal{H}^{\kappa}$. Second, we define the double cover Mp and exhibit its action on $\mathcal{H}^{\kappa}$. Finally we show that this action is unitary.

3.3.1. *Construction of the Hilbertian space.* We need to specify the following data

- A square root $\Upsilon = (Det^{1/2}, \alpha) \in \mathfrak{Det}^{1/2}(Lag(\mathbb{R}))$.
- A non-trivial class $\gamma \in H_N(Lag(\mathbb{R}), \mathbb{C})$.
- Assume the central weight κ is purely imaginary, that is $\kappa \in i\mathbb{R}$.

We define the vector space $\mathcal{H}^{\kappa}$ to be

$$\mathcal{H}^{\infty}(\kappa) = \mathsf{Sol}(\mathcal{M}^{\kappa}, \mathcal{F}^{\infty}(\kappa)).$$

Theorem 14 implies that there exists a canonical pairing

$$B_{\gamma} : \mathcal{H}^{\infty}(\kappa) \times \mathcal{H}^{\infty}(-\kappa) \longrightarrow \mathbb{C}.$$

We have the following simple lemma

Lemma 4. *The Galois action defines an anti-linear isomorphism*

$$\epsilon : \mathcal{H}^{\infty}(\kappa) \xrightarrow{\simeq} \mathcal{H}^{\infty}(-\kappa), \tag{3.3}$$

For every $\nu, \varphi \in \mathcal{H}^{\infty}(\kappa)$, we define the Hermitian product $\langle \nu, \varphi \rangle_{\gamma}$ to be

$$\langle \nu, \varphi \rangle_{\gamma} \triangleq B_{\gamma}(\nu, \epsilon(\varphi)),$$

3.3.2. *Construction of the metaplectic group.* The metaplectic group Mp consists of pairs $(g, \theta_g^{1/2})$ where g is an element in $Sp(\mathbb{R})$ and $\theta_g^{1/2}$ is an isomorphism

$$\theta_g^{-1/2} : Det^{1/2} \xrightarrow{\simeq} g^* Det^{1/2},$$

satisfying

$$g^*\alpha \circ \left(\theta_g^{-1/2} \otimes \theta_g^{-1/2}\right) \circ \alpha^{-1} = can_g,$$

where can_g is the canonical isomorphism $can_g : Det \xrightarrow{\simeq} g^* Det$ coming from the natural $Sp(\mathbb{R})$-equivariance structure of Det. The group structure is given by

$$(g, \theta_g^{1/2}) \circ (h, \theta_h^{1/2}) = (gh, h^*\theta_g \circ \theta_h),$$

Forgetting the isomorphism $\theta_g^{1/2}$ we obtain a cover $Mp \longrightarrow Sp_{\mathbb{R}} \longrightarrow 1$.The kernel is the group consisting of isomorphisms

$$\theta^{1/2} : Det^{1/2} \xrightarrow{\simeq} Det^{1/2}$$

satisfying $\alpha \circ \left(\theta^{1/2} \otimes \theta^{1/2}\right) \circ \alpha^{-1} = Id$, hence it is isomorphic to $\mathbb{Z}_2$. Concluding, Mp is a double cover of the group $Sp(\mathbb{R})$.

3.3.3. *Construction of the metaplectic action.* The group Mp naturally acts on the vector space $\mathcal{H}^{\kappa}$ preserving the Hermitian product. The action is obtained using the standard construction described in 1.5 using the following two simple facts

- The $\mathcal{D}^{\kappa}_{\mathsf{tot}}$ module $\mathcal{F}^{\infty}(\kappa)$ is equipped with a natural Mp-equivariant structure. This fact principally follows from the definition of Mp.
- The $\mathcal{D}^{\kappa}_{\mathsf{tot}}$ module $\mathcal{M}^{\kappa}$ is equipped with a natural Mp-equivariant structure. This is a concequence of the fact that $\mathcal{M}^{\kappa}$ is already $Sp(\mathbb{C})$-equivariant.

As a result we obtain a map

$$\rho : \mathsf{Sol}(\mathcal{M}^{\kappa}, \mathcal{F}^{\infty}(1/2, \kappa)) \longrightarrow p^{\cdot}_{Lag}\mathsf{Sol}(\mathcal{M}^{\kappa}, \mathcal{F}^{\infty}(1/2, \kappa)),$$

where p_{Lag} is the projection from $Mp \times Lag$ on Lag. The map ρ gives the required action.

Proposition 6. *The action ρ is unitary.*

Proof. The Mp-invariance of the Hermitian product $\langle \cdot, \cdot \rangle_{\gamma}$ is a consequence of the following two simple statements.

- The homology class $\gamma \in H_N(Lag(\mathbb{R}), \mathbb{C})$ is fixed by $Sp(\mathbb{R})$.
- The Green class $G_{\mathcal{M}^{\kappa}} \in H^N(DR^{\bullet}(\mathcal{M}^{-\kappa} \bigotimes_{\mathcal{O}} \mathcal{M}^{\kappa}))$ is fixed by the action of the complex group Sp.

Concluding the proof. □

Appendix A. Proof of statements

A.1. **Proofs for Section 2.**

A.1.1. *Proof of Lemma 2.* In order to define $\mathcal{W}^{\kappa}$ as an holomorphic vector bundle, we have to exhibit a cover

$$Lag = \bigcup_i U_i,$$

and trivializations $\varphi_i : \mathcal{W}^{\kappa}_{|U_i} \xrightarrow{\simeq} \mathcal{O}\left(U \times \mathbb{A}^N\right)$ with holomorphic transition isomorphisms

$$\varphi_{ji} = \varphi_j \circ \varphi_i : \mathcal{O}\left(U_i \cap U_j \times \mathbb{A}^N\right) \longrightarrow \mathcal{O}\left(U_i \cap U_j \times \mathbb{A}^N\right).$$

Defining the cover. For every Lagrangian $L \in Lag$, we consider the open set $U_L \subset Lag$ consisting Lagrangian subspaces M such that $L \cap M = 0$. Clearly $\{U_L\}_{L \in Lag}$ is an open cover of Lag. For every $M \in U_L$, the fiber $\mathcal{W}^{\kappa}_{|M}$ can be naturally identified by restriction with $\mathcal{O}(L)$, this yields the trivialization

$$\varphi_L : \mathcal{W}^{\kappa}_{|U_L} \xrightarrow{\simeq} \mathcal{O}(U_L \times L).$$

Note that if we choose in addition a basis of L then $\mathcal{O}(U_L \times L)$ can be further identified with $\mathcal{O}\left(U_L \times \mathbb{A}^N\right)$ but we will skip this additional step.
Computing the transition maps. Let $L_1, L_2 \in Lag$ be a pair of Lagrangian subspaces. Consider $M \in U_{L_1} \cap U_{L_2}$. We will proceed to compute

$$\left(\varphi_{L_2, L_1}\right)_{|M} : \mathcal{O}(L_1) \xrightarrow{\simeq} \mathcal{O}(L_2).$$

The computation is based on the following general construction from linear algebra.

(1) *Construction from Linear algebra.* Given a triple of Lagrangian subspaces $L_1, L_2, M \in Lag$, such that

$$L_1 \cap M = L_2 \cap M = 0,$$

then there exists a linear map

$$R^M_{L_1,L_2} : L_2 \longrightarrow L_1,$$

associated with this configuration, where $R^M_{L_1,L_2}(l_2)$ is defined as the unique element in L_1 such that

$$\omega\left(l_2, m\right) = \omega\left(R^M_{L_1,L_2}(l_2), m\right),$$

for every $m \in M$.

(2) *Computation of the transition isomorphism.* A direct computation reveals that the transition isomorphism is given by

$$\left(\varphi_{L_2,L_1}\right)_{|M}[f]\left(l_2\right) = \psi_\kappa\left(\tfrac{1}{2}\omega\left(R^M_{L_1,L_2}(l_2), l_2\right)\right) f\left(R^M_{L_1,L_2}(l_2)\right), \tag{A.1}$$

for $f \in \mathcal{O}\left(L_1\right)$. It is evident that (A.1) defines an holomorphic map

$$\varphi_{L_2,L_1} : \mathcal{O}\left(U_{L_1} \cap U_{L_2} \times L_1\right) \xrightarrow{\simeq} \mathcal{O}\left(U_{L_1} \cap U_{L_2} \times L_1\right).$$

This concludes the proof of the lemma.

A.1.2. *Proof of Proposition 1.* Let us denote $\tau = \tau_L$.
Independence of basis. First we will prove that formula (2.2) does not depend on the choice of the orthonormal basis. Assume $\beta \in \mathsf{Hom}^{sym}(L, V)$ is non-degenerate and choose a pair of orthonormal bases

$$\begin{aligned} \overrightarrow{e} &= \left(e_1, e_2, .., e_N\right), \\ \overrightarrow{f} &= \left(f_1, f_2, .., f_N\right). \end{aligned}$$

Denote

$$\begin{aligned} \widetilde{\tau}_{\overrightarrow{e}}(\beta) &= \frac{1}{2\kappa}\sum_{i=1}^{N}\beta(e_i)^2, \\ \widetilde{\tau}_{\overrightarrow{f}}(\beta) &= \frac{1}{2\kappa}\sum_{i=1}^{N}\beta(f_i)^2, \end{aligned}$$

where, both terms considered as vectors in $\mathcal{U}^\kappa\left(\mathfrak{h}\right)$. We will denote by $\tau_{\overrightarrow{e}}(\beta)$ and $\tau_{\overrightarrow{f}}(\beta)$ the corresponding elements in the quotient $\mathcal{U}^\kappa\left(\mathfrak{h}\right)/L \cdot \mathcal{U}^\kappa\left(\mathfrak{h}\right)$. It is easy to verify that

$$[\widetilde{\tau}_{\overrightarrow{e}}(\beta), l] = [\widetilde{\tau}_{\overrightarrow{f}}(\beta), l] = \beta\left(l\right),$$

for every $l \in L$, which implies that

$$\tau_{\overrightarrow{f}}(\beta) - \tau_{\overrightarrow{e}}(\beta) \in \mathbb{C} \subset \mathcal{U}^\kappa\left(\mathfrak{h}\right)/L \cdot \mathcal{U}^\kappa\left(\mathfrak{h}\right).$$

We can assume that $\overrightarrow{f} = \overrightarrow{e} \cdot g$ for some $g \in O\left(L, \omega_\beta\right)$ and denote the difference $\tau_{\overrightarrow{f}}(\beta) - \tau_{\overrightarrow{e}}(\beta)$ by $\iota(g)$. We are left to show that $\iota(g) = 0$. The argument works as follows. For every $g \in O\left(L, \omega_\beta\right)$ we defined the scalar

$$\iota(g) = \tau_{\overrightarrow{e}\cdot g}(\beta) - \tau_{\overrightarrow{e}}(\beta).$$

It is easy to verify that in fact we constructed an homomorphism of groups

$$\iota : O\left(L, \omega_\beta\right) \longrightarrow \left(\mathbb{C}, +\right),$$

However for the orthogonal group, there exits no non-trivial such homomorphisms, hence $\iota(g) = 0$ for every $g \in O(L, \omega_\beta)$.
Uniqueness of τ. Uniqueness follows from the fact that every morphism $\beta \in \mathsf{Hom}^{sym}(L, V)$ can be written as a sum

$$\beta = \beta_1 + \beta_2,$$

of two non-degenerate morphisms $\beta_1,\, \beta_2 \in \mathsf{Hom}^{sym}(L, V)$.
Existence of τ. Let us fix some notations first. Let $W = \mathsf{Hom}^{sym}(L, V)$ and let $W^\circ \subset W$ the open set consisting of non-degenerate morphisms. Let us denote by τ° the morphism

$$\tau^\circ : W^\circ \longrightarrow \mathcal{U}^\kappa(\mathfrak{h}) / L \cdot \mathcal{U}^\kappa(\mathfrak{h}),$$

which, for every $\beta \in W^\circ$, is given by the explicit formula

$$\tau^\circ(\beta) = \frac{1}{2\kappa} \sum_{i=1}^{N} \beta(e_i)^2.$$

Our plan is to show that τ° extends to a morphism

$$\tau : W \longrightarrow \mathcal{U}^\kappa(\mathfrak{h}) / L \cdot \mathcal{U}^\kappa(\mathfrak{h}),$$

which satisfies the linearity condition $\tau\left(\beta + \beta'\right) = \tau(\beta) + \tau\left(\beta'\right)$. As it turns out, the existence of the extension is strongly tied with the linearity property. We will need to consider one additional open set $U \subset W^\circ \times W^\circ$. The set U consists of non-degenerate pairs (β, β') such that $\beta + \beta'$ is non-degenerate as well. Our argument is based on the following technical statements

Lemma 5. *For every* $\left(\beta, \beta'\right) \in U$

$$\tau^\circ_{\beta+\beta'} = \tau^\circ_\beta + \tau^\circ_{\beta'}. \tag{A.2}$$

Granting the validity of (A.2) we can finish the proof. Fix a functional

$$\xi : \mathcal{U}^\kappa(\mathfrak{h}) / L \cdot \mathcal{U}^\kappa(\mathfrak{h}) \longrightarrow \mathbb{A}^1,$$

and denote by τ°_ξ the composition

$$\tau^\circ_\xi = \xi \circ \tau^\circ : W^\circ \longrightarrow \mathbb{A}^1.$$

The morphism τ°_ξ extends to a regular morphism

$$\tau^\circ_\xi : W \longrightarrow \mathbb{P}^1.$$

(1) We will show that $\operatorname{Im} \tau^\circ_\xi$ in fact lies in $\mathbb{A}^1$. Denote by $m : W^\circ \times W^\circ \longrightarrow W$ the addition map $m(\beta, \beta') = \beta + \beta'$. It is easy to show that m is smooth and surjective. It is enough to show that

$$\operatorname{Im}\left(m^* \tau^\circ_\xi\right) \subset \mathbb{A}^1,$$

The last statement follows from the fact that $m^*\tau^\circ_\xi = +\left(\tau^\circ_\xi \times \tau^\circ_\xi\right)$, where $+ : \mathbb{A}^1 \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ is the addition morphism. This is true since both $m^*\tau^\circ_\xi$ and $+\left(\tau^\circ_\xi \times \tau^\circ_\xi\right)$ are algebraic morphisms from $W^\circ \times W^\circ$ to $\mathbb{P}^1$ which, by (A.2) agree on the open set U, therefore the coincide everywhere. Now $\operatorname{Im}\left(+\left(\tau^\circ_\xi \times \tau^\circ_\xi\right)\right)$ lies in $\mathbb{A}^1$ hence $\operatorname{Im}\left(m^*\tau^\circ_\xi\right)$ lies in $\mathbb{A}^1$ as well.

(2) Having shown that τ°_ξ extends to a regular morphism $\tau^\circ_\xi : W \longrightarrow \mathbb{A}^1$ for every functional ξ implies that, in fact, τ° extends to a morphism

$$\tau : W \to \mathcal{U}^\kappa(\mathfrak{h}) / L \cdot \mathcal{U}^\kappa(\mathfrak{h}).$$

(3) We are left to show that τ satisfies linearity. Consider the multiplication morphism $m : W \times W \longrightarrow W$. We would like to show that

$$m^{*}\tau = +(\tau, \tau). \tag{A.3}$$

This follows from the fact that both sides of (A.3) are algebraic morphisms form $W \times W$ to $\mathcal{U}^{\kappa}(\mathfrak{h})/L \cdot \mathcal{U}^{\kappa}(\mathfrak{h})$ which coincide by (A.2) on an open subvariety $U \subset W \times W$ hence they coincide everywhere.

(4) We are left to prove (A.2). Let $(\beta, \beta') \in U$ be a pair of non-degenerate symmetric maps. Let $\overrightarrow{e} = (e_1, .., e_N)$ be an orthonormal basis with respect to ω_{β} such that $\omega_{\beta'}$ is diagonal in this basis, namely

$$\omega_{\beta'} = \sum_{i=1}^{N} a_i e_i^{*} e_i^{*}.$$

then $e_i' = \frac{1}{\sqrt{a_i}}, i = 1, .., N$ form an orthonormal basis with respect to $\omega_{\beta'}$. In addition

$$\omega_{\beta+\beta'} = \sum_{i=1}^{N} (a_i + 1) e_i^{*} e_i^{*}.$$

Let $\overrightarrow{f} = (f_1, .., f_N)$, where $f_i = \frac{1}{\sqrt{a_i+1}} e_i$ be an orthonormal basis with respect to $\omega_{\beta+\beta'}$. We have

$$\begin{aligned} \tau^{\circ}(\beta + \beta') &= \frac{1}{2\kappa} \sum_{i=1}^{N} (\beta + \beta')(f_i)^2 \\ &= \frac{1}{2\kappa} \sum_{i=1}^{N} \left[\beta(f_i)^2 + \beta'(f_i)^2 + \beta(f_i)\beta'(f_i) + \beta'(f_i)\beta(f_i) \right]. \end{aligned} \tag{A.4}$$

Now, explicit computation reveals

$$\begin{aligned} \beta(f_i)^2 + \beta'(f_i)^2 &= \frac{1}{a_i+1}\beta(e_i)^2 + \frac{a_i}{a_i+1}\beta'(e_i'), \\ \beta(f_i)\beta'(f_i) &= \frac{a_i}{a_i+1}\beta(e_i)^2, \\ \beta'(f_i)\beta(f_i) &= \frac{1}{a_i+1}\beta'(e_i')^2. \end{aligned}$$

Substituting in (A.4) we obtain

$$\begin{aligned} \tau^{\circ}(\beta + \beta') &= \frac{1}{2\kappa} \sum_{i=1}^{N} \left[\beta(e_i)^2 + \beta'(e_i')^2 \right] \\ &= \frac{1}{2\kappa} \sum_{i=1}^{N} \beta(e_i)^2 + \frac{1}{2\kappa} \sum_{i=1}^{N} \beta'(e_i')^2 \\ &= \tau^{\circ}(\beta) + \tau^{\circ}(\beta'). \end{aligned}$$

This concludes the argument.

Concluding the proof of the proposition.

A.1.3. *Proof of Proposition 2.* **Step 1.** First, we have to show that $\nabla_\beta(s) \in \mathcal{W}^\kappa$ for every $s \in \mathcal{W}^\kappa$. Given a section $s \in \mathcal{W}^\kappa$ we will show that

$$c^\flat \triangleright \nabla_\beta(s) = 0,$$

for every $c \in C$. This follows from a direct computation

$$\begin{aligned} c^\flat \triangleright \nabla_\beta(s) &= c^\flat \left(\beta \triangleright s + \tau(\beta)^\flat \triangleright s\right) \\ &= [c^\flat, \beta] \triangleright s + [c^\flat, \tau(\beta)^\flat] \triangleright s \\ &= -[\beta, c^\flat] \triangleright s + [c^\flat, \tau(\beta)^\flat] \triangleright s \\ &= -(\beta \triangleright c)^\flat \triangleright s - \beta(c)^\flat \triangleright s + [\tau(\beta), c]^\flat \triangleright s \\ &= -(\beta \triangleright c)^\flat \triangleright s - \beta(c)^\flat \triangleright s + \beta(c)^\flat \triangleright s \\ &= -(\beta \triangleright c)^\flat \triangleright s = 0. \end{aligned}$$

Step 2. Second, we have to show that

$$\nabla_{[\beta,\beta']} = \left[\nabla_\beta, \nabla_{\beta'}\right], \tag{A.5}$$

for every $\beta, \beta' \in \mathcal{T}_{Fr}$. In order to do that, we will introduce an auxiliary Lie algebroid.

The symplectic Lie algebroid. The group Sp acts on Lag. Let $\mathfrak{sp}_{Lag} = \mathfrak{sp} \otimes_{\mathbb{C}} \mathcal{O}_{Lag}$. (see Example 5) be the Lie algebroid associated with this action. Since Sp acts on the frame bundle Fr there exists a surjective morphism of Lie algebroids

$$\Theta : \mathfrak{sp}_{Lag} \longrightarrow \mathcal{T}_{Fr}.$$

The morphism Θ is determined by its values on the constant Lie subalgebra $\mathfrak{sp} \subset \mathfrak{sp}_{Lag}$ on which it is given by

$$\Theta(\alpha) = -\alpha^\#.$$

Here $\alpha^\#$ stands for the GL_N-invariant vector field on Fr associated to the element α. The main reason we consider $\mathfrak{sp}_{Lag}$ is because the map τ admits a simpler form when lifted to the level of $\mathfrak{sp}_{Lag}$.

Proposition 7. *There exists a unique morphism*

$$\widetilde{\tau} : \mathfrak{sp}_{Lag} \to \mathcal{U}^\kappa(\mathfrak{h})_{Lag}, \tag{A.6}$$

satisfying

1. *Lifting.*
$$p \circ \widetilde{\tau} = \tau \circ \Theta,$$
where p *is the natural projection from* $\mathcal{U}^\kappa(\mathfrak{h})_{Lag}$ *on* $\mathcal{U}^\kappa(\mathfrak{h})_{Lag} / C \cdot \mathcal{U}^\kappa(\mathfrak{h})_{Lag}$.
2. *Formula. For every non-degenerate*[11] $\alpha \in \mathfrak{sp}$, *the element* $\widetilde{\tau}(\alpha)$ *is given by the the following explicit formula*
$$\widetilde{\tau}(\alpha) = -\frac{1}{2\kappa} \sum_{i=1}^{2N} \alpha(e_i)^2, \tag{A.7}$$
where $\overrightarrow{e} = (e_1, .., e_{2N})$ *is an orthonormal basis with respect to the non-degenerate symmetric form* $\omega_\alpha(u,v) = \omega(\alpha(u), v)$.

[11] Here non-degenerate means that the symmetric form $\omega_\alpha(u,v) = \omega(\alpha(u), v)$ on V is non-degenerate.

Proof. The argument is very similar to the one in the proof of Proposition 1, therefore we omit it. We explain only the lifting property.

It is sufficient to prove the lifting property on fibers. Fix a point $L \in Lag$ and denote by $\widetilde{\tau}$ and τ the fibers $\widetilde{\tau}_L$ and τ_L respectively. It is sufficient to consider non-degenerate element $\alpha \in \mathfrak{sp}$ such that ω_α restricted to L is non-degenerate. For such an element we will show

$$p\left(\widetilde{\tau}\left(\alpha\right)\right) = \tau\left(\alpha_{|L}\right). \tag{A.8}$$

Let $\overrightarrow{e} = (e_1, .., e_N)$ be an orthonormal basis of L with respect to $\omega_{\alpha|L}$. The basis $\overrightarrow{e}$ can be completed to an orthonormal basis of V with respect to ω_α. Let us denote the remaining elements by $\overrightarrow{f} = (f_1, .., f_N)$. In order to prove (A.8) it is enough to show

$$\alpha\left(f_i\right) \in L.$$

for every $f_i \in \overrightarrow{f}$. But this is evident since $0 = \omega_\alpha(f_i, e_j) = \omega(\alpha\left(f_i\right), e_i)$ for every $e_i \in \overrightarrow{e}$ and using the fact that $\overrightarrow{e}$ is a basis of L and L is Lagrangian.

This concludes the proof of the lifting property. □

Verification of formula (A.5). It is enough to verify (A.5) for elements β, β' of the form

$$\begin{aligned} \beta &= \Theta\left(\alpha\right), \\ \beta' &= \Theta\left(\alpha'\right), \end{aligned}$$

where $\alpha, \alpha' \in \mathfrak{sp}$ are non-degenerate. In this situation, the right side of (A.5) becomes

$$\begin{aligned} \left[\nabla_\beta, \nabla_{\beta'}\right](s) &= \left[\beta, \beta'\right] + [\tau\left(\beta\right)^\flat, \tau\left(\beta'\right)^\flat] \\ &= \left[\beta, \beta'\right] + [\widetilde{\tau}\left(\alpha\right)^\flat, \widetilde{\tau}\left(\alpha'\right)^\flat]. \end{aligned}$$

Now, we have

$$\begin{aligned} \left[\widetilde{\tau}\left(\alpha\right)^\flat, \widetilde{\tau}\left(\alpha'\right)^\flat\right] &= \left[\widetilde{\tau}\left(\alpha'\right), \widetilde{\tau}\left(\alpha\right)\right]^\flat = \widetilde{\tau}\left(\left[\alpha, \alpha'\right]\right)^\flat \\ &= \tau\left(\Theta\left(\left[\alpha, \alpha'\right]\right)\right)^\flat = \tau\left(\left[\Theta\left(\alpha\right), \Theta\left(\alpha'\right)\right]\right)^\flat \\ &= \tau\left(\left[\beta, \beta'\right]\right)^\flat. \end{aligned}$$

All the steps in the above computation are direct therefore we omit any further explanation. Concluding we obtained

$$\left[\nabla_\beta, \nabla_{\beta'}\right] = \left[\beta, \beta'\right] + \tau\left(\left[\beta, \beta'\right]\right)^\flat = \nabla_{[\beta, \beta']}.$$

This concludes the proof of the proposition.

A.1.4. *Proof of Proposition 3.* Recall that we have a surjective morphism of Lie algebroids

$$\phi : \mathcal{T}_{Fr} \twoheadrightarrow \mathcal{T}_{Det^\times},$$

sending the vertical subalgebra $\mathcal{T}_{Fr}^{\mathsf{v}} = \mathcal{H}om(C, C)$ onto the vertical subalgebra $\mathcal{T}_{Det^\times}^{\mathsf{v}} = \mathcal{H}om(\wedge^N C, \wedge^N C)$ and it is given by

$$\phi\left(\beta\right) = Tr\left(\beta\right) = \sum_{i=1}^{N} Id \wedge .. \wedge \overset{i}{\beta} \wedge .. \wedge Id.$$

for any $\beta \in \mathcal{H}om(C, C)$. From this description we can deduce the following

- The kernel of the map ϕ consists of $\beta \in \mathcal{H}om(C,C)$ such that $Tr(\beta) = 0$.
- The morphism ϕ sends the canonical section $Id \in \mathcal{H}om(C,C)$ to $N \cdot Id \in \mathsf{Hom}\left(\wedge^N L, \wedge^N L\right)$.

In order to show that the $\mathcal{D}_{Fr}$-action factors through a $\mathcal{D}^{1/2}_{Det^\times}$-action it is enough to show that the connection ∇ factors through $\mathcal{T}_{Det^\times}$ and $\nabla_{Id} = -\frac{N}{2}$.

Factorization through $\mathcal{T}_{Det^\times}$. It is enough to show that

$$\tau_L(\beta) = 0,$$

for every $\beta \in \mathsf{Hom}(L,L)$ such that $Tr(\beta) = 0$. Fix such β and choose, in addition, a non-degenerate $\gamma \in \mathsf{Hom}^{sym}(L,V)$. We have

$$\begin{aligned}
\tau_L^\kappa(\beta) &= \tau_L(\gamma+\beta) - \tau_L(\gamma) \\
&= \frac{1}{2\kappa}\sum_{i=1}^N (\gamma+\beta)(e_i)^2 - \frac{1}{2\kappa}\sum_{i=1}^N \gamma(e_i)^2 \\
&= \frac{1}{2\kappa}\sum_{i=1}^N \gamma(e_i)\beta(e_i) = \frac{1}{2\kappa}\sum_{i=1}^N [\gamma(e_i), \beta(e_i)] \\
&= \frac{\kappa}{2\kappa}\sum_{i=1}^N \omega(\gamma(e_i), \beta(e_i)) = \frac{1}{2}\sum_{i=1}^N \omega_\gamma(e_i, \beta(e_i)) \\
&= \frac{1}{2} Tr(\beta) = 0.
\end{aligned} \tag{A.9}$$

where the first equality is by the linearity property of τ_L and all the other equalities are standard manipulations in $\mathcal{U}^\kappa(\mathfrak{h})$.

Verification that $\nabla_{Id} = -\frac{N}{2}$. The fact that $\nabla_{Id} = -\frac{N}{2}$ is a direct consequence of (A.9) if we substitute $\beta = Id$.

This concludes the proof of the proposition.

A.1.5. *Proof of Theorem 12.* We define an auxiliary Lie algebroid.

Auxiliary Lie algebroid. Denote by $\mathcal{T}$ the following Lie algebroid on $Lag \times V$

$$\mathcal{T} = \mathfrak{sp}_{Lag} \oplus \mathcal{T}_H,$$

and by $\mathcal{D}$ the corresponding universal enveloping algebra, $\mathcal{D} = \mathcal{U}(\mathcal{T})$. The action of Sp on the frame bundle Fr induces a surjective morphism of Lie algebroids $\varepsilon : \mathfrak{sp}_{Lag} \twoheadrightarrow \mathcal{T}_{Fr}$, hence there exists a morphism of Lie algebroids $\mathcal{T} \longrightarrow \mathcal{T}_{Fr\times H}$, which in turns yields a surjective homomorphism of D-algebras

$$\begin{aligned}
\phi &: \mathcal{D} \longrightarrow \mathcal{D}_{\mathsf{tot}}, \\
\phi^\kappa &: \mathcal{D}^\kappa \longrightarrow \mathcal{D}^\kappa_{\mathsf{tot}}.
\end{aligned}$$

Using ϕ^κ we can consider the module $\mathcal{M}^\kappa$ as a $\mathcal{D}^\kappa$-module, namely as an object in $\mathsf{DCoh}(\mathcal{D}^\kappa)$. The advantage of doing that is that in this category we can effectively construct a free resolution of $\mathcal{M}^\kappa$.

It will be convenient to consider the pushforward $pr_{Lag_*}(\mathcal{M}^\kappa)$ which is a sheaf of modules over $pr_{Lag_*}(\mathcal{D}^\kappa)$ living on the variety Lag. Since V is affine, no information is lost in this step. We will continue to denote by $\mathcal{M}^\kappa$ and $\mathcal{D}^\kappa$ the sheaves $pr_{Lag_*}(\mathcal{M}^\kappa)$ and $pr_{Lag_*}(\mathcal{D}^\kappa)$ respectively.

Resolution of $\mathcal{M}^\kappa$. We will construct a "Koszul" like resolution of $\mathcal{M}^\kappa$. In order to do that we need to define an appropriate Lie subalgebroid of $\mathcal{D}^\kappa$.

(1) We consider the sheaf $\mathcal{T}^{\circ}$=$\mathfrak{sp}_{Lag} \ltimes C$, with the following Lie algebroid structure
   - The commutator of two elements $(\xi, c), (\xi', c') \in \mathfrak{sp}_{Lag} \ltimes C$ is given by
   $$\left[(\xi, c), (\xi', c')\right] = \left([\xi, \xi'], \varepsilon_{\xi} \triangleright c' - \varepsilon_{\xi'} \triangleright c\right),$$
   - The standard morphism $\sigma : \mathfrak{sp}_{Lag} \ltimes C \longrightarrow Tan_{Lag}$ is defined by $\sigma(\xi, c) = -\xi^{\#}$, where $\xi^{\#}$ is the vector field associated to the element ξ via the Sp-action on Lag.

   We have an injective morphism of Lie algebras $r^{\kappa} : \mathcal{T}^{\circ} \hookrightarrow \mathcal{D}^{\kappa}$ which is determined by
   $$\begin{aligned} r^{\kappa}(c) &= c^{\flat}, \\ r^{\kappa}(\xi) &= \xi + \widetilde{\tau}^{\kappa}(\xi)^{\flat}, \end{aligned}$$
   where $\widetilde{\tau}^{\kappa}$ is the map defined in (A.6). Here we intentionally remember the superscript $(\cdot)^{\kappa}$.

(2) Let $\mathcal{P}^{\kappa,\bullet} = Koz^{\bullet}(\mathcal{D}^{\kappa}, \mathcal{T}^{\circ}, r^{\kappa})$ be the complex of free $\mathcal{D}^{\kappa}$-modules given by $\mathcal{P}^{\kappa,-n} = \mathcal{D}^{\kappa} \bigotimes_{\mathcal{O}} \bigwedge^{n} \mathcal{T}^{\circ}$ for $n = 0, ..., M = rank(\mathcal{T}^{\circ})$, with differential $\delta : \mathcal{P}^{\kappa,-n-1} \to \mathcal{P}^{\kappa,-n}$ given by
   $$\begin{aligned} \delta(a \otimes \partial_0 \wedge .. \wedge \partial_n) &= \sum_{i} (-1)^{i} a \cdot r^{\kappa}(\partial_i) \otimes \partial_0 \wedge ..\widehat{\partial}_i.. \wedge \partial_n \\ &\quad + \sum_{i<j} (-1)^{i+j} a \otimes [\partial_i, \partial_j] \wedge \partial_0 \wedge ..\widehat{\partial}_i..\widehat{\partial}_j.. \wedge \partial_n. \end{aligned}$$
   The complex $\mathcal{P}^{\kappa,\bullet}$ yields a free resolution of $\mathcal{M}^{\kappa}$, that is we have a quasi-isomorphism $\mathcal{P}^{\kappa,\bullet} \xrightarrow{q.i} \mathcal{M}^{\kappa}$. The proof of the last assertion is standard yet tedious and therefore it is omitted.

Computing the Verdier dual of $\mathcal{M}^{\kappa}$. We would like to show
$$\mathbb{D}(\mathcal{M}^{\kappa}) = \mathcal{M}^{-\kappa} \bigotimes_{\mathcal{O}} Det^{-1}.$$
We can write
$$\begin{aligned} \mathbb{D}(\mathcal{M}^{\kappa}) &= \mathsf{Hom}_{\mathcal{D}^{\kappa}}\left(\mathcal{P}^{\kappa,\bullet}, \mathcal{D}^{\kappa,\Omega^{-\mathsf{top}}}\right)[rank(\mathcal{T}) - 1] \\ &= \mathsf{Hom}_{\mathcal{D}^{\kappa}}\left(\mathcal{P}^{\kappa,\bullet}, \mathcal{D}^{\kappa,\Omega^{-\mathsf{top}}}\right)[M + N], \end{aligned}$$
We will compute $\mathsf{Hom}_{\mathcal{D}^{\kappa}}\left(\mathcal{P}^{\kappa,\bullet}, \mathcal{D}^{\kappa,\Omega^{-\mathsf{top}}}\right)[M + N]$ in two stages.

(1) First, we compute
   $$Q^{\kappa,\bullet} = \mathsf{Hom}_{\mathcal{D}^{\kappa}}(\mathcal{P}^{\kappa,\bullet}, \mathcal{D}^{\kappa})[M + N].$$
   The complex $Q^{\kappa,\bullet}$ is a "de-Rham" like complex of right $\mathcal{D}^{\kappa}$-modules given by $Q^{\kappa,n} = \bigwedge^{n} \mathcal{T}^{\circ,*} \bigotimes_{\mathcal{O}} \mathcal{D}^{\kappa}$ for $n = -N - M, .., -N$, with differential $d : Q^{\kappa,n-1} \to Q^{\kappa,n}$ given by

$$d(\omega \otimes a) = d\omega \otimes a + \sum_{i} \partial_i^{*} \wedge \omega \otimes r^{\kappa}(\partial_i) \cdot a. \tag{A.10}$$

Here $(\partial_0, .., \partial_M)$ is an arbitrary local basis of $\mathcal{T}^\circ$ and $(\partial_0^*, .., \partial_M^*)$ is the corresponding dual basis[12] and $d\omega$ means

$$d\omega\left(\partial_0, .., \partial_n\right) = \sum_i (-1)^i \sigma_{\partial_i} \triangleright \omega\left(\partial_0, .\widehat{\partial}_i., \partial_n\right) + \sum_{i<j} (-1)^{i+j} \omega\left([\partial_i, \partial_j], \partial_0, .\widehat{\partial}_i.\widehat{\partial}_j., \partial_n\right),$$

where $\sigma : \mathcal{T}^\circ \to Tan_{Lag}$ is the standard map.

(2) Second, compute the tensor product

$$Q^{\kappa,\bullet} \bigotimes_{\mathcal{O}} \mathcal{D}^{\kappa,\Omega^{-\mathsf{top}}} \simeq Q^{\kappa,\bullet} \bigotimes_{\mathcal{O}} \left(\mathcal{T}^{\circ\wedge\mathsf{top}} \bigotimes_{\mathcal{O}} C^{*\wedge\mathsf{N}}\right),$$

which translates the right $\mathcal{D}^\kappa$-action to a left $\mathcal{D}^\kappa$-action. We have to show

$$Q^{\kappa,\bullet} \bigotimes_{\mathcal{O}} \mathcal{D}^{\kappa,\Omega^{-\mathsf{top}}} \xrightarrow{q.i} \mathcal{M}^{-\kappa} \bigotimes_{\mathcal{O}} Det^{-1}[N].$$

In fact, we will exhibit a quasi-isomorphism

$$\varphi : Q^{\kappa,\bullet} \bigotimes_{\mathcal{O}} \mathcal{D}^{\kappa,\Omega^{-\mathsf{top}}} \xrightarrow{q.i} Koz^\bullet\left(\mathcal{D}^{-\kappa}, \mathcal{T}^\circ, r^{-\kappa}\right) \bigotimes_{\mathcal{O}} Det^{-1}[N].$$

We will use the following identification of the dualizing module $\mathcal{D}^{\kappa,\Omega^{-\mathsf{top}}}$ with

$$\mathcal{D}^\kappa \bigotimes_{\mathcal{O}} \left(\mathfrak{sp}_{Lag}^{\wedge\mathsf{top}} \bigotimes_{\mathcal{O}} C^{\wedge\mathsf{N}} \bigotimes_{\mathcal{O}} C^{*\wedge\mathsf{N}}\right) \simeq \mathcal{D}^\kappa \bigotimes_{\mathcal{O}} \left(\mathcal{T}^{\circ\wedge\mathsf{M}} \bigotimes_{\mathcal{O}} C^{*\wedge\mathsf{N}}\right).$$

For $n = -M - N, ..., -N$, the morphism

$$\varphi^n : \bigwedge^{M+N+n} \mathcal{T}^{\circ,*} \bigotimes_{\mathcal{O}} \mathcal{D}^\kappa \bigotimes_{\mathcal{O}} \left(\mathcal{T}^{\circ\wedge\mathsf{M}} \bigotimes_{\mathcal{O}} C^{*\wedge\mathsf{N}}\right) \longrightarrow \mathcal{D}^{-\kappa} \bigotimes_{\mathcal{O}} \mathcal{T}^{\circ\wedge-\mathsf{N}-\mathsf{n}} \bigotimes_{\mathcal{O}} C^{*\wedge\mathsf{N}},$$

is determined by $\varphi^{-n}\left(\omega \otimes 1 \otimes \alpha \otimes s\right) = 1 \otimes \iota_\alpha\left(\omega\right) \otimes s$, where ι is the standard contraction operation. The proof that φ yields a morphism of complexes is by direct computation.

(3) As a conclusion, we obtained that

$$\mathbb{D}\left(\mathcal{M}^\kappa\right) \xrightarrow{q.i} Koz^\bullet\left(\mathcal{D}^{-\kappa}, \mathcal{T}^\circ, r^{-\kappa}\right) \bigotimes_{\mathcal{O}} Det^{-1}[N],$$

which is quasi-isomorphic to $\mathcal{M}^{-\kappa} \bigotimes_{\mathcal{O}} Det^{-1}[N]$.

This concludes the proof of the theorem.

A.1.6. *Proof of proposition 5.* Recall

$$\mathcal{M}^\kappa = \mathcal{D}^\kappa_{\mathsf{tot}}/\mathcal{I}^\kappa,$$

where $\mathcal{I}^\kappa \subset \mathcal{D}^\kappa_{\mathsf{tot}}$ is a sheaf of left ideals generated by elements of the form $c^\#$ for $c \in C$ and $\beta + \tau(\beta)^\flat$ for $\beta \in \mathcal{T}_{Fr}$. It is enough to show that the ideal $\mathcal{I}^\kappa$ is Sp-invariant, namely $g(\mathcal{I}^\kappa) \subset \mathcal{I}^\kappa$ for every $g \in Sp$. We show this separately for the two kind of generators.

(1) Given elements of the form $c^\flat$ we have

$$g(c^\flat) = \left[\mathsf{g}\left(c \circ g^{-1}\right)\right]^\flat,$$

which is clearly an element of the form $d^\flat$ for some $d \in C$.

[12] It is a standard argument showing that formula (A.10) does not depend on the choice of the basis $(\partial_0, .., \partial_M)$.

(2) We are left to show that $g(\beta+\tau(\beta)^{\flat})\in\mathcal{I}^{\kappa}$. It is enough to show that the map
$$\tau:\mathcal{T}_{Fr}\longrightarrow\mathcal{U}^{\kappa}(\mathfrak{h})_{Lag}/C\cdot\mathcal{U}^{\kappa}(\mathfrak{h})_{Lag},$$
commutes with the Sp-action on both sides. Fix a point $L\in Lag$. We show
$$\tau(g(\beta))_{|L}=g(\tau(\beta))_{|L},$$
for every $\beta\in\mathcal{T}_{Fr|g^{-1}L}$. Recall $\mathcal{T}_{Fr|L}$ is naturally identified with $\mathsf{Hom}^{\mathsf{sym}}(L,V)$. Let $\beta\in\mathcal{T}_{Fr|g^{-1}L}=\mathsf{Hom}^{\mathsf{sym}}(g^{-1}L,V)$. We can assume in addition that β is non-degenerate since any other element in $\mathsf{Hom}^{\mathsf{sym}}(g^{-1}L,V)$ can be written as a sum of non-degenerate ones. We have
$$\tau(g(\beta))_{|L}=\tau_L(g(\beta)_{|L})=\frac{1}{2\kappa}\sum_{i=1}^{N}g(\beta)_{|L}(e_i)^2,$$
where $\overrightarrow{e}=(e_1,..,e_N)$ is an orthonormal basis with respect to $\omega_{g(\beta)}$. Now $g(\beta)_{|L}$ is given by
$$g(\beta)_{|L}:L\overset{g^{-1}}{\longrightarrow}g^{-1}L\overset{\beta}{\longrightarrow}V\overset{g}{\longrightarrow}V.$$
Hence
$$\tau(g(\beta))_{|L}=\frac{1}{2\kappa}\sum_{i=1}^{N}g\left(\beta(g^{-1}e_i)\right)^2=\frac{1}{2\kappa}\sum_{i=1}^{N}g\left(\beta(f_i)\right)^2,$$
where $\overrightarrow{f}=g^{-1}\overrightarrow{e}$ is an orthonormal basis of $g^{-1}L$ with respect to ω_β. Finally
$$\frac{1}{2\kappa}\sum_{i=1}^{N}g\left(\beta(f_i)\right)^2=g(\tau(\beta))_{|L},$$

This concludes the proof of the proposition.

A.2. **Proofs for Section 3.**

A.2.1. *Proof of Theorem 13.* Let us denote by $X=Lag\times V$. Recall
$$\begin{aligned}\mathcal{H}^{an}(\kappa)&=\mathsf{Sol}\left(\mathcal{M}^{\kappa},\mathcal{F}^{an}(1/2,\kappa)\right),\\ \mathcal{H}^{an}(-\kappa)&=\mathsf{Sol}\left(\mathcal{M}^{-\kappa},\mathcal{F}^{an}(1/2,-\kappa)\right).\end{aligned}$$
We have
$$\begin{aligned}\mathcal{M}^{-\kappa}&\simeq\widetilde{\mathbb{D}}(\mathcal{M}^{\kappa})[-N]\\ &\simeq pr_{Lag}^{*}Det\bigotimes_{\mathcal{O}}\mathbb{D}\mathcal{M}^{\kappa}[-N]\end{aligned}$$
which implies
$$\begin{aligned}\mathcal{H}^{an}(-\kappa)&=R^N\mathsf{Hom}_{\mathcal{D}_{\det}^{-\kappa}}\left(pr_{Lag}^{*}Det\bigotimes_{\mathcal{O}}\mathbb{D}\mathcal{M}^{\kappa},\mathcal{F}^{an}(1/2,-\kappa)\right)\\ &=R^N\mathsf{Hom}_{\mathcal{D}_{\det}^{-\kappa}}\left(\mathbb{D}\mathcal{M}^{\kappa},pr_{Lag}^{*}Det^{-1}\bigotimes_{\mathcal{O}}\mathcal{F}^{an}(1/2,-\kappa)\right)\\ &=R^N\mathsf{Hom}_{\mathcal{D}_{\det}^{-\kappa}}\left(\mathbb{D}\mathcal{M}^{\kappa},\mathcal{F}^{an}(-1/2,-\kappa)\right),\end{aligned}$$
The pairing is now evident. Given $\nu\in\mathcal{H}^{an}(\kappa)$ and $\varphi\in\mathcal{H}^{an}(-\kappa)$ we have
$$\begin{aligned}\varphi\otimes\nu\quad&\in\quad R^N\mathsf{Hom}_{\mathcal{D}_X}\left(\mathbb{D}\mathcal{M}^{\kappa}\bigotimes_{\mathcal{O}}\mathcal{M}^{\kappa},\mathcal{F}^{an}(-1/2,-\kappa)\bigotimes_{\mathcal{O}}\mathcal{F}^{an}(1/2,\kappa)\right)\\ &\overset{m}{\longrightarrow}R^N\mathsf{Hom}_{\mathcal{D}_X}\left(\mathbb{D}\mathcal{M}^{\kappa}\bigotimes_{\mathcal{O}}\mathcal{M}^{\kappa},j_*\mathcal{O}_U^{an}\boxtimes\mathcal{O}_{V(\mathbb{C})}^{an}\right),\end{aligned}$$

where the map $m:\mathcal{F}^{an}(-1/2,-\kappa)\bigotimes_{\mathcal{O}}\mathcal{F}^{an}(1/2,\kappa)\rightarrow j_{*}\mathcal{O}_{U}^{an}\boxtimes\mathcal{O}_{V}^{an}$ is the canonical pairing. Applying $m\circ(\varphi\otimes\nu)$ to the Green class $G_{\mathcal{M}}$ we obtain an honest cohomology class

$$m\circ\varphi\otimes\nu(G_{\mathcal{M}})\in H^{N}(U\times V,\mathbb{C})=H^{N}(U,\mathbb{C}).$$

We define

$$B_{\gamma}(\nu,\varphi)=\left\langle m\circ\varphi\otimes\nu(G_{\mathcal{M}}),\gamma\right\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the pairing between homology and cohomology. We are left to show that B_{γ} is Heisenberg invariant. It is enough to show that

$$B_{\gamma}(d\pi^{\kappa}\left(\xi\right)\nu,\varphi)+B_{\gamma}(\nu,d\pi^{\kappa}\left(\xi\right)\varphi)=0,$$

for every $\xi\in\mathfrak{h}=Lie(H)$. Using Lemma 3 we have

$$\begin{aligned}&B_{\gamma}(d\pi^{\kappa}\left(\xi\right)\nu,\varphi)+B_{\gamma}(\nu,d\pi^{\kappa}\left(\xi\right)\varphi)\\=\;&B_{\gamma}(\Pi^{\kappa}\left(\xi\right)\nu,\varphi)+B_{\gamma}(\nu,\Pi^{\kappa}\left(\xi\right)\varphi,\nu)\\=\;&\left\langle\varphi\otimes\nu((\Pi^{\kappa}\left(\xi\right)\otimes1+1\otimes\Pi^{\kappa}\left(\xi\right))\,G_{\mathcal{M}}),\gamma\right\rangle.\end{aligned}$$

Now it is evident that

$$(\Pi^{\kappa}\left(\xi\right)\otimes1+1\otimes\Pi^{\kappa}\left(\xi\right))\,G_{\mathcal{M}}=0,$$

which concludes the argument.

A.2.2. *Proof of Theorem 14.* Let us denote by $X=Lag\times V$. We repeat step by step the proof of Theorem 13. Recall

$$\begin{aligned}\mathcal{H}^{\infty}\left(\kappa\right)&=\mathsf{Sol}\left(\mathcal{M}^{\kappa},\mathcal{F}^{\infty}(1/2,\kappa)\right),\\\mathcal{H}^{\infty}\left(-\kappa\right)&=\mathsf{Sol}\left(\mathcal{M}^{-\kappa},\mathcal{F}^{\infty}(1/2,-\kappa)\right).\end{aligned}$$

Since

$$\begin{aligned}\mathcal{M}^{-\kappa}&\simeq\widetilde{\mathbb{D}}(\mathcal{M}^{\kappa})[-N]\\&\simeq pr_{Lag}^{*}Det\bigotimes_{\mathcal{O}}\mathbb{D}\mathcal{M}^{\kappa}[-N]\end{aligned}$$

we can write

$$\begin{aligned}\mathcal{H}^{\infty}\left(-\kappa\right)&=R^{N}\mathsf{Hom}_{\mathcal{D}_{\det}^{-\kappa}}\left(pr_{Lag}^{*}Det\bigotimes_{\mathcal{O}}\mathbb{D}\mathcal{M}^{\kappa},\mathcal{F}^{\infty}(1/2,-\kappa)\right)\\&=R^{N}\mathsf{Hom}_{\mathcal{D}_{\det}^{-\kappa}}\left(\mathbb{D}\mathcal{M}^{\kappa},pr_{Lag}^{*}Det^{-1}\bigotimes_{\mathcal{O}}\mathcal{F}^{\infty}(1/2,-\kappa)\right)\\&=R^{N}\mathsf{Hom}_{\mathcal{D}_{\det}^{-\kappa}}\left(\mathbb{D}\mathcal{M}^{\kappa},\mathcal{F}^{\infty}(-1/2,-\kappa)\right),\end{aligned}$$

Given $\nu\in\mathcal{H}^{\infty}\left(\kappa\right)$ and $\varphi\in\mathcal{H}^{\infty}\left(-\kappa\right)$ we have

$$\begin{aligned}\varphi\otimes\nu\;\;&\in\;\;R^{N}\mathsf{Hom}_{\mathcal{D}_{X}}\left(\mathbb{D}\mathcal{M}^{\kappa}\bigotimes_{\mathcal{O}}\mathcal{M}^{\kappa},\mathcal{F}^{\infty}(-1/2,-\kappa)\bigotimes_{\mathcal{O}}\mathcal{F}^{\infty}(1/2,\kappa)\right)\\&\xrightarrow{m}R^{N}\mathsf{Hom}_{\mathcal{D}_{X}}\left(\mathbb{D}\mathcal{M}^{\kappa}\bigotimes_{\mathcal{O}}\mathcal{M}^{\kappa},\mathsf{ex}^{*}C_{Lag(\mathbb{R})}^{\infty}\right),\end{aligned}$$

where the map $m:\mathcal{F}^{\infty}(-1/2,-\kappa)\bigotimes_{\mathcal{O}}\mathcal{F}^{\infty}(1/2,\kappa)\rightarrow\mathsf{ex}^{*}C_{Lag(\mathbb{R})}^{\infty}$ is the pairing map (2.4). Applying $m\circ(\varphi\otimes\nu)$ to the Green class $G_{\mathcal{M}}$ we obtain an honest cohomology class

$$m\circ\varphi\otimes\nu(G_{\mathcal{M}})\in H^{N}(Lag\left(\mathbb{R}\right)\times V\left(\mathbb{R}\right),\mathbb{C})=H^{N}(Lag\left(\mathbb{R}\right),\mathbb{C}).$$

We define

$$B_{\gamma}(\varphi,\nu)=\left\langle m\circ\varphi\otimes\nu(G_{\mathcal{M}}),\gamma\right\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the pairing between homology and cohomology. The Heisenberg invariance is shown using the same argument as in the proof of Theorem 13. The proof is concluded.

A.2.3. *Proof of Theorem 15.* Let us denote by $X = Lag \times V$. Recall $\Upsilon^{\pm} = \left(Det_{\pm}^{1/2}, \alpha_{\pm}\right) \in \mathfrak{Der}^{1/2}(U_{\pm})$ and we have an isomorphism of square roots

$$\theta : \Upsilon^{+}_{|U_+\cap U_-} \xrightarrow{\simeq} \Upsilon^{-}_{|U_+\cap U_-}.$$

Write

$$\begin{aligned} \mathcal{H}^{an}_{\Upsilon^+}(\kappa) &= \mathsf{Sol}\left(\mathcal{M}^{\kappa}, \mathcal{F}^{an}_{\Upsilon^+}(1/2,\kappa)\right), \\ \mathcal{H}^{an}_{\Upsilon^-}(-\kappa) &= \mathsf{Sol}\left(\mathcal{M}^{-\kappa}, \mathcal{F}^{an}_{\Upsilon^-}(1/2,-\kappa)\right). \end{aligned}$$

Since

$$\begin{aligned} \mathcal{M}^{-\kappa} &\simeq \widetilde{\mathbb{D}}(\mathcal{M}^{\kappa})[-N] \\ &\simeq pr^{*}_{Lag}Det \bigotimes_{\mathcal{O}} \mathbb{D}\mathcal{M}^{\kappa}[-N], \end{aligned}$$

we can write $\mathcal{H}^{an}_{\Upsilon^-}(-\kappa)$ as follows

$$\begin{aligned} \mathcal{H}^{an}_{\Upsilon^-}(-\kappa) &= R^N\mathsf{Hom}_{\mathcal{D}^{-\kappa}_{\det}}\left(pr^{*}_{Lag}Det \bigotimes_{\mathcal{O}} \mathbb{D}\mathcal{M}^{\kappa}, \mathcal{F}^{an}_{\Upsilon^-}(1/2,-\kappa)\right) \\ &= R^N\mathsf{Hom}_{\mathcal{D}^{-\kappa}_{\det}}\left(\mathbb{D}\mathcal{M}^{\kappa}, pr^{*}_{Lag}Det^{-1} \bigotimes_{\mathcal{O}} \mathcal{F}^{an}_{\Upsilon^-}(1/2,-\kappa)\right) \\ &= R^N\mathsf{Hom}_{\mathcal{D}^{-\kappa}_{\det}}\left(\mathbb{D}\mathcal{M}^{\kappa}, \mathcal{F}^{an}_{\widetilde{\Upsilon}^-}(-1/2,-\kappa)\right) \end{aligned}$$

where $\widetilde{\Upsilon}^-$ is the dual of Υ^-. Given $\nu \in \mathcal{H}^{an}_{\Upsilon^+}(\kappa)$ and $\varphi \in \mathcal{H}^{an}_{\Upsilon^-}(-\kappa)$, we have

$$\begin{aligned} \varphi\otimes\nu \in{}& R^N\mathsf{Hom}_{\mathcal{D}_X}\left(\mathbb{D}\mathcal{M}^{\kappa}\bigotimes_{\mathcal{O}}\mathcal{M}^{\kappa}, \mathcal{F}^{an}_{\widetilde{\Upsilon}^-}(-1/2,-\kappa)\bigotimes_{\mathcal{O}}\mathcal{F}^{an}_{\Upsilon^+}(1/2,\kappa)\right) \\ &\xrightarrow{m} R^N\mathsf{Hom}_{\mathcal{D}_X}\left(\mathbb{D}\mathcal{M}^{\kappa}\bigotimes_{\mathcal{O}}\mathcal{M}^{\kappa}, j_*\mathcal{O}^{an}_{U_+\cap U_-}\boxtimes\mathcal{O}^{an}_{V(\mathbb{C})}\right), \end{aligned}$$

where the map

$$m : \mathcal{F}^{an}(-1/2,-\kappa)\bigotimes_{\mathcal{O}}\mathcal{F}^{an}(1/2,\kappa) \to j_*\mathcal{O}^{an}_{U_+\cap U_-}\boxtimes\mathcal{O}^{an}_{V}$$

is the canonical pairing and we used the isomorphism θ in order to identify $\widetilde{\Upsilon}^-$ with the dual of Υ^+ on the intersection $U_+\cap U_-$.

Applying $m\circ(\varphi\otimes\nu)$ to the Green class $G_{\mathcal{M}}$ we obtain an honest cohomology class

$$m\circ\varphi\otimes\nu(G_{\mathcal{M}}) \in H^N(U_+\cap U_-\times V,\mathbb{C}) = H^N(U_+\cap U_-,\mathbb{C}).$$

We define

$$B_{\gamma}(\nu,\varphi) = \langle m\circ\varphi\otimes\nu(G_{\mathcal{M}}),\gamma\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the pairing between homology and cohomology. The fact that B_{γ} is $H(\mathbb{C})$-invariant is proved in the same manner as for the proof of Theorem 13.

which concludes the proof of the theorem.

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