RESEARCH STATEMENT

1. Overview

Broadly speaking, I am interested in the development and application of Floer homology to geometric questions regarding knots, surfaces, and manifolds in three and four dimensions. Originating out of advances in Yang-Mills theory and symplectic geometry, Floer homology provides a powerful set of tools for understanding these objects and studying related problems. A unifying theme throughout my work is the use of *equivariant* Floer theory to obtain more refined invariants or to leverage additional geometric information.

My recent research has focused on utilizing Floer homology in the presence of an external symmetry on a 3-manifold or a knot. This represents an emerging program which seeks to combine techniques from equivariant Floer theory with new geometric insights in order to shed light on a surprisingly broad array of topological questions:

- In joint work with Kang, Mallick, Park, and Stoffregen [DKM$^+$22], I proved that the $(2,1)$-cable of the figure-eight knot is not slice, resolving a forty-year old question of Kawauchi [Kaw80]. This represented one of the simplest possible counterexample to the slice-ribbon conjecture [Miy94]. Our proof proceeds by showing that the branched double cover of $(4_1)_{2,1}$ does not bound an equivariant $\mathbb{Z}/2\mathbb{Z}$-homology ball.
- In joint work with Hedden and Mallick [DHM23], I introduced a suite of Floer-theoretic invariants aimed at detecting corks. These are a fundamental set of objects whose symmetries are essential for understanding the difference between smooth and continuous topology in dimension four. We give flexible methods for finding new and interesting families of corks; some of our examples have been used in recent constructions of exotic pairs of closed 4-manifolds by Levine, Lidman, and Piccirillo [LLP23].

I am also interested in the connection between Floer theory and other topological invariants. Chief among these is lattice homology, a combinatorial invariant for plumbed manifolds due to Ozsváth and Szabó [OS03] and Némethi [Ném08]. My research utilizes lattice homology to perform calculations of powerful homotopy-theoretic or gauge-theoretic invariants such as Manolecu's Seiberg-Witten-Floer spectrum [Man16] or Kronheimer and Mrowka's framed instanton Floer homology [KM11]. Although intensely studied, these are often extremely difficult to compute.

- In joint work with Sasahira and Stoffregen [DSS23], I used lattice homology to help compute Manolescu's Seiberg-Witten-Floer homotopy type [Man16] for all Seifert fibered homology spheres. This represents the first broad, non-trivial class of homology spheres for which the Seiberg-Witten-Floer homotopy type has effectively been computed.
- In joint work with Alfieri, Baldwin, and Sivek [ABDS22], I used lattice homology to compute the framed instanton Floer homology of Kronheimer and Mrowka [KM11] for all Seifert fibered homology spheres. As a byproduct, we obtained an isomorphism between instanton and Heegaard Floer homology for such manifolds. This is important in light of a general conjecture that these theories coincide in all cases [KM10].

Finally, Floer homology has proven extremely effective at answering structural questions involving the homology cobordism group $\Theta^3_{\mathbb{Z}}$ and knot concordance group $\mathcal{C}$. My coauthors and I have carried out significant work aimed and understanding and utilizing the output of these invariants:

- In joint work with Hom, Stoffregen, and Truong [DHST23], I showed that $\Theta^3_{\mathbb{Z}}$ admits a $\mathbb{Z}^\infty$-summand by constructing an epimorphism $\Theta^3_{\mathbb{Z}} \to \mathbb{Z}^\infty$. This was previously an open question [Man18]. Our proof utilized involutive Heegaard Floer homology [HM17], a refinement of the Heegaard Floer package of Ozsváth and Szabó [OS04c, OS04b].
- In [DHST21], we carried out a similar construction in the context of the concordance group, using knot Floer homology [Ras03, OS04a]. This gave a re-proof of the fact that the subgroup of topologically slice knots admits a $\mathbb{Z}^\infty$-summand (see also [Hom15, OSS17]).

2. Corks

Broad Goals. Use Floer homology to study automorphisms of 3-manifolds. Develop invariants to help detect and construct corks.

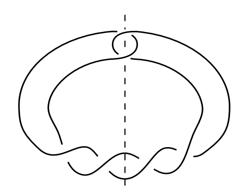
A *cork* is a contractible 4-manifold W equipped with an involution τ on $Y = \partial W$ which does not extend over W as a diffeomorphism [Akb91]. (By work of Freedman, τ always extends as a homeomorphism [FQ90].) It is a fundamental result that any two smooth structures on the same simply-connected 4-manifold are related by cutting out such a W and re-gluing along τ [Mat96, CFHS96]. The theory of corks thus forms an essential part of smooth 4-manifold topology and significant research has been devoted to finding and establishing examples of corks [AM97, AY08, Gom17, LRS23, MS21b].

In joint work with Hedden and Mallick, I used techniques inspired by involutive Heegaard Floer homology to study the set of *strong corks.* Introduced by Lin, Ruberman, and Saveliev [LRS23], these are a generalization of the usual notion of a cork in which the boundary involution τ does not extend as a diffeomorphism over *any* homology ball which Y bounds, rather than a specific contractible manifold. Building on naturality results of Juhász, Thurston, and Zemke [JTZ21], we defined the following invariants aimed at detecting (strong) corks:

Theorem 2.1. [DHM23, Theorem 1.1] *Let Y be a homology sphere with an involution τ. Then there are two Floer-theoretic invariants $h_\tau(Y)$ and $h_{\iota\tau}(Y)$ associated to (Y, τ). If either of these are non-zero, then τ does not extend to a self-diffeomorphism of any homology ball bounded by Y.*

Our approach in [DHM23] is quite different from previous strategies for studying corks in the literature. There, W (rather than its boundary) is naturally the focal point of analysis. In contrast, in our formalism the role of W is almost entirely absent: the non-extendability of τ is inherent to the boundary Y, rather than a property of W. This allows for a great deal of flexibility in terms of finding and constructing new examples.

For instance, we produced many novel families of (strong) corks via $1/n$-surgeries on classes of symmetric slice knots. (See the adjacent figure.) Each of these generates multiple new corks, as different slice disks for a knot K give different contractible manifolds with boundary $S^3_{1/n}(K)$. Recently, some of our examples and computations were used by Levine, Lidman, and Piccirillo to give new pairs of exotic closed 4-manifolds [LLP23].



In order to quantify the profusion of (strong) corks provided by our invariants, we defined an *involutive homology bordism group* $\Theta^\tau_{\mathbb{Z}}$. This is a refinement of the usual homology cobordism group which takes into account an involution on each end. We showed:

Theorem 2.2. [DHM23, Theorem 1.2, Theorem 1.3] *There exists an infinite linearly independent set of strong corks, generating a $\mathbb{Z}^\infty$-subgroup of $\Theta^\tau_{\mathbb{Z}}$.*

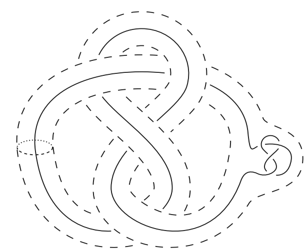
In joint work with Alfieri, Mallick, and Taniguchi, I utilized instanton Floer homology and the Chern-Simons filtration to investigate further questions involving corks and equivariant bounding [ADMT23]. Unlike in Heegaard Floer or monopole Floer homology, it has long been understood that the set of critical values of the Chern-Simons functional contains interesting topological information. This gives the instanton Floer groups additional structure that has been leveraged to obtain new results concerning bordism and bounding [Dae20, NST19].

In [ADMT23], we combine the techniques of [NST19] and [DHM23] to derive new results involving corks and equivariant bounding that were previously out of reach using other Floer-theoretic tools. While Yang-Mills theory has been heavily utilized in smooth 4-manifold topology, this is the first time that the critical *values* of the Chern-Simons functional have had any bearing on corks or exotic phenomena. A sample result is:

Theorem 2.3. [ADMT23, Corollary 1.4] *There exists a strong cork (Y, τ) such that τ does not extend as a diffeomorphism over either of $W \# n\mathbb{CP}^2$ or $W \# n\overline{\mathbb{CP}}^2$ for any $n \in \mathbb{Z}$ and homology ball W that Y bounds.*

Theorem 2.3 is somewhat similar in spirit to recent stabilization results regarding corks [Kan22], except that we work in the setting of strong corks and deal with stabilization by $n\mathbb{CP}^2$ and $n\overline{\mathbb{CP}}^2$ (rather than spin manifolds such as $S^2 \times S^2$). There are many remaining structural questions involving equivariant bounding that I am studying: for instance, one can ask whether there is a strong cork such that τ does not extend over any definite manifold. Together with my coauthors, I am developing further Floer-theoretic tools to attack such problems.

The application of Floer homology to the study of corks has proven to be very robust, and the techniques of [DHM23] can be used to shed light on many interesting geometric constructions. In joint work with Mallick and Zemke [DMZ23], I study a particularly prominent class of (infinite-order) corks constructed by Gompf [Gom17]. These are built using the *swallow-follow operation.* Given a knot $K \# J$, the swallow-follow operation consists of a longitudinal Dehn twist on a torus that engulfs K, as displayed in the figure on the right.



In [Gom17], Gompf showed that for K in a restricted class of double-twist knots, the swallow-follow operation induces a self-diffeomorphism of $S^3_{1/n}(K \# -K)$ which makes it into a cork boundary. Gompf asked for which further families of knots this holds. While various negative results have appeared [RR17], so far there have been no other positive examples. In [DMZ23], we derive a Floer-theoretic condition on K that provides an affirmative answer to Gompf's question:

Theorem 2.4. [DMZ23, Theorem 1.2] *Let K be a knot such that the action of the Sarkar map is homotopically nontrivial on the connected complex $CFK^{\mathrm{conn}}(K)$. Then the swallow-follow operation makes $S^3_{1/n}(K \# -K)$ into a strong cork for any n odd.*

The condition of Theorem 2.4 is quite general: for instance, Gompf's original paper answers his question affirmatively for four knots up to eight crossings, while Theorem 2.4 applies to seventeen such knots. Gompf's construction is especially interesting since every power of the boundary automorphism is non-extendable. We expect that our Floer-theoretic methods can similarly be used to establish the infinite-order nature of these corks.

3. Applications to Sliceness and Exotic Surfaces

Broad Goals. Apply equivariant Floer homology to obstruct sliceness of knots via their branched double covers. Use symmetries of knots to produce exotic slice disks and slice surfaces.

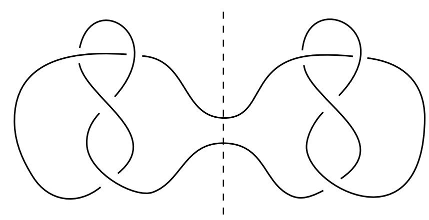
Studying automorphisms of knots and manifolds has resulted in several surprising applications to (seemingly) non-equivariant questions in low-dimensional topology. Chief among these is:

Theorem 3.1. [DKM$^+$22, Theorem 1.1] *The $(2,1)$-cable of the figure-eight knot is not slice.*

The non-sliceness of $(4_1)_{2,1}$ was conjectured by Kawauchi [Kaw80] over forty years ago and has attracted considerable attention due to its connection to the slice-ribbon conjecture. Explicitly, Miyazaki showed that if K is a fibered, negative amphichiral knot with irreducible Alexander polynomial, then the $(2n, 1)$-cable of K is not (homotopy) ribbon for any $n \neq 0$ [Miy94]. On the other hand, these knots are known to be (strongly) rationally slice (and thus algebraically slice). While such cables are generally believed not to be slice, the fact that no prior argument has appeared in the literature has left open the possibility that these generate counterexamples to the slice-ribbon conjecture.

The strategy of [DKM$^+$22] is based on understanding the branched double cover of K. If K is slice with slice disk D, then the branched double cover $\Sigma_2(K)$ bounds a $\mathbb{Z}/2\mathbb{Z}$-homology ball given by $\Sigma_2(D)$, which gives a well-known obstruction to the sliceness of K. The new ingredient used in [DKM$^+$22] is to remember the data of the branching involution on $\Sigma_2(K)$. Since this evidently extends over $\Sigma_2(D)$, we obtain a more refined obstruction to the sliceness of K by asking for the existence of an *equivariant* homology ball. This is precisely the kind of phenomenon detected by the cork-theoretic tools of [DHM23]. In the setting of Heegaard Floer homology, similar ideas were first employed in [AKS20] by Alfieri, Kang, and Stipsicz, who studied branched covers over torus knots and Montesinos knots.

This is especially effective for studying $(2,1)$-cables. For any knot K, the branched double cover $\Sigma_2(K_{2,1})$ is homeomorphic to $S^3_{+1}(K\#K^r)$. Moreover, the branching action on $\Sigma_2(K_{2,1})$ is given by the involution on $S^3_{+1}(K\#K^r)$ induced by the inversion interchanging K and K^r. This explicit identification allows for a Floer-theoretic understanding of the branching action [Mal22, DMS23] to which the methods of [DHM23] can be applied. Note that $S^3_{+1}(K\#K^r)$ does in fact bound a (non-equivariant) contractible manifold for $K = 4_1$; hence the additional analysis of the branching involution is essential.



The approach of [DKM$^+$22] can be used in any situation in which the branching symmetry is sufficiently understood. I am interested in several further questions regarding the application of this method, including the higher cables $(4_1)_{2n,1}$ for $n > 1$.

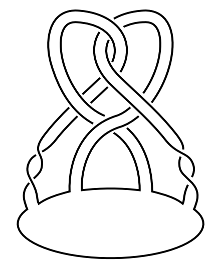
In joint work with Mallick and Stoffregen, I extended the approach in [DHM23] to study automorphisms of knots. We defined a suite of Floer-theoretic invariants aimed at bounding the *equivariant slice genus* $\widetilde{g}_4(K)$ from below. Given a strongly invertible or 2-periodic knot (K,τ), this is the minimum genus of a smoothly embedded slice surface $\Sigma \subseteq B^4$ which is invariant under the standard symmetry τ_{B^4} on B^4 extending τ. In [BI22], Boyle and Issa exhibited families of 2-periodic knots for which $\widetilde{g}_4(K) - g_4(K)$ grows arbitrarily large, and conjectured the same should be true for strongly invertible knots. As an initial application, we used our invariants to provide an affirmative answer to their question:

Theorem 3.2. [DMS23, Theorem 1.4] *There exist families of strongly invertible slice knots whose equivariant slice genus grows arbitrarily large.*

In fact, we showed that our invariants bound a rather stronger quantity that the equivariant genus. We say that a slice surface Σ for K is *isotopy equivariant* if its image $\tau_{B^4}(\Sigma)$ is (smoothly) isotopic to Σ rel boundary. The *isotopy equivariant slice genus* $\widetilde{ig}_4(K)$ of K is then defined to be the minimal genus over all such Σ.

Although the notion of isotopy equivariance may initially seem rather contrived, a slight shift in perspective demonstrates its usefulness. Indeed, if Σ is any slice surface for K with $g(\Sigma) < \widetilde{ig}_4(K)$, then we may immediately conclude that the two surfaces Σ and $\tau_{B^4}(\Sigma)$ are not isotopic rel K. The calculation of $\widetilde{ig}_4(K)$ thus provides an easy method for generating non-isotopic slice surfaces in the presence of a symmetry on K. For example, if K is an equivariant slice knot with $\widetilde{ig}_4(K) > 0$, then we may take *any* slice disk Σ for K and form its image under any extension of τ; the resulting pair of slice disks are automatically non-isotopic rel K. This is in marked contrast to the usual approach taken in the literature, where in order to deploy various invariants, one has in mind a specific family of slice disks or surfaces that are conjectured to be non-isotopic.

Theorem 3.3. [DMS23, Theorem 1.5] *The knot K displayed below has $\widetilde{ig}_4(K) > 0$. Thus, if Σ is any slice disk for K, then Σ and $\tau_{B^4}(\Sigma)$ are not isotopic rel boundary.*



In [Hay21], Hayden showed that K admits a particular (symmetric) pair of slice disks which are topologically but not smoothly isotopic rel boundary. (The fact that the disks are topologically isotopic follows immediately from work of Conway and Powell [CP21].) Theorem 3.3 thus provides a Floer-theoretic re-proof of Hayden's result, and in fact shows that *any* pair of symmetric disks for K are non-isotopic rel boundary. Knot Floer homology has previously been used to distinguish exotic higher-genus surfaces (see the work of Miller, Juhász, and Zemke [JMZ21]); Theorem 3.3 gives the first example of the use of knot Floer homology in the genus-zero case.

In [HS21], Hayden and Sundberg used Khovanov homology to obtain an alternative proof of the fact that K admits a pair of exotic slice disks. This suggests that one should be able to construct an analogous formalism to the one in [DMS23] on the Khovanov side. Together with Borodzik, Mallick, and Stoffregen, I am currently pursuing this line of research.

4. Lattice Homology

Broad Goals. Develop effective methods for computing Floer homotopy. Understand the relationship between different Floer theories and lattice homology.

Beginning with Furuta's proof of the 10/8-theorem [Fur01], Floer homotopy has assumed an increasingly prominent role in low-dimensional topology. In [Man16], Manolescu defined the *Seiberg-Witten-Floer stable homotopy type* $SWF(Y, \mathfrak{s})$ for Y a rational homology sphere, leading to his disproof of the triangulation conjecture. The construction of a homotopy type for other invariants from low-dimensional has likewise attracted a great deal of attention and forms part of a general program of Cohen, Jones, and Segal [CJS95]; see [Flo89, LS14, MS21a, KL22].

Despite this, there are few explicit computations of Seiberg-Witten-Floer spectra in the literature. Indeed, an (essentially) exhaustive list of all previously known non-trivial Seiberg-Witten-Floer homotopy types is presented in [Man14]: these consist of Brieskorn spheres of the form $\Sigma(2, 3, k)$. Such examples follow from the work of Mrowka, Ozsváth, and Yu [MOY97], who provided an explicit description of the critical points of the Chern-Simons-Dirac functional for Seifert fibered spaces. In a handful of particularly simple cases, the critical points are such that the Seiberg-Witten-Floer spectrum is completely determined, giving the computations in [Man14].

In joint work with Sasahira and Stoffregen [DSS23], I computed the Seiberg-Witten-Floer homotopy type for all *almost-rational (AR) plumbed homology spheres*. These are rational homology spheres which occur as boundaries of a certain restricted class of plumbings [Ném05]. Given such a plumbing Γ, we construct a combinatorially-defined *lattice spectrum* $\mathcal{H}(\Gamma, [k])$ purely using the intersection form of Γ. We prove that this computes the Floer homotopy type of the boundary Y_Γ:

Theorem 4.1. [DSS23, Theorem 1.2] *Let Y_Γ be an AR homology sphere and $\mathfrak{s}$ be a self-conjugate* spin^c*-structure on Y_Γ. Then we have a* Pin(2)*-equivariant homotopy equivalence*

$$\mathcal{H}(\Gamma, [k]) = SWF(Y_\Gamma, \mathfrak{s}).$$

The class of AR homology spheres includes (for example) all Seifert fibered homology spheres over S^2. Theorem 4.1 thus provides the first non-trivial computation of the Seiberg-Witten-Floer spectrum for a general class of rational homology spheres.

The construction of $\mathcal{H}(\Gamma, [k])$ and proof of Theorem 4.1 relies heavily on work of Némethi regarding *lattice homology* [Ném08]. This is a combinatorial invariant for plumbed manifolds first introduced by Ozsváth and Szabó [OS03], which is known to be isomorphic to Heegaard Floer homology by work of Zemke [Zem21]. Lattice homology has also been used by Némethi to study algebro-geometric invariants of certain surface singularities [Ném08]. In [DSS23], we likewise show that in favorable circumstances, $SWF(Y_\Gamma, \mathfrak{s})$ may be expressed in terms of certain sequences of sheaf cohomology groups associated to these singularities. Many exciting questions remain, both in terms of applications and generalizations of our calculations, as well as understanding their relation to other areas of mathematics.

I have also used lattice homology to help perform computations for a wide variety of other Floer theories. In joint work with Alfieri, Baldwin, and Sivek, I showed that lattice homology can be used to compute the framed instanton Floer homology of Kronheimer and Mrowka [KM11]. As a result, we obtained:

Theorem 4.2. [ABDS22, Corollary 1.3] *Let Y be an AR plumbed homology sphere. Then*

$$I^{\#}(Y) \cong \widehat{HF}(Y;\mathbb{C})$$

as $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces.

Theorem 4.2 is important in light of the conjecture that Heegaard Floer and instanton Floer homology are isomorphic in all cases; see [KM10, Conjecture 7.24]. The application of lattice homology to computing different Floer homologies is very robust: I have used lattice homology to help calculate Pin(2)-equivariant monopole Floer homology [Dai18] and, in joint work with Manolescu, involutive Heegaard Floer homology [DM19].

5. Cobordism and Concordance Homomorphisms

Broad Goals. Understand the structure of various cobordism and concordance groups. Do these have (interesting) torsion?

A common construction in low-dimensional topology is to quotient out a set of topological objects by a restricted form of bordism in order to obtain a group. For example, taking the set of integer homology 3-spheres and quotienting out by the relation of homology cobordism gives the homology cobordism group $\Theta^3_{\mathbb{Z}}$. Similarly, the knot concordance group $\mathcal{C}$ is defined by taking the set of knots in S^3 and declaring two knots to be concordant if they cobound a smoothly embedded annulus in $S^3 \times I$. In each case, the operation of connected sum gives the set in question a group structure, the study of which forms an active area of research.

In a joint series of papers with Hom, Stoffregen, and Truong, I developed a general program for constructing families of $\mathbb{Z}$-valued homomorphisms for use in the study of homology cobordism and knot concordance. Using the involutive Heegaard Floer homology package of Hendricks and Manolescu [HM17], we proved:

Theorem 5.1. [DHST23, Theorem 1.1, Section 7] *There exists an infinite family of surjective, linearly independent homomorphisms*

$$\phi_i \colon \Theta^3_{\mathbb{Z}} \to \mathbb{Z}.$$

In particular, $\Theta^3_{\mathbb{Z}}$ admits a $\mathbb{Z}^\infty$-summand.

In general, understanding the structure of $\Theta^3_{\mathbb{Z}}$ is very difficult, and techniques for doing so have relied on a surprisingly wide array of tools. Using Yang-Mills theory, Furuta [Fur90] and Fintushel and Stern [FS90] showed that $\Theta^3_{\mathbb{Z}}$ has a $\mathbb{Z}^\infty$-subgroup; Frøyshov [Frø02] proved that $\Theta^3_{\mathbb{Z}}$ admits a $\mathbb{Z}$-summand. Prior to our work, the existence of a $\mathbb{Z}^\infty$-summand was an open question [Man18]. It is still unknown whether or not $\Theta^3_{\mathbb{Z}}$ contains any torsion. The most general result in this direction is due to Manolescu [Man16], who used Pin(2)-equivariant Seiberg-Witten-Floer homology to show that there is no 2-torsion in $\Theta^3_{\mathbb{Z}}$ with Rokhlin invariant one. My coauthors and I hope to use the techniques of [DHST23] to further rule out all torsion with Rokhlin invariant one.

Using knot Floer homology, we constructed a similar family of homomorphisms in the context of knot concordance:

Theorem 5.2. [DHST21, Theorem 1.1, Theorem 1.12] *There exists an infinite family of surjective, linearly independent homomorphisms*

$$\varphi_i \colon \mathcal{C} \to \mathbb{Z}.$$

These can be used to establish the existence of a $\mathbb{Z}^\infty$-summand of $\mathcal{C}_{TS}$, where $\mathcal{C}_{TS}$ is the subgroup of $\mathcal{C}$ generated by topologically slice knots.

The result regarding $\mathcal{C}_{TS}$ was previously shown by Ozsváth, Stipsicz, and Szabó using the Υ-invariant [OSS17], which can similarly be used to construct an infinite family of linearly independent homomorphisms. (See also the work of Hom [Hom15], on which [DHST21] is based.) Previously, Endo showed that $\mathcal{C}_{TS}$ contains a $\mathbb{Z}^\infty$-subgroup [End95]; Manolescu and Owens [MO07] and Livingston [Liv08] showed that $\mathcal{C}_{TS}$ admits a $\mathbb{Z}^3$-summand.

6. Introductory Research Problems

Here, I list some introductory research problems suitable for interested graduate students. One circle of ideas is to generalize the work of [DHM23, DMS23, DKM+22] to different kinds of symmetries, such as higher-order symmetries or orientation-reversing involutions on 3-manifolds:

- In [HP20], Hayden and Piccirillo gave the first examples of orientation-reversing corks, while Boyle and Issa [BI21] and Boyle and Chen [BC22] have recently initiated a systematic study of strongly negative amphichiral symmetries on knots. How can we adapt the formalism of [DHM23, DMS23] to these situations?
- In order to rigorously extend the formalism of [DHM23, DMS23] beyond symmetries of order two, it is necessary to establish naturality results in Heegaard Floer homology for coefficient fields/rings other than $\mathbb{Z}/2\mathbb{Z}$. There has already been some research in this direction; see work of Gartner [Gar23].
- There are a wealth of examples of higher-order corks on the topological side; see the work of Tange [Tan17] and Auckly, Kim, Melvin, and Ruberman [AKMR17]. What can Floer homology say about these? Can we compute the induced actions of these symmetries? Can we re-prove that the $\mathbb{Z}$-corks of Gompf [Gom17] are infinite-order?
- We can also obstruct sliceness via studying branching actions on arbitrary prime-power branched covers (rather than just two-fold covers, as in [DKM+22]). What new examples can be obtained in this manner? In [AMM+21], Aceto, Meier, A. Miller, M. Miller, Park, and Stipsicz provided examples of non-slice knots whose prime-power branched covers all bound rational homology balls. What can we say about this question in the equivariant category?

Many of these questions serve as an introduction to broad areas of low-dimensional topology (such Heegaard Floer homology), while still being based on recent research. Other problems involve finding further examples and applications of existing work:

- Levine, Lidman, and Piccirillo [LLP23] found an effective embedding of the cork $S^3_{+1}(6_1)$ established in [DHM23]. Can other examples of interesting exotic 4-manifolds be constructed using the new corks from [DHM23] or [DMZ23]?
- What other knots from Miyazaki's list does the argument of [DKM+22] apply to? What about the higher cables $(4_1)_{2n,1}$? Theorem 3.1 holds for several other $(2,1)$-cables; the smallest knot in Miyazaki's list for which the argument of Theorem 3.1 fails is $(10_{17})_{2,1}$. Is this slice?

There are also questions surrounding our computations of the Seiberg-Witten-Floer homotopy type:

- Recent work of Zemke [Zem21] has shown that lattice homology and Heegaard Floer homology are isomorphic in all cases. Can we likewise prove that the lattice homotopy type $\mathcal{H}(\Gamma,[k])$ computes $SWF(Y,\mathfrak{s})$ in all cases?
- One can define an equivariant Seiberg-Witten-Floer homotopy type in the presence of a geometric symmetry on Y; see work of Montague [Mon22]. Can we compute this using the techniques of [DSS23]?

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