THE NUMBER OF TORSION DIVISORS IN A STRONGLY F-REGULAR RING IS BOUNDED BY THE RECIPROCAL OF F-SIGNATURE

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ABSTRACT. Polstra showed that the cardinality of the torsion subgroup of the divisor class group of a local strongly F-regular ring is finite. We expand upon this result and prove that the reciprocal of the F-signature of a local strongly F-regular ring R bounds the cardinality of the torsion subgroup of the divisor class group of R.

1. INTRODUCTION

Throughout this article, R is a commutative Noetherian ring of prime characteristic $p>0$ and $F^e : R \to R$ to is the eth iterate of the Frobenius endomorphism. We also assume that the Frobenius endomophism is a finite map, i.e. that R is F-finite. Given an R-module M, we denote by F^e_*M the R-module obtained from M by restricting scalars along the F^e. That is, the endofunctor F^e_* : R-mod $\to$ R-mod takes M to the R-module F^e_*M, which is precisely M as an Abelian group and whose R-action is defined according to the R-action on M by $r \cdot F^e_* m := r^{pe}m$ (here, if $m \in M$, we use F^e_*m to denote the corresponding element in F^e_*M). It is clear that F^e_* is exact.

Associated to F-finite local rings is an invariant known as F-signature. This was first introduced by Smith and Van den Bergh [14], was formally defined by Huneke and Leuschke [8], and was proven to exist under general hypotheses by Tucker [16]. Because we work only with integral domains, for our purposes we define the F-signature of R to be the limit

$$s(R) := \lim_{e\to\infty} \frac{\operatorname{frk} F^e_*R}{\operatorname{rank}_R F^e_*R}.$$

Here, $\operatorname{frk} F^e_*R$ denotes the free-rank of F^e_*R, the maximal rank of a free-module appearing in a direct sum decomposition of F^e_*R.

The ring R is said to be strongly F-regular if for each nonzero $r \in R$ there is some $e \in \mathbb{N}$ and $\varphi \in \operatorname{Hom}_R(F^e_*R, R)$ such that $\varphi(F^e_*r) = 1$. Aberbach and Leuschke proved that a local ring of prime characteristic is strongly F-regular if and only if its F-signature is positive [1]. Every stongly F-regular ring is a normal domain and therefore has a well-defined divisor class group on $\operatorname{Spec}(R)$, which we call $\operatorname{Cl}(R)$. Polstra showed that if R is strongly F-regular, then the torsion subgroup of $\operatorname{Cl}(R)$ is finite [10]. Together, these results lend plausibility to the following theorem, the primary contribution of this paper:

Theorem. *Let $(R,\mathfrak{m},k)$ be a local F-finite and strongly F-regular ring of prime characteristic $p>0$. Then the cardinality of the torsion subgroup of the divisor class group of R is bounded by $1/s(R)$ where $s(R)$ is the F-signature of R.*

The author notes that $1/s(R)$ has previously been used to establish upper bounds on other related invariants, notably on the order of the étale fundamental group of a strongly F-regular ring [4] and on the order of an individual torsion divisor D in a strongly F-regular ring [3]. These results further motivate this article. We further note that the techniques employed here are largely inspired by the novel proof in [11, Theorem 3.8] of the classic result first proven in [8]: $s(R) = 1$ if and only if R is regular.

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2. Preliminary Results and Notation

For R-modules M and N, denote by $a^M(N)$ the maximal number of M summands appearing in a direct sum decomposition of N. In the case that $N = F^e_* R$, we say that $a^M_e(R) := a^M(F^e_* R)$. We use $\mathrm{T}(\mathrm{Cl}(R))$ to denote the torsion subgroup of $\mathrm{Cl}(R)$, the divisor class group of R.

2.1. Divisorial Ideals.

This section is included for convenience, and readers may choose to skip it. The results in this section are not new, but rather a collection of proofs for commonly used tricks involving divisorial ideals. We establish Lemma 2.1 before moving onto the primary result of this section, Proposition 2.2, which is used throughout this document to manipulate divisorial ideals.

Recall that if R is a Noetherian normal domain with $X = \mathrm{Spec}(R)$, then we let $\mathrm{Div}(X)$ (or sometimes $\mathrm{Div}(R)$) be the free Abelian group on the height 1 primes of R. Denote by K the fraction field of R. If we fix a height 1 prime $\mathfrak{p}$ in R, then $R_\mathfrak{p}$ is a regular local ring of Krull dimension 1, and is therefore a principal ideal domain with fraction field K. It's maximal ideal is $\mathfrak{p}R_\mathfrak{p}$, and is generated by some element $\pi_\mathfrak{p} \in R_\mathfrak{p}$. If $0 \neq f \in K$, then we may uniquely write f as $u\pi_\mathfrak{p}^N$ for some unit $u \in R_\mathfrak{p}$ and integer N. Thus, for each height 1 prime, we have a valuation $\nu_\mathfrak{p} : K^\times \to \mathbb{Z}$ defined

$$\nu_\mathfrak{p}(f) = N.$$

There are only finitely many height 1 primes $\mathfrak{p}$ such that $\nu_\mathfrak{p}(f) \neq 0$, so

$$\mathrm{div}(f) = \sum_{\substack{\mathfrak{p} \in \mathrm{Spec}(R) \\ \mathrm{ht}\,\mathfrak{p} = 1}} \nu_\mathfrak{p}(f) \cdot \mathfrak{p}$$

is a divisor. We call divisors of the form $\mathrm{div}(f)$ *principal divisors*, and since $\mathrm{div}(f \cdot g) = \mathrm{div}(f) + \mathrm{div}(g)$, the set of principal divisors forms a subgroup in $\mathrm{Div}(R)$. The divisor class group of R, denoted $\mathrm{Cl}(R)$, is defined to be the quotient of $\mathrm{Div}(R)$ by this subgroup of principal divisors.

If all the coefficients of the terms in a divisor D are nonnegative, then we say D is *effective* and write $D \geq 0$. Given a Weil divisor D, we define the *divisorial ideal* of D to be

$$R(D) = \{f \in K^\times \mid \mathrm{div}(f) + D \geq 0\} \cup \{0\}.$$

Every divisorial ideal is a finitely generated, rank 1 R-module which satisfies Serre's condition (S_2), and conversely, every rank 1 R-module which satisfies (S_2) is isomorphic to a divisorial ideal. In particular, this means $R(D)$ is a reflexive module [7]. We will be particularly interested in how divisorial ideals interact with restriction along Frobenius $F^e_*(-)$, and note here that because F^e_* commutes with $\mathrm{Hom}(-,R)$ it also commutes with the reflexification functor $\mathrm{Hom}_R(\mathrm{Hom}_R(-,R),R) = (-)^{**}$.

Recall that for a prime $P \in \mathrm{Spec}(R)$, the nth symbolic power of P is defined $P^{(n)} = P^n R_P \cap R$. Divisorial ideals can be realized as the intersections of symbolic powers of primes. For a divisor $D = N_1\mathfrak{p}_1 + \ldots + N_\ell \mathfrak{p}_\ell$,

$$R(D) = R(N_1\mathfrak{p}_1) \cap \ldots \cap R(N_\ell\mathfrak{p}_\ell) = \mathfrak{p}_1^{(-N_1)} \cap \ldots \cap \mathfrak{p}_\ell^{(-N_\ell)}. \tag{1}$$

Note that if $N \geq 0$ and $\mathfrak{p} \in \mathrm{Spec}(R)$ is a prime, then

$$\mathfrak{p}^{(-N)} := \{f \in K \mid \nu_\mathfrak{p}(f) \geq -N\} \cup \{0\}.$$

This means $P^{(-N)}$ consists only of elements in k which have at most an Nth power of π_P in their denominator. We use the following lemma in the proof of Proposition 2.2 (c), and its proof is included for convenience.

Lemma 2.1. *Suppose $(R, \mathfrak{m})$ is a local principal ideal domain of prime characteristic $p > 0$. Denote by $\langle \pi \rangle$ the maximal ideal $\mathfrak{m}$. Then for any integers $n, m \in R$,*

$$F^e_* \langle \pi^n \rangle \otimes_R \langle \pi^m \rangle \cong F^e_* \langle \pi^{n+mp^e} \rangle$$

via the isomorphism $\varphi : F^e_* x \otimes y \mapsto F^e_*(xy^{p^e})$.

Proof. We first establish that this map is a R-module homomorphism. It is R-multiplicative: if $r \in R$, $x \in \langle \pi^n \rangle$ and $y \in \langle \pi^m \rangle$, then

$$\begin{aligned}\varphi\big(r \cdot (F^e_* x \otimes_R y)\big) &= \varphi(F^e_* r^{p^e} x \otimes_R y) \\ &= F^e_*(r^{p^e} x y^{p^e}) \\ &= r \cdot F^e_*(xy^{p^e}) = r \cdot \varphi(F^e_* x \otimes_R y),\end{aligned}$$

and by extending additively to arbitrary tensors we have that φ is R-linear. To see that it is an isomorphism, we define a map

$$\psi : F^e_* \langle \pi^{n+mp^e} \rangle \to F^e_* \langle \pi^n \rangle \otimes_R \langle \pi^m \rangle,\ F^e_*(xy^{p^e}) \mapsto F^e_* x \otimes_R y$$

Every element of $\langle \pi^{n+mp^e} \rangle = \langle \pi^{mp^e} \cdot \pi^n \rangle = \langle \pi^n \rangle \cdot \langle \pi^m \rangle^{p^e}$ may be realized as a product $x \cdot y^{p^e}$ where $x \in \langle \pi^n \rangle$ and $y \in \langle \pi^m \rangle$, so this map is well-defined and is easily seen to be a morphism of R-modules. We then have

$$\varphi \circ \psi(F^e_*(xy^{p^e})) = \varphi(F^e_* x \otimes_R y) = F^e_*(xy^{p^e})$$

and

$$\psi \circ \varphi(F^e_* x \otimes_R y) = \psi(F^e_*(xy^{p^e})) = F^e_* x \otimes_R y,$$

so we conclude that φ is an isomorphism. ☐

We now proceed to the following proposition, which provides a means of manipulating expressions involving tensor products, reflexifications, and scalar-restrictions of divisorial ideals.

Proposition 2.2. *Suppose $(R, \mathfrak{m}, k)$ is a Noetherian normal domain of prime characteristic $p > 0$. Let D_1 and D_2 be Weil divisors, and note that $M^{**} = \operatorname{Hom}_R(\operatorname{Hom}_R(M,R),R)$. The following are true:*

(a) $\operatorname{Hom}_R(R(D_1), R(D_2)) \cong R(D_2 - D_1)$
(b) $(R(D_1) \otimes R(D_2))^{**} \cong R(D_1 + D_2)$
(c) $(F^e_* R(D_1) \otimes_R R(D_2))^{**} \cong F^e_* R(D_1 + p^e D_2)$.

Proof. We first prove (a). Suppose $f \in R(D_2 - D_1)$, and define a map $\varphi_f : R(D_1) \to K^\times$ by $g \mapsto f \cdot g$. Since $f \in R(D_2 - D_1)$, $\operatorname{div}(f) + D_2 \geq D_1$, and so for any $g \in R(D_1)$,

$$\operatorname{div}(f \cdot g) + D_2 = \operatorname{div}(f) + \operatorname{div}(g) + D_2 \geq \operatorname{div}(g) + D_1 \geq 0,$$

hence $\varphi_f(g) = f \cdot g \in R(D_2)$. Each $f \in R(D_2 - D_1)$ therefore defines a map $\varphi_f : R(D_1) \to R(D_2)$, so $R(D_2 - D_1) \subseteq \operatorname{Hom}_R(R(D_1), R(D_2))$.

Now fix a map $\varphi \in \operatorname{Hom}_R(R(D_1), R(D_2))$. Each divisorial ideal $R(D)$ is rank 1, so tensoring $\varphi : R(D_1) \to R(D_2)$ gives us a commutative diagram

$$\begin{array}{ccc} R(D_1) & \xrightarrow{\quad\varphi\quad} & R(D_2) \\ \downarrow & & \downarrow \\ K \cong R(D_1) \otimes_R K & \xrightarrow{\ \varphi'\ } & R(D_2) \otimes_R K \cong K. \end{array}$$

The map φ' is linear as a map of k-vector spaces, so there is some element $f \in k$ such that $\varphi'(x) = xf$ for every $x \in k$. Tracing through the diagram and using the fact that each divisorial ideal is a submodule of k, we realize $\varphi(x) = xf$ as well. This means $R(D_1 - \operatorname{div}(f)) = f \cdot R(D_1) \subseteq R(D_2)$, so $D_1 - \operatorname{div}(f) \leq D_2 \implies D_2 - D_1 + \operatorname{div}(f) \geq 0$, giving us the second inclusion.

Given (a), the proof of (b) follows from the fact that $\operatorname{Hom}(M, -)$ and $- \otimes M$ form an adjoint pair, i.e. that $\operatorname{Hom}(A \otimes B, C) = \operatorname{Hom}(A, \operatorname{Hom}(B, C))$. Indeed,

$$\begin{aligned}\operatorname{Hom}_R\big(\operatorname{Hom}_R(R(D_1) \otimes R(D_2), R), R\big) &\cong \operatorname{Hom}_R\big(\operatorname{Hom}_R(R(D_1), \operatorname{Hom}(R(D_2), R)), R\big) \\ &\cong \operatorname{Hom}_R\big(\operatorname{Hom}_R(R(D_1), R(-D_2)), R\big) \\ &\cong \operatorname{Hom}_R\big(R(-(D_2 + D_1)), R\big) \\ &\cong R(D_1 + D_2).\end{aligned}$$

To prove (c), for two divisors D_1 and D_2 we first notice that the map

$$\varphi : F^e_* R(D_1) \otimes_R R(D_2) \to F^e_* R(D_1 + p^e D_2), \quad F^e_* x \otimes y \mapsto F^e_*(x \cdot y^{p^e})$$

is a homomorphism. Indeed, if $x \in R(D_1)$ and $y \in R(D_2)$, then

$$\operatorname{div}(x \cdot y^{p^e}) + D_1 + p^e D_2 = \operatorname{div}(x) + D_1 + p^e(\operatorname{div}(y) + D_2)) \geq 0,$$

so $F^e_* x \otimes y$ lands in $F^e_* R(D_1 + p^e D_2)$. It's R-multiplicative: taking $r \in R$, we see

$$\varphi\big(r \cdot (F^e_* x \otimes y)\big) = \varphi\big(F^e_* x \otimes r \cdot y\big) = F^e_*(x \cdot r^{p^e} y^{p^e}) = r \cdot F^e_*(x \cdot y^{p^e}) = r \cdot \varphi\big(F^e_* x \otimes y\big),$$

and by extending additive to arbitrary tensors we have that φ is R-linear. By localizing at some height 1 prime $\mathfrak{p} \in \operatorname{Spec}(R)$, we get a map

$$\varphi_{\mathfrak{p}} : F^e_* R(D_1)_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} R(D_2)_{\mathfrak{p}} \to F^e_* R(D_1 + p^e D_2)_{\mathfrak{p}}$$

where we have taken advantage of the fact $(F^e_* R(D_1) \otimes_R R(D_2))_{\mathfrak{p}} \cong F^e_* R(D_1)_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} R(D_2)_{\mathfrak{p}}$. We claim $\varphi_{\mathfrak{p}}$ is an isomorphism.

Let $n\mathfrak{p}$ and $m\mathfrak{p}$ be the components of $\mathfrak{p}$ in D_1 and D_2 respectively, where n and m are integers. Because $\mathfrak{p}$ is height 1, we see $R(D_1)_{\mathfrak{p}} \cong \mathfrak{p}^{-n} R_{\mathfrak{p}} \cong \langle \pi^{-n} \rangle$ and $R(D_2)_{\mathfrak{p}} \cong \mathfrak{p}^{-m} R_{\mathfrak{p}} \cong \langle \pi^{-m} \rangle$, where $\langle \pi \rangle$ is the maximal ideal $\mathfrak{p} R_{\mathfrak{p}}$ in $R_{\mathfrak{p}}$. After localization and composition with the above isomorphisms, the map $\varphi_{\mathfrak{p}}$ is defined

$$\varphi_{\mathfrak{p}} : F^e_* \langle \pi^{-n} \rangle \otimes_R \langle \pi^{-m} \rangle \to F^e_* \langle \pi^{-n-mp^e} \rangle, \; F^e_* x \otimes y \mapsto F^e_*(xy^{p^e}),$$

and applying Lemma 2.1 tells us it is an isomorphism.

Since $\mathfrak{p}$ was chosen arbitrarily, φ is an isomorphism after localizing at *any* height 1 prime. Thus, since φ is an isomorphism at the level of height 1 primes, by reflexifying, we see that

$$(F^e_* R(D_1) \otimes_R R(D_2))^{**} \xrightarrow{\varphi^{**}} \big(F^e_* R(D_1 + p^e D_2)\big)^{**}$$

is an isomorphism by [7, Theorem 1.12]. Since reflexification commutes with $F^e_*(-)$ and every divisorial ideal is reflexive, $F^e_* R(D_1 + p^e D_2)$ is reflexive as well. This gives us

$$(F^e_* R(D_1) \otimes_R R(D_2))^{**} \cong \big(F^e_* R(D_1 + p^e D_2)\big)^{**} \cong F^e_* R(D_1 + p^e D_2)$$

as desired. ☐

2.2. **Strongly *F*-regular rings.** We now present a refinement of [10, Corollary 2.2], stated as Lemma 2.4, which features the same techniques employed by Polstra. We state [10, Lemma 2.1] for convenience.

Lemma 2.3 ([10, Lemma 2.1])**.** *Let $(R, \mathfrak{m}, k)$ be a local normal domain. Let C be a finitely generated (S_2)-module, M a rank 1 module, and suppose that $C \cong M^{\oplus a_1} \oplus N_1 \cong M^{\oplus a_2} \oplus N_2$ are choices of direct sum decompositions of C so that M cannot be realized as a direct summand of either N_1 or N_2. Then $a_1 = a_2$.*

Lemma 2.4. *Let $(R, \mathfrak{m}, k)$ be a local normal domain and C a finitely generated (S_2)-module. If $D_1, \ldots, D_t$ are divisors representing distinct elements of the divisor class group and $R(D_i)$ is a direct summand of C for each $1 \leq i \leq t$, then*

$$R(D_1)^{a^{R(D_1)}(C)} \oplus \ldots \oplus R(D_t)^{a^{R(D_t)}(C)}$$

is a direct summand of C.

Proof. Suppose we have found a decomposition

$$C \cong R(D_1)^{n_1} \oplus \ldots \oplus R(D_t)^{n_t} \oplus N \tag{2}$$

where n_i is a positive integer for $1 \leq i \leq t$. One such decomposition is given by [10, Corollary 2.2], in which each $n_i = 1$. Fix $i \in \{1, \ldots, n\}$. We prove that if $n_i < a^{R(D_i)}(C)$ then $R(D_i)$ must necessarily be a summand of N, and in this way, refine N until

$$C \cong R(D_1)^{a^{R(D_1)}(C)} \oplus \ldots \oplus R(D_t)^{a^{R(D_t)}(C)} \oplus N.$$

There exists a decomposition of C such that $C \cong R(D_i)^{a^{R(D_i)}(C)} \oplus P$ by definition; hence, it suffices to show that $R(D_i)$ is not a summand of $R(D_1)^{n_1} \oplus \ldots \oplus R(D_t)^{n_t}$ by Lemma 2.3. We proceed by contradiction and assume instead that $R(D_1)^{n_1} \oplus \ldots \oplus R(D_t)^{n_t}$ has n_i+1 many $R(D_i)$ summands, that is, the n_i summands already present in addition to one extra. Passing our decomposition through $\mathrm{Hom}_R(-,R(D_i))$ and applying Proposition 2.2 means

$$R(D_i-D_1)^{n_1} \oplus \ldots \oplus R^{\oplus n_i} \oplus \ldots \oplus R(D_i-D_t)^{n_t}$$

necessarily has n_i+1 R summands. There then exists a surjective R-linear map

$$R(D_i-D_1)^{n_1} \oplus \ldots \oplus R^{n_i} \oplus \ldots \oplus R(D_i-D_t)^{n_t} \to R^{n_i+1}.$$

Quotienting by $R^{\oplus n_i}$ induces a map

$$\bigoplus_{1\le j\le t, j\neq j} R(D_i-D_j)^{n_j} \to R,$$

and by the locality of R there must be some j not equal to i such that the image of $R(D_i-D_j)$ contains a unit. This means $R(D_i-D_j)$ must have rank free rank 1, but because every divisorial ideal has rank 1, $R(D_i-D_j) \cong R$ as R-modules. Thus, D_i and D_j are linearly equivalent, a contradiction. It must then be the case that $R(D_i)$ is a summand of N by Lemma 2.3. ☐

It is known that the divisorial ideals of torsion divisors in strongly F-regular rings are maximal Cohen-Macaulay modules due to [9] and [5], but we present a novel proof here. If M is a finitely generated module over a local ring $(R,\mathfrak{m},k)$ of prime characteristic p and $e\in\mathbb{N}$, then we let

$$I_e(M)=\{\eta\in M \mid \varphi(F^e_*\eta)\in\mathfrak{m}, \forall\varphi\in \mathrm{Hom}_R(F^e_*M,R)\}.$$

Lemma 2.5. *Let $(R,\mathfrak{m},k)$ be an F-finite strongly F-regular ring and M_i a finitely generated torsion free R-module for $1\le i\le n$. Then there exists an $e_0\in\mathbb{N}$ such that $F^e_*M_i$ has a free summand for all $1\le i\le n$ and $e>e_0$.*

Proof. Observe that F^e_*M has a free summand exactly when there is some $\varphi\in\mathrm{Hom}_R(F^e_*M,R)$ such that $\varphi(m)=1$. To see this, suppose we have such a $\varphi(m)=1$. If this is the case, then the map $\alpha: R\to M$ defined $\alpha(1)=m$ is a morphism such that $\varphi\circ\alpha=\mathrm{id}_R$, so the exact sequence

$$0\to\ker\varphi\to M\to R\to 0$$

splits and $M\cong\ker\varphi\oplus R$.

Assume that M is a torsion free R-module. Lemma 2.3 (4) in [10] gives us that

$$\bigcap_{e\in\mathbb{N}} I_e(M)=0. \tag{3}$$

Thus, for every $0\neq\eta\in M$, there is some $e(\eta)\in\mathbb{N}$ such that $\eta\notin I_{e(\eta)}(M)$ and therefore some $\varphi\in\mathrm{Hom}_R(F^{e(\eta)}_*M,R)$ such that $\varphi(\eta)\notin\mathfrak{m}$. Without loss of generality we take $\varphi(\eta)=1$.

Now suppose $M_1,\ldots,M_n$ are torsion free R-modules. For each M_i, choose $0\neq\eta_i\in M_i$ and let $e(\eta_i)$ be a natural number depending on η_i such that $\eta_i\notin I_{e(\eta_i)}(M_i)$. Set

$$e_0=\max\{e(\eta_1),\ldots,e(\eta_n)\}.$$

By part (3) of Lemma 2.3 in [10], $I_{e_0}(M_i)\subseteq I_{e(\eta_i)}(M_i)$ since $e_0\ge e(\eta_i)$. Thus, for each $1\le i\le n$ we may find a $\varphi_i\in\mathrm{Hom}_R(F^{e_0}_*M_i,R)$ such that $\varphi_i(\eta_i)=1$, and conclude that $F^{e_0}_*M_i$ has a free summand for each $1\le i\le n$. ☐

Proposition 2.6. *Let $(R,\mathfrak{m},k)$ be an F-finite strongly F-regular ring. If D is a torsion divisor, then $R(D)$ is a maximal Macaulay module.*

Proof. Since $R(D)\subseteq K$, $R(D)$ is torsion free. Furthermore, since D is a torsion divisor, up to linear equivalence $nD=0$ for some $0\neq n\in\mathbb{Z}$ and the list $\{nD\}_{n\in\mathbb{Z}}$ is finite. By Lemma (2.5) there is some $e\in\mathbb{N}$ such that for all $n\in\mathbb{Z}$, $F^e_*R(nD)$ has a free summand. This means we may write $F^e_*R(-p^eD)=R\oplus M$ for some module M. Tensoring with $R(D)$ and reflexifying yields

$$F^e_*R\cong R(D)\oplus\mathrm{Hom}_R(\mathrm{Hom}_R(M\otimes_R R(D),R),R)$$

after applying Proposition 2.2. Thus, $R(D)$ is a summand of the maximal Cohen-Macaulay R-module (F^e_*R) so we conclude that $R(D)$ is a maximal Cohen-Macaulay R-module. ☐

3. Main Result

Throughout this section, $(R,\mathfrak{m},k)$ is a local F-finite strongly F-regular ring.

Lemma 3.1. *Let D be any torsion divisor. There exists an e_0 such that if $e \geq e_0$, then $a_e^{R(D)}(R) \geq 1$.*

Proof. Follows immediately from the proof of Proposition 2.6. ☐

Lemma 3.2. *Let D be a torsion divisor. Then*

$$\lim_{e\to\infty} \frac{a_e^{R(D)}(R)}{\operatorname{rank} F_*^e R} = s(R),$$

where $s(R)$ is the F-signature of R.

Proof. This proof consists of two parts. We first show that $\operatorname{frk}_R F_*^e R(-p^e D) = a_e^{R(D)}(R)$, and then we calculate the limit.

First $e \in \mathbb{N}$ and let $n = a_e^{R(D)}(R)$. We have $F_*^e R \cong R(D)^n \oplus M$, where M is a finitely generated R-module without an $R(D)$ summand. By Proposition 2.2, applying $-\otimes_R R(-D)$ and then $\operatorname{Hom}_R(\operatorname{Hom}_R(-,R),R)$ to this isomorphism we obtain

$$F_*^e R(-p^e D) \cong R^n \oplus N \tag{4}$$

where $N = \operatorname{Hom}_R(\operatorname{Hom}_R(M\otimes_R R(-D),R),R)$. We claim $n = \operatorname{frk} F_*^e R(-p^e D)$. Suppose for the sake of contradiction that N had a free summand, i.e. that $N \cong R\oplus P$ for some R-module P. Tensoring equation 4 by $R(D)$ and reflexifying gives us

$$F_*^e R \cong R(D)^n \oplus R(D) \oplus \operatorname{Hom}_R(\operatorname{Hom}_R(P\otimes_R R(D),R),R).$$

This means $R(D)^{n+1}$ appears as a summand in a direct sum decomposition of $F_*^e R$, which contradicts the maximality of n. Thus, $\operatorname{frk}_R F_*^e R(-p^e D) = a_e^{R(D)}(R)$.

For the second part of the proof, we first establish notation. Polstra proved that the torsion subgroup $\mathrm{T}(\mathrm{Cl}(R))$ of the divisor class group of a strongly F-regular ring is finite [10], so we may enumerate them: $\mathrm{T}(\mathrm{Cl}(R)) = \{D_1,\dots,D_k\}$. We denote the e-th term in the sequence defining the F-signature of $R(D_i)$ as follows:

$$s_e(R(D_i)) = \frac{\operatorname{frk} F_*^e R(D_i)}{\operatorname{rank} F_*^e R}.$$

Since each divisorial ideal is a finitely generated rank 1 module, Tucker tells us [16, Theorem 4.11]

$$\lim_{e\to\infty} s_e(R(D_i)) = s(R(D_i)) = s(R)\cdot \operatorname{rank} R(D_i) = s(R)$$

for each $1 \leq i \leq k$. In particular, $s_e(R(D_i))$ and $s_e(R(D_j))$ are equivalent Cauchy sequences for each $1 \leq i,j \leq k$. Now set

$$b_e = \frac{a_e^{R(D)}(R)}{\operatorname{rank} F_*^e R}$$

for sake of clarity. We show that the sequence $\{b_e\}$ is equivalent to $\{s_e(R(D_1))\}$ as a Cauchy sequence and conclude that $\lim b_e = s(R)$.

Fix $\varepsilon > 0$. By the equivalence of Cauchy sequences, for each $1 \leq i \leq k$, we may find $N_i \in \mathbb{N}$ such that for all $e \geq N_i$, $|s_e(R(D_i)) - s_e(R(D_j))| < \varepsilon$. Notice that since $a_e^{R(D)}(R) = \operatorname{frk} F_*^e R(-p^e D)$ and $-p^e D$ is a torsion divisor, b_e is equal to $s_e(R(D_i))$ for some $1 \leq i \leq k$. If we let $N = \max\{N_1,\dots,N_k\}$, then for all $e \geq N$, we have

$$|s_e(R(D_1)) - b_e| \leq \max\left\{ |s_e(R(D_1)) - s_e(R(D_i))| \ : \ 1 \leq i \leq k \right\} < \varepsilon.$$

Thus, $\{s_e(R(D_1)) - b_e\}$ is equivalent to the 0 sequence, so $\{b_e\}$ is equivalent to $\{s_e(R(D_1))\}$ as a Cauchy sequence. We conclude that $\lim_{e\to\infty} b_e = s(R)$. ☐

Theorem 3.3. *Let $(R,\mathfrak{m},k)$ be a local F-finite and strongly F-regular ring of prime characteristic $p > 0$. Then*

$$|\mathrm{T}(\mathrm{Cl}(R))| \leq 1/s(R),$$

where $\mathrm{T}(\mathrm{Cl}(R))$ is the torsion subgroup of the divisor class group of R.

Proof. Set

$$n_e = \sum_{D \in \mathrm{T}(\mathrm{Cl}(R))} a_e^{R(D)} x(R).$$

Fix e_0 as in Lemma 3.1, and let $e \geq e_0$. For each torsion divisor D, $R(D)$ is a summand of $F_*^e R$, so by Lemma 2.4 and the fact that $R(D)$ is rank 1 for any torsion divisor, we have that

$$n_e = \sum_{D \in \mathrm{T}(\mathrm{Cl}(R))} a_e^{R(D)}(R) \cdot \operatorname{rank} R(D) \leq \operatorname{rank} F_*^e R.$$

By Lemma 3.2,

$$\begin{aligned}
\lim_{e \to \infty} \frac{n_e}{\operatorname{rank} F_*^e R} &= \lim_{e \to \infty} \sum_{D \in \mathrm{T}(\mathrm{Cl}(R))} \frac{a_e^{R(D)}(R)}{\operatorname{rank} F_*^e R} \\
&= \sum_{D \in \mathrm{T}(\mathrm{Cl}(R))} \lim_{e \to \infty} \frac{a_e^{R(D)}(R)}{\operatorname{rank} F_*^e R} \\
&= \sum_{D \in \mathrm{T}(\mathrm{Cl}(R))} s(R) \\
&= |\mathrm{T}(\mathrm{Cl}(R))| \cdot s(R).
\end{aligned}$$

The limit commutes with the sum since $|\mathrm{T}(\mathrm{Cl}(R))| < \infty$ by Corollary 3.3 in [10]. Because $n_e \leq \operatorname{rank} F_*^e R$,

$$|\mathrm{T}(\mathrm{Cl}(R))| \cdot s(R) = \lim_{e \to \infty} \frac{n_e}{\operatorname{rank} F_*^e R} \leq 1,$$

and we conclude

$$|\mathrm{T}(\mathrm{Cl}(R))| \leq \frac{1}{s(R)}.$$

□

We immediately obtain the following corollary to Theorem 3.3. Local results often extend to graded rings via localization at the maximal ideal, so this is not surprising.

Corollary 3.4. *Let R be a $\mathbb{N}$-graded F-finite and strongly F-regular ring of prime characteristic $p > 0$ such that R_0 is a field. Then*

$$|\mathrm{T}(\mathrm{Cl}(R))| \leq \frac{1}{s(R)}.$$

Proof. Let $\mathfrak{m}$ denote the unique homogeneous maximal ideal of R. Strong F-regularity is a local property, so the localization $R_\mathfrak{m}$ is strongly F-regular and therefore $|\mathrm{T}(\mathrm{Cl}(R_\mathfrak{m}))| \leq \frac{1}{s(R_\mathfrak{m})}$ by Theorem 3.3.

We know $\mathrm{Cl}(R) \to \mathrm{Cl}(R_\mathfrak{m})$ is a bijection by [6, Corollary 10.3] and that $s(R) = s(R_\mathfrak{m})$ by [15, Corollary 6.19], so we have the desired result. □

3.1. **Examples.** Here we provide two examples of graded F-finite strongly F-regular rings R of prime characteristic $p > 0$ to illustrate that the inequality in 3.3 is indeed not strict. We note the F-signature may be computed via the formula $s(R) = 2 - e_{HK}(R)$ in both examples since $e(R) = 2$, but we opt instead for arguments which do not invoke the Hilbert-Kunz multiplicity.

Example 3.5. Suppose $p > 0$ is prime and $R = \frac{\mathbb{F}_p[w,x,y,z]}{(wx-yz)}$. This is a determinantal ring with $r = s = 2$, in the notation of Singh [12, Example 3.1], and therefore has dimension $d = r + s - 1 = 3$. By Singh's example, we have that

$$s(R) = \frac{1}{d!} \sum_{i=0}^{s} (-1)^i \binom{d+1}{i} (s-i)^d = \frac{1}{3!} \sum_{i=0}^{2} (-1)^i \binom{4}{i} (2-i)^3 = \frac{2}{3}.$$

Since R is a determinant ring satisfying the hypotheses of [2, 7.3.5], we have that $\mathrm{Cl}(R) = \mathbb{Z}$ and hence $|\mathrm{T}(\mathrm{Cl}(R))| = 1$, so $|\mathrm{T}(\mathrm{Cl}(R))| < \frac{1}{s(R)}$.

For a local example, let $\mathfrak{m} = (w,x,y,z)$ and consider $R_\mathfrak{m}$. By [6, Corollary 10.3] we immediately see $\mathrm{Cl}(R) = \mathrm{Cl}(R_\mathfrak{m})$ and by [15, Corollary 6.19] $s(R_\mathfrak{m}) = s(R)$, so

$$|\,\mathrm{T}(\mathrm{Cl}(R_\mathfrak{m}))| < \frac{1}{s(R_\mathfrak{m})}.$$

Example 3.6. Suppose $p > 0$ is prime and $n \geq 2$ and set $R = \frac{\mathbb{F}_p[x,y,z]}{xy-z^n}$. The class group of R is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ by [13, Corollary 3.4], so it remains to find $s(R)$. Notice that we have the isomorphism $\frac{\mathbb{F}_p[x,y,z]}{xy-z^n} \cong \mathbb{F}_p[x^n,xy,y^n]$. The latter ring lends itself well to the calculation of F-signature as it is an affine semigroup ring, so we redefine $R = \mathbb{F}_p[x^n,xy,y^n]$, set $A = \mathbb{F}_p[x,y]$ and note that $R \subseteq A$.

Fix $e \in \mathbb{N}$ and set $q = p^e$ and let $\mathfrak{m} \subseteq A$ denote the homogeneous maximal ideal. By [12, Lemma 4],

$$a_e(R) = \ell\left(\frac{R}{\mathfrak{m}^{[q]} \cap R}\right).$$

Let S_e denote the ring $\frac{R}{\mathfrak{m}^{[p^e]} \cap R}$. We can form a maximal chain of submodules of S_e entirely from ideals generated by monomials. To see this, let T denote the collection of distinct monomials in S_e and let

$$(0) = I_0 \subsetneq I_1 \subsetneq \ldots \subsetneq I_n = S_e$$

be a maximal chain of ideals in S_e whose generators are in T. Suppose $0 \leq i \leq n$, and choose elements $f_1,\ldots,f_m \in T$ so that $(f_1,\ldots,f_m) = I_i$. If I_{i+1} contained two monomials not in I_i, then the above chain would not be maximal, so we can find a monomial $f_m \in T$ so that $(f_1,\ldots,f_m,f_{m+1}) = I_{i+1}$.

Now suppose we have a nonzero coset $\overline{g} \in I_{i+1}/I_i$. The representative g must be a nonzero element in $I_{i+1} \setminus I_i$, and therefore $g = a_1f_1 + \ldots + a_{m+1}f_{m+1}$ with $a_{m+1} \neq 0$. This means the set $\{f_1,\ldots,f_m,g\}$ generates I_{i+1} as an ideal, and so $\langle \overline{g} \rangle = I_{i+1}/I_i$. Since any nonzero element of I_{i+1}/I_i generates the entire group, I_{i+1}/I_i is simple. This means the above maximal sequence is a composition series, and it therefore suffices to count the number of distinct monomials in S_e to determine $\ell(S_e)$.

The nonzero monomials x^ay^b in S_e are precisely those monomials in R which are not killed by $\mathfrak{m}^{[q]}$. A monomial $x^ay^b \in R$ must satisfy $x^ay^b = x^{ni}(xy)^jy^{nk} = x^{ni+j}y^{nk+j}$ for some positive integers i,j and k, which implies that $a \equiv b \mod (n)$. If x^ay^b is nonzero in S_e then it is not contained in $\mathfrak{m}^{[q]} = F^e((x,y))A = (x^q,y^q)$ and hence $a < q$ and $b < q$. Likewise, it can be easily seen that any monomial x^ay^b in A for which $a < q$, $b < q$, and $a \equiv b \mod (n)$ is a monomial in S_e, hence there is a bijection between the set of distinct monomials in S_e and pairs of nonnegative integers (a,b) satisfying these conditions.

Suppose for a moment that $q = mn$ for some $m \in \mathbb{N}$, and fix a so that $0 \leq a \leq q-1$. The integers congruent to a modulo n are of the form $ni + a$ for some $i \in \mathbb{N}$, and there are exactly m such distinct integers b such that $0 \leq b \leq q-1$. As there are mn choices for a and m choices for b given a, there are exactly m^2n pairs of integers (a,b) such that $a < q$, $b < q$, and $a \equiv b \mod (n)$. Therefore $a_e(R) = m^2n$.

Now suppose q is once again arbitrary and pick m_q to be the maximal integer such that $m_qn \leq q$. By the special case addressed above we know $m_q^2n \leq a_e(R) \leq (m_q+1)^2n$. The ring R has Krull dimension 2, therefore $\mathrm{rank} F_*^eR = p^{ed} = q^2$. We have the equality

$$q^2 - (m_qn)^2 = 2q(q - m_qn) - (q - m_qn)^2$$

from which we obtain

$$q^2 - (m_qn)^2 \leq 2qn - (q - m_qn)^2 \leq 2qn.$$

Using this inequality we see

$$\frac{1}{n} - \frac{m_q^2n}{q^2} \leq \frac{2}{q}$$

and

$$\frac{(m_q+1)^2n}{q^2} - \frac{1}{n} \leq \frac{2qn + n^2 + 2q}{q^2}.$$

The rightmost terms in both of the above inequalities approach 0 as $q \to \infty$, hence

$$\frac{1}{n} = \lim_{q\to\infty} \frac{(m_q+1)^2n}{q^2} \leq \lim_{q\to\infty} \frac{a_e(R)}{q^2} \leq \lim_{q\to\infty} \frac{(m_q+1)^2n}{q^2} = \frac{1}{n}$$

and $\lim_{e\to\infty}\frac{a_e(R)}{p^{2e}}=\frac{1}{n}$. We conclude that $s(R)=1/n$ and $|\mathrm{T}(\mathrm{Cl}(R))|=n$, and in particular, that

$$|\mathrm{T}(\mathrm{Cl}(R))|=1/s(R).$$

This result easily localizes as in Example 3.5.

References

[1] Ian M. Aberbach and Graham J. Leuschke. The F-signature and strong F-regularity. *Math. Res. Lett.*, 10(1):51–56, 2003.

[2] Winfried Bruns and Jürgen Herzog. *Cohen-Macaulay rings*, volume 39 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1993.

[3] Javier Carvajal-Rojas. Finite torsors over strongly F-regular singularities, 2017. Preprint, arXiv:1710.06887v5 [math.AG].

[4] Javier Carvajal-Rojas, Karl Schwede, and Kevin Tucker. Fundamental groups of F-regular singularities via F-signature. *Ann. Sci. Éc. Norm. Supér. (4)*, 51(4):993–1016, 2018.

[5] Hailong Dao and Tony Se. Finite F-type and F-abundant modules, 2016. Preprint, arXiv:1603.00334v1 [math.AC].

[6] Robert M. Fossum. *The divisor class group of a Krull domain*. Springer-Verlag, New York-Heidelberg, 1973. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 74.

[7] Robin Hartshorne. Generalized divisors on Gorenstein schemes. In *Proceedings of Conference on Algebraic Geometry and Ring Theory in honor of Michael Artin, Part III (Antwerp, 1992)*, volume 8, pages 287–339, 1994.

[8] Craig Huneke and Graham J. Leuschke. Two theorems about maximal Cohen-Macaulay modules. *Math. Ann.*, 324(2):391–404, 2002.

[9] Zsolt Patakfalvi and Karl Schwede. Depth of F-singularities and base change of relative canonical sheaves. *J. Inst. Math. Jussieu*, 13(1):43–63, 2014.

[10] Thomas Polstra. A theorem about maximal Cohen-Macaulay modules, 2020. Preprint, arXiv:2002.04661v2 [math.AC].

[11] Thomas Polstra and Ilya Smirnov. Equimultiplicity theory of strongly F-regular rings, 2019. Preprint, arXiv:1906.01162v2 [math.AC].

[12] Anurag K. Singh. The F-signature of an affine semigroup ring. *J. Pure Appl. Algebra*, 196(2-3):313–321, 2005.

[13] Anurag K. Singh and Sandra Spiroff. Divisor class groups of graded hypersurfaces. In *Algebra, geometry and their interactions*, volume 448 of *Contemp. Math.*, pages 237–243. Amer. Math. Soc., Providence, RI, 2007.

[14] Karen E. Smith and Michel Van den Bergh. Simplicity of rings of differential operators in prime characteristic. *Proc. London Math. Soc. (3)*, 75(1):32–62, 1997.

[15] Alessandro De Stefani, Thomas Polstra, and Yongwei Yao. Global frobenius betti numbers and F-splitting ratio, 2018. Preprint, arXiv:1811.11022v1 [math.AC].

[16] Kevin Tucker. F-signature exists. *Invent. Math.*, 190(3):743–765, 2012.

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