Conventions:

- $\mathbb{k}$ be alg. closed, $R \sim$ complete DVR with residue field $\mathbb{k}$, fraction field $\mathbb{F}$

  $\overline{\mathbb{F}} \sim$ separable closure of $\mathbb{F}$, $\nu \sim$ normalized valuation on $\overline{\mathbb{F}}$

  s.t. $\nu(\mathbb{F}^*) = \mathbb{Z}$

- Graphs: ~ finite connected with the usual notations.
- Curves: $(C, D) \sim$ complete curve with marked pts $(q_1, \dots, q_{|D|})$

  over $\overline{\mathbb{F}}$, & $(C_{R_{\mathbb{L}}}, D_{R_{\mathbb{L}}})$ be the nodal model.

  $C_{R_{\mathbb{L}}} \longrightarrow \operatorname{Spec} R_{\mathbb{L}}$ proper curve $\mathbb{L}/\mathbb{F}$ finite sep.

  $D_{R_{\mathbb{L}}} \sim$ fin. set of $R_{\mathbb{L}}$ pts, total space $C_{R_{\mathbb{L}}}$ normal,

  $(C_{R_{\mathbb{L}}}, D_{R_{\mathbb{L}}}) \times_{\operatorname{Spec} R_{\mathbb{L}}} \operatorname{Spec} \mathbb{k}$ is nodal &

  $(C_{R_{\mathbb{L}}}, D_{R_{\mathbb{L}}}) \times_{\operatorname{Spec} R_{\mathbb{L}}} \operatorname{Spec} \overline{\mathbb{F}} \cong (C, D)$

Tropical curve Γ is a topological graph w/ complete possibly deg. metric:

(S1) vertices of Γ are divided into finite vertices & infinite vertices

(S2) $V^{\infty}(\Gamma)$ is equipped with a total order & $V^f(\Gamma)$ is just a set.

(P1) Γ has finitely many vertices & edges

(P2) any infinite vertex has valency 1 & is connected to a finite vertex by an edge called "unbounded edges"

$E^b(\Gamma)$ edges b/w finite vertices

$E^{\infty}(\Gamma)$ edges b/w a finite & an infinite vertex -

(P_3) any bounded edge e is isometric to $[0, |e|]$;

$|e| \in \mathbb{R}_{\geq 0}$ & an unbounded edge is isometric

to $[0, \infty]$ where $0 \longmapsto$ finite vertex

$\infty \longmapsto$ infinite vertex

- $\mathbb{Q}$ tropical curve $\sim$ $|e| \in \mathbb{Q}$ for any $e \in E^b(\Gamma)$
- irreducible if Γ is connected
- genus $\underline{g(\Gamma) = 1 - |V(\Gamma)| + |E(\Gamma)|}$
- stable if all finite vertices have valency at least 3.

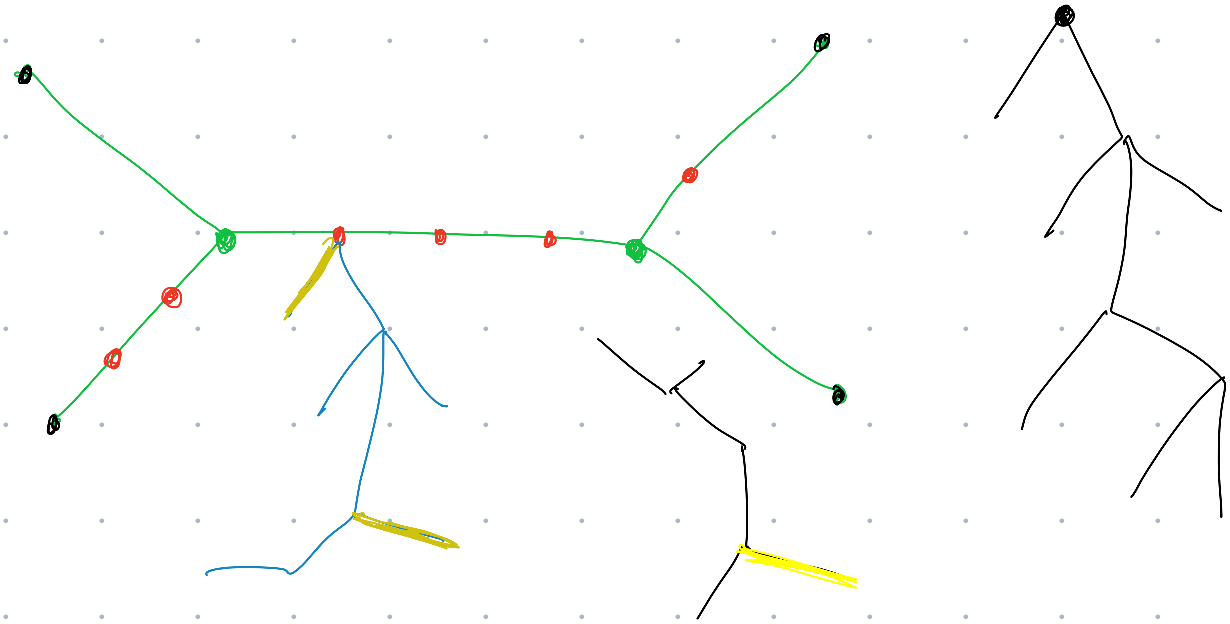
Algorithm:

$\Gamma \rightsquigarrow \Gamma'$ by

step 1: divide each bounded edge e into finitely many pieces

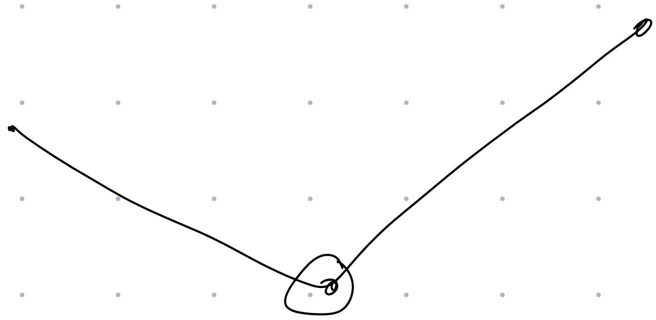
step 2: divide each unbounded edge into finitely many pieces.

step 3: attach rooted metric trees at some finite vertices s.t. all edges but maybe some leaves of that metric tree are bounded



Claim: Let Γ be an irreducible tropical curve satisfying $g(\Gamma) + \frac{|V^\infty(\Gamma)|+1}{2} \geq 2$

then $\exists!$ stable tropical curve Γ^{st} s.t. $V^\infty(\Gamma) = V^\infty(\Gamma^{st})$ & Γ can be obtained from Γ^{st} by the above algorithm.



Γ^{st} is called the stabilization of Γ

- $\mathbb{Q}$ tropical curve attached to (C, D)

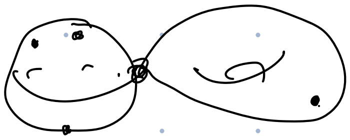
$(C_{R_{\mathbb{L}}}, \underline{D_{R_{\mathbb{L}}}})$ be a nodal model

$C_{R_{\mathbb{L}}} \times_{\operatorname{Spec} R_{\mathbb{L}}} \operatorname{Spec} k$ $\qquad C_{R_{\mathbb{L}}} \longrightarrow \operatorname{Spec} R_{\mathbb{L}}$

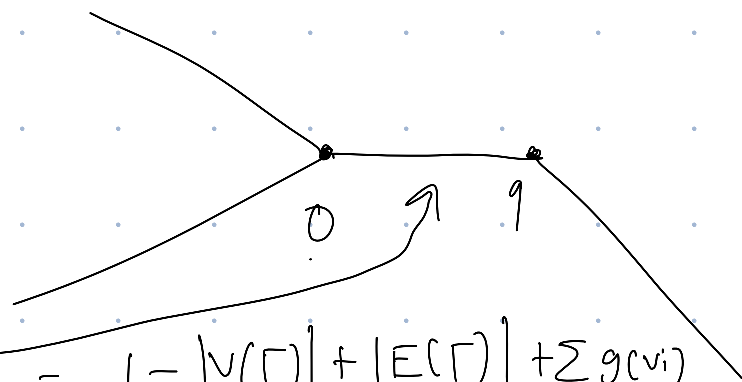
irreducible comps C_v $\rightsquigarrow$ finite vertices v

nodes $\sim$ edges b/w finite vertices

$\underline{D_{R_{\mathbb{L}}}}$'s that specialize to C_v $\sim$ infinite edges from v.



$\sim$



$$g(\Gamma) = 1 - |V(\Gamma)| + |E(\Gamma)| + \sum_{v_i \in V(\Gamma)} g(v_i)$$

$\underline{\underline{A_{r_e}}}$ $\sim$ $x^2 + y^2 + w^{1+r_e} \longleftrightarrow xy = t^{r_e+1}$

$\dfrac{1+r_e}{e_{\mathbb{L}}}$

$\mathbb{L}$ $\qquad R_{\mathbb{L}}$

$e_{\mathbb{L}} = [v(\mathbb{L}^*) : v(\mathbb{F}^*)]$

$\boxed{(C, D) \text{ is stable}}$ if $\exists$ a nodal model $(\mathcal{R}_L, \mathcal{D}_{R_L})$ s.t. its special fiber is stable

$\Gamma^{st}_{(C,D)}$ to be the tropical curve associated to the distinguished stable model.

Parametrized tropical curves.

N be lattice

$N_{\mathbb{R}}$ parametrized tropical curve

(Γ, h_Γ) $\quad \underline{h_\Gamma} : V(\Gamma) \longrightarrow N_{\mathbb{R}}$

s.t. $\underline{h_\Gamma}(v) \in N$ for $v \in V^\infty(\Gamma)$

$$\frac{1}{|e|}\left(h_\Gamma(v) - h_\Gamma(v')\right) \in N$$

balancing condition:

$v \in V^f(\Gamma)$

$$\sum_{v' \in V^f(\Gamma),\ e \in E_{vv'}(\Gamma)} \frac{1}{|e|}\left(h_\Gamma(v) - h_\Gamma(v')\right) + \sum_{\substack{v' \in V^\infty(\Gamma) \\ e \in E_{vv'}(\Gamma)}} h_\Gamma(v') = 0$$

$h_\Gamma(v) \in N_{\mathbb{Q}} \quad \forall v$ then Γ is called an $N_{\mathbb{Q}}$ parametrized $\mathbb{Q}$-tropical cure

$f : C \setminus D \longrightarrow T_{N, \mathbb{F}}$ & let $(C_{R_{\mathbb{L}}}, D_{R_{\mathbb{L}}})$ be a nodal model

$$\begin{array}{ccc} C \setminus D & \xrightarrow{f} & T_{N, \mathbb{F}} \\ \downarrow & & \downarrow \\ C_{R_{\mathbb{L}}} \setminus D_{R_{\mathbb{L}}} & \xrightarrow{f} & T_{N, R_{\mathbb{L}}} \end{array}$$

χ^m

$h_\Gamma(v) : \frac{1}{e_{\mathbb{L}}} \operatorname{ord}_v f^*(\chi^m) \in \mathbb{Q}$ $\qquad m \in M = N^\vee$

is a linear function of m

$\frac{1}{e_{\mathbb{L}}} \operatorname{ord}_v f^*(\chi^\bullet) \in N_{\mathbb{Q}}$

$$\operatorname{ord}_\gamma f^*(\chi^{m_1 + m_2}) = \operatorname{ord}_\gamma f^*(\chi^{m_1}) + \operatorname{ord}_\gamma f^*(\chi^{m_2})$$

$\frac{1}{|e|}\left(h_\Gamma(v) - h_\Gamma(v')\right) \in N_{\mathbb{Q}}$ for any bounded edge

$C_{R_{\mathbb{L}}}, D_{R_{\mathbb{L}}}$ is regular

C_v is meeting

$t_{\mathbb{L}}^{-(e_{\mathbb{L}} h_\Gamma(v), m)}$ $\quad f^*(x^m)\Big|_{C^v}$ $\quad C_{v'}$

$$\sum_{\substack{v' \in V^f \\ e \in E_{v,v'}}} \left(e_{\mathbb{L}} \cdot h_\Gamma(v') - e_{\mathbb{L}} h_\Gamma(v), m\right) + \sum_{v' \in V^\infty, e \in E_{v,v'}} \left(h_\Gamma(v'), m\right) = 0$$

$$\sum_{\substack{v' \in V^f(\Gamma) \\ e \in E_{vv'}(\Gamma)}} e_{\mathbb{L}} \cdot h_\Gamma(v') - e_{\mathbb{L}} h_\Gamma(v) + \sum_{\substack{v' \in V^\infty \\ e \in E_{vv'}}} h_\Gamma(v') = 0$$

$\square$