1st Talk of Tropical Geometry Seminar V. 2

Setup: $K = \overline{K}$, $\mathrm{val}: K^\times \to \mathbb{R} = \Gamma_{\mathrm{val}}$, $M = \mathbb{Z}^n$, $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$, $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$

- $R = K[x_1^{\pm}, \ldots, x_n^{\pm}]$ ring of Laurent poly over K, $\rightsquigarrow$ coordinate ring of $T^n = (K^\times)^n$
- $f \in R$, $f = \sum_{\substack{n \in \underline{S} \subset \underline{N} \\ |S| < \infty}} a_n z^n$, $\rightsquigarrow \mathrm{trop}(f) : \mathbb{R}^n \to \mathbb{R}$,

$$\mathrm{trop}(f)(w) = \min_{n \in S} \left\{ \mathrm{val}(a_n) + \langle n, w \rangle \right\} \quad n(w)$$

Notice: $\mathrm{trop}(f)$ is akin to replacing $+$ w/ $\oplus$, $\cdot$ w/ $\odot$, and then replacing each a_n w/ $\mathrm{val}(a_n)$.

- $V(f)$ is a subvariety of $T^n \rightsquigarrow$

$$\mathrm{trop}(V(f)) = \left\{ w \in \mathbb{R}^n \,\middle|\, \mathrm{trop}(f) \text{ is not linear } @ w \right\}$$

$=$ the "nonlinear locus" of $\mathrm{trop}(f)$.

Lots of ways to think of $\mathrm{trop}(V(f))$

TFAE

(i) $\mathrm{trop}(V(f))$ as above

(ii) closure in $\mathbb{R}^n$ of $\left\{ w \in \mathbb{R}^n \,\middle|\, \mathrm{in}_w(f) \text{ is not a monomial} \right\}$

initial form of f @ w — elements of $\underline{K[x_1, \ldots, x_n]}$

— track the monomials of f which are the minimizers @ $w \in \mathbb{R}^n$, roughly

$\mathrm{trop}(f)$ linear @ w $\iff$ $\mathrm{in}_w(f)$ is a monomial

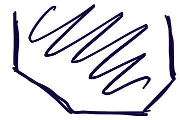
(iii) closure in $\mathbb{R}^n$ of $\left\{ (\mathrm{val}(z_1), \ldots, \mathrm{val}(z_n)) \in \mathbb{R}^n \,\middle|\, z \in V(f) \right\}$

This is Krapanov's Theorem — it's basically the Fundamental Theorem of Tropical Geometry in the case of curves. Proof in $\S$ 3.3 of Sturmfels, Maclagan

There's a lot of combinatorial data associated to tropical varieties

Prop: $f \in R$. Then $\mathrm{trop}(V(f))$ is the support of a rational polyhedral complex of dimension $(n-1)$ in $\mathbb{R}^{n-1}$. It is the $(n-1)$ skeleton of the polyhedral complex dual to the Newton polytope of f.

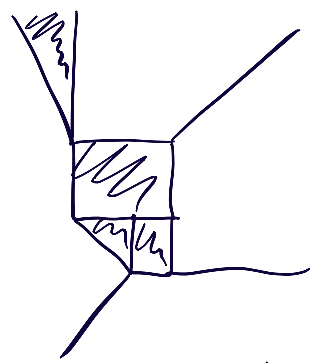
Definitions from Polyhedral Geometry



- Polyhedron $P \subseteq \mathbb{R}^n$ = finite intersection of halfspaces
- Polytope $P \subseteq \mathbb{R}^n$ = bounded Polyhedron (compact Polyhedron)
- Face of P (Polyhedron) $\rightsquigarrow$ determined (non uniquely) by $w \in (\mathbb{R}^n)^\vee = N_{\mathbb{R}}$

$$\text{face}_w(P) = \{ x \in P \mid \langle w, x \rangle \leq \langle w, y \rangle ,\ \forall y \in P \}$$

- Facet of P = face not contained properly in any other face
- Polyhedral Complex Σ = collection of polyhedra s.t.
  - (a) $P \in \Sigma \implies \text{face}_w(P) \in \Sigma$
  - (b) $P, Q \in \Sigma \implies P \cap Q = \emptyset$ or is a face of both P and Q



$|\Sigma|$

polyhedra are called cells

- Support of Σ, $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma \subset M_{\mathbb{R}} = \mathbb{R}^n$
- Σ is of pure-dimension d if every facet of Σ is of dimension d.

$$f \in R \rightsquigarrow \underbrace{(\Delta_S, \mathcal{P}, \varphi)}\ \xrightarrow[\text{dual}]{\text{discrete Legendre}}\ (M_{\mathbb{R}}, \check{\mathcal{P}}, \check{\varphi})$$

Newton polytope of f

Polyhedral complex subdividing Δ_S

PL-function on Δ_S, whose graph gives us the means of forming $\mathcal{P}$

dual of $\mathcal{P}$, polyhedral complex of $M_{\mathbb{R}} = \mathbb{R}^n$ whose $(n-1)$ skeleton is $\mathrm{trop}(V(f))$

$\mathrm{trop}(f)$

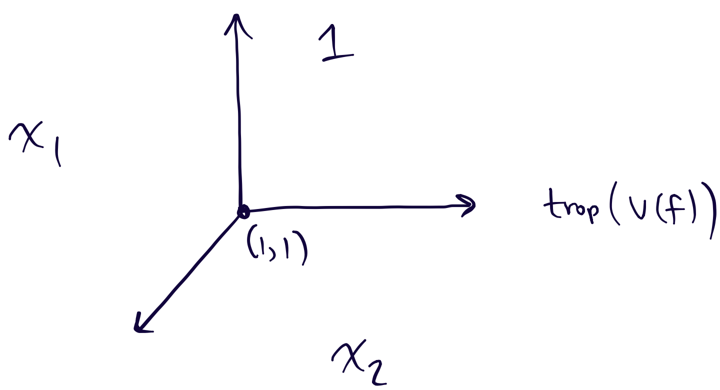
$$\Delta_S := \mathrm{conv}(S) \text{ in } N_{\mathbb{R}} \quad (\text{convex hull})$$

Ex 1:

$\in R = K[x_1, x_2]$,

$$\underbrace{F = \mathrm{trop}(f)} = \underbrace{1 \oplus (0 \odot x_1) \oplus (0 \odot x_2)}$$

$$\underbrace{S = \{(0,0), (1,0), (0,1)\}}$$



$$S = \{(1,1), (1,-1), (-1,1)\}$$

$$f = x_1 x_2 + x_1 \frac{1}{x_2} + \frac{1}{x_1} \cdot x_2$$

$$\mathrm{trop}(f)(w) = \sum_{u \in S} \mathrm{val}(a_u) + \langle u, w \rangle$$

General Procedure $\underbrace{(\Delta_S, \mathcal{P}, \varphi)} \longrightarrow (M_{\mathbb{R}}, \check{\mathcal{P}}, \check{\varphi})$

Start with a tropical polynomial $f = \sum_{n \in S} a_n z^n$, $S \subset N$, $|S| < \infty$

- Δ_S = Newton polytope of f $= \bigcap_{\substack{C \subset N_{\mathbb{R}} \\ C \text{ is convex} \\ \text{and} \\ S \subset C}} C = \operatorname{conv}(S) \subset N_{\mathbb{R}}$

- Define $\varphi : \Delta_S \longrightarrow \mathbb{R}$. Convenient to move to dim $+=1$;
  - $\tilde{S} = (\text{"upper" } S) = \{(n, a_n) \mid n \in S\}$
  - $\tilde{\Delta}_S = (\text{"upper" convex hull}) = \left\{ (n,x) \in N_{\mathbb{R}} \times \mathbb{R} \;\middle|\; \begin{array}{l} n \in \Delta_S \text{ and } x \geq a \\ \text{for some } a \in \mathbb{R}, \text{ s.t.} \\ (n,a) \in \operatorname{conv}(\tilde{S}) \end{array} \right\}$

  choose $\varphi : \Delta_S \longrightarrow \mathbb{R}$ to be the PL-function s.t.

  $$\operatorname{graph}(\varphi) = \partial \tilde{\Delta}_S \cap \operatorname{conv}(\tilde{S}) = \text{"lower boundary"}$$

- Get $\mathcal{P}$ = polyhedral decomp Δ_S, take

  $$\mathcal{P} = \{ \text{images of proper faces of } \tilde{\Delta}_S \text{ under } p_1 : N_{\mathbb{R}} \times \mathbb{R} \longrightarrow N_{\mathbb{R}} \}$$

Ex: $f = 1 \oplus (0 \odot x) \oplus (0 \odot x^2) \oplus (2 \odot x^3)$, $M = \mathbb{Z}$, $N \cong \mathbb{Z}$

- $S = \{0, 1, 2, 3\}$

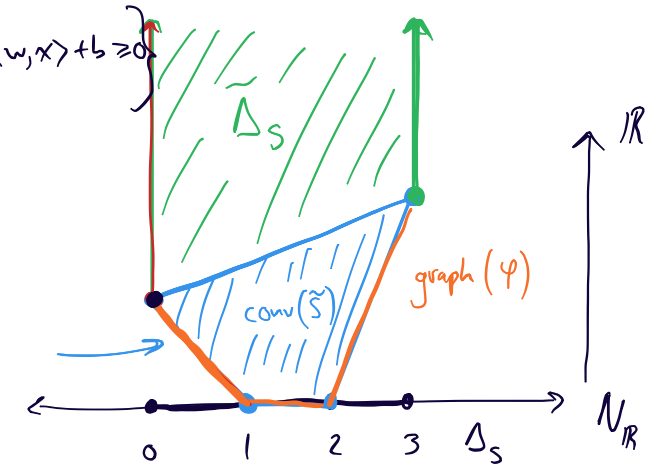
$\tilde{\Delta}_S \subset \mathbb{R}^2$ $\qquad H \cap \tilde{\Delta}_S = \operatorname{face}_w(\tilde{\Delta}_S)$

$H_{\vec{w},b} = \{x \in \mathbb{R}^2 \mid \langle w, x \rangle + b \geq 0\}$

- $\Delta_S = [0,3] \subseteq N_{\mathbb{R}}$

- $\tilde{S} = \{(0,1), (1,0), (2,0), (3,2)\}$

$\operatorname{conv}(\tilde{S})$



$- \tilde{\Delta}_S$

$$\varphi = \begin{cases} -x+1 & x \in [0,1] \\ 0 & x \in [1,2] \\ 2x-4 & x \in [2,3] \end{cases}$$

$- \mathcal{P} =$ $[0,1]$ $[1,2]$ $[2,3]$

$\{0\}$ $\{1\}$ $\{2\}$ $\{3\}$

$\rightsquigarrow$ polyhedral decomp of Δ_S

$$(\Delta_S, \mathcal{P}, \varphi)$$

<u>General Procedure</u> (for dualization) $\qquad (M_{\mathbb{R}}, \check{\mathcal{P}}, \check{\varphi})$

- $M_{\mathbb{R}}$ ✓
- $\check{\mathcal{P}}$ = dual of $\mathcal{P}$ $= \{ \check{\tau} \mid \tau \in \mathcal{P} \}$

$$\check{\tau} = \left\{ m \in M_{\mathbb{R}} \;\middle|\; \begin{array}{l} \exists\, a \in \mathbb{R},\ \text{s.t. } a \le \varphi(n) + \langle m, n \rangle \\ \text{for all } n \in \Delta_S \text{ w/ equality} \iff n \in \tau \end{array} \right\}$$

- $\check{\varphi} : M_{\mathbb{R}} \longrightarrow \mathbb{R}$, $\quad \check{\varphi}(m) = \max \left\{ a \in \mathbb{R} \;\middle|\; \begin{array}{l} a \le \varphi(n) + \langle n, m \rangle \\ \forall\, n \in \Delta_S \end{array} \right\}$

<u>Ex:</u>

$\check{\mathcal{P}}$: $[0,1] \longmapsto \{-1\}$, $[1,2] \longmapsto \{0\}$, $[2,3] \longmapsto \{2\}$

$\{0\} \longmapsto (-\infty,-1]$; $\{1\} \longmapsto [-1,0]$, $\{2\} \longmapsto [0,2]$, $\{3\} \longmapsto [2,\infty)$

$$\check{\varphi}\big|_{(-\infty,-1]}(x) = 3x+2 \rightsquigarrow 2 \odot x^3$$

$$\check{\varphi}\big|_{[-1,0]}(x) = 2x + 0 \qquad \check{\varphi}\big|_{[0,2]}(x) = x + 0$$

$$\check{\varphi}\big|_{[2,\infty)}(x) = 0 \cdot x + 1$$

$$\check{\varphi} = 1 \oplus (0 \odot x) \oplus (0 \odot x^2) \oplus (2 \odot x^3)$$