Tropicalisation of $\overline{\mathcal{M}}_{g,n}$.

First we recall the definition of stable curves.

Let us fix an algebraically closed field k. An *n-pointed nodal curve* $(C; p_1, \dots, p_n)$ of genus g over k is a projective curve C with arithmetic genus $g = g(C)$ with at most nodal singularities. The curve is *stable* if it is connected and the automorphism group $\mathrm{Aut}(C, p_1, \dots, p_n)$ of C fixing the points p_i is finite.

The dual graph of a pointed curve. (Definition 1)

For each n-pointed curve $(C; p_1, \dots, p_n)$ with at most nodes as singularities over $\bar{k}$, we can assign its *weighted dual graph* $G_C = G = (V, E, L, h)$, where

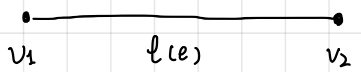
1. the set of vertices $V = V(G)$ is the set of irreducible components C;
2. the set of edges $E = E(G)$ is the set of nodes of C, where an edge $e \in E$ is incident to vertices v_1, v_2 if the corresponding node lies in the intersection of the corresponding components;
3. the ordered set of legs of $L = L(G)$ corresponds the marked points, where a marking is incident to the component on which it lies;
4. the function $h: V \to \mathbb{N}$ is the genus function, where $h(v)$ is the geometric genus of the component corresponding to v.

Note that a node of C that is contained in only one irreducible component corresponds to a loop.

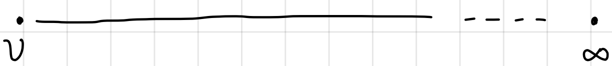
Tropical curves and their moduli

A *tropical curve* is a metric weighted graph $\Gamma = (G, \ell) = (V, E, L, h, \ell)$ where $\ell: E \to \mathbb{R}_{>0}$.

One realize a tropical curve as an "extended" metric space by realizing an edge e as an interval of length $\ell(e)$,



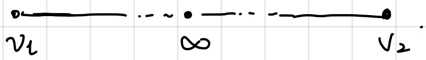
and realize a leg as a copy of $\mathbb{R}_{\geq 0} \sqcup \{\infty\}$, where 0 is attached to its incident vertex:



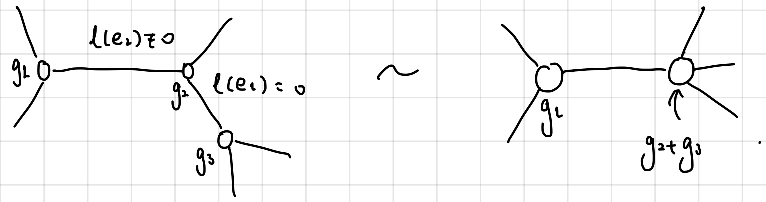
We denote $\mathrm{Aut}(\Gamma) \subseteq \mathrm{Aut}(G)$ as the subgroup of symmetries preserving the length function ℓ.

An *extended tropical curve* is an *extended metric weighted graph* $\Gamma = (G, \ell) = (V, E, L, h, \ell)$ where this time $\ell: E \to \mathbb{R}_{>0} \sqcup \{\infty\}$; we realize an extended tropical curve as an extended metric space by realizing an edge with $\ell(e) = \infty$ as $(\mathbb{R}_{\geq 0} \sqcup \{\infty\}) \cup (\{-\infty\} \sqcup \mathbb{R}_{\leq 0})$

where the points at infinity are identified



The extended cone $\overline{\sigma_G^\circ} = (\mathbb{R}_{>0} \sqcup \{\infty\})^E$ is a fine moduli space for extended tropical curves whose underlying graph is identified with G, and the quotient $\overline{M_G^{trop}} = \overline{\sigma_G^\circ} / \mathrm{Aut}(G)$. And the moduli space of tropical curve is nothing but $\overline{\mathcal{M}_{g,n}^{trop}} = \coprod \overline{\mathcal{M}_G^{trop}} / \sim$. The equivalence relation can be described as the following picture.



Proposition 2: The dimension of $\overline{\mathcal{M}_{g,n}^{trop}}$ is $3g-3+n$.

Proof: left as an exercise.

Reminder: Tropicalization is a functor:

$$\begin{array}{ccc} \overline{\mathcal{M}_{g,n}} & \longmapsto & \overline{\mathcal{M}_{g,n}^{trop}} \\ \cup\!\shortmid & & \cup\!\shortmid \\ C & \longrightarrow & C^{trop} \end{array}$$

, C^{trop} is an extend metric graph, the graph is just the dual graph of C, the length information comes from a semistable model $\mathfrak{X} \to \mathrm{Spec}\, R$, where $\mathrm{Quot}(R)$ is a complete valuation ring (to $\mathbb{R}$). One a nodal point $x \in C$, the local model of x (étale neighborhood) $xy - f = 0$. $\ell(e_x) = \mathrm{val}(f) \in \mathbb{R}_{>0}$.

Now we introduce three natural morphisms: universal morphism, clutching morphism and gluing morphisms.

Instead of giving wordy and length definition: we draw the pictures of these morphism. And it is also more convenient for latter purpose.

Universal morphisms: $\overline{\mathcal{M}_{g,n+1}} \xrightarrow{\pi} \overline{\mathcal{M}_{g,n}}$, For a curve $[C] \in \overline{\mathcal{M}_{g,n+1}}$, if removing the $(n+1)$-th point, $[C']$ remains to be stable, then $\pi([C]) = [C']$.

But if after removing the $(n+1)$-th point, C' is not stable. Then we have the following cases.

1. 

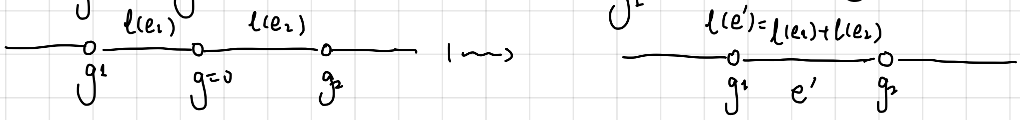
2. 

Correspondingly, the universal morphisms for $\overline{\mathcal{M}_{g,n+1}^{trop}} \longmapsto \overline{\mathcal{M}_{g,n}^{trop}}$ on the level of dual graphs, they are exactly the same. And moreover for non-contracting cases, we just need to keep all the length of edges.

For case 1.

, we just delete $l(e_1)$.

For case 2.



From above, it is not hard to see that

Theorem 3 (i) $\Leftarrow$

$$\begin{tikzcd}
\overline{\mathcal{M}}_{g,n+1} \arrow[r, "\mathrm{trop}"] \arrow[d, "\pi^{an}"] & \overline{\mathcal{M}}^{\mathrm{trop}}_{g,n+1} \arrow[d, "\pi^{\mathrm{trop}}"] \\
\overline{\mathcal{M}}_{g,n} \arrow[r, "\mathrm{trop}."] & \overline{\mathcal{M}}^{\mathrm{trop}}_{g,n}.
\end{tikzcd}$$

The only thing missing here is for case (2): e_1 corresponds to a node p_1 and e_2 corresponds to another node p_2. The local model for p_1: $xy - f_1 = 0$ and for p_2 $xy - f_2 = 0$, after "merging" p_1 and p_2, The local model can be descriped $xy - f_1 f_2 = 0$. $\Rightarrow l(e') = l(e_1) + l(e_2)$ by the uniqueness.

Clutching morphism:

$$\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \xmapsto{\ \kappa\ } \overline{\mathcal{M}}_{g,n} \qquad g = g_1 + g_2, \quad n = n_1 + n_2.$$

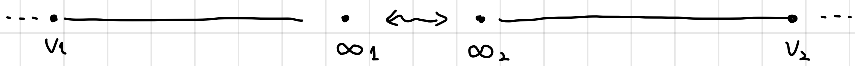
$$(C_1, p_1^1, \cdots, p_{n_1+1}^1) \times (C_2, p_1^2, \cdots, p_{n_2+1}^2) \longmapsto (C, p_1, \cdots, p_n)$$

This is obtained by identifying $p_{n_1+1}^1 = p_{n_2+1}^2$.

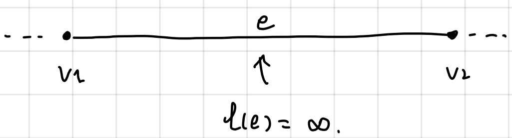
For moduli space of tropical curves, the clutching morphism κ^{trop}: $\overline{M}^{\mathrm{trop}}_{g_1,n_1+1} \times \overline{M}^{\mathrm{trop}}_{g_2,n_2+1} \longrightarrow \overline{M}^{\mathrm{trop}}_{g,n}$

$$\Gamma_1 \times \Gamma_2 \longmapsto \Gamma.$$

we attach the last leg of Γ_1 to the last leg of Γ_2 by identifying their infinite points



to form an edge e of Γ with endpoints v_1 and v_2:



Theorem 3 (ii)

$$\begin{tikzcd}
\overline{M}^{an}_{g_1,n_1+1} \times \overline{M}^{an}_{g_2,n_2+1} \arrow[r, "\mathrm{Trop} \times \mathrm{Trop}"] \arrow[d, "\kappa^{an}"] & \overline{M}^{\mathrm{trop}}_{g_1,n_1+1} \times \overline{M}^{\mathrm{trop}}_{g_2,n_2+1}. \arrow[d, "\kappa^{\mathrm{trop}}."] \\
\overline{M}^{an}_{g,n} \arrow[r, "\mathrm{Trop}"] & \overline{M}^{\mathrm{trop}}_{g,n}.
\end{tikzcd}$$

Proof: At the dual graph level, the communitivity is trivial. The rest is only about length of the cluthting morphism. For (n_1+1)-th point, it correspondes to $\operatorname{Spec} R \xrightarrow{u_i} M_{g_i, n_i+1}$. Since $\kappa^{an}([C^1, C^2])$
(n_2+1)
$= [C] \in \overline{M_{g,n}}^{an}$; it is represented by composition $\kappa \circ (u_1 \times u_2) : \operatorname{Spec} R \longrightarrow \overline{M_{g,n}}$.

Hence $\ell(e) = \infty$, since the local model around $[C]$ is $xy - f = 0$, where $f = 0 \in R$.

3. Gluing morphisms.

In the algebraic setting, for $g > 0$ there is a map $r: \overline{\mathcal{M}_{g-1,n+2}} \longrightarrow \overline{\mathcal{M}_{g,n}}$ obtained by gluing the last two marked points. Now we define the tropical gluing maps $r^{trop}: \overline{\mathcal{M}_{g-1,n+2}}^{trop} \longrightarrow \overline{M_{g,n}}^{trop}$. The procedure is similier to the definition of the tropical clutching maps. r^{trop} maps a tropical curve Γ with $n+2$ legs to the tropical curve Γ' obtained by attaching the last two legs of Γ. The new edge e' has infinite length.

Similier to Theorem 3 (ii), we have the following commutative diagram.

Theorem 3 (iii)

$$\begin{tikzcd}
\overline{\mathcal{M}}_{g-1,n+2}^{an} \arrow[r, "\mathrm{Trop}"] \arrow[d, "r^{an}"] & \overline{\mathcal{M}}_{g-1,n+2}^{trop} \arrow[d] \\
\overline{\mathcal{M}}_{g,n}^{an} \arrow[r] & \overline{\mathcal{M}}_{g,n}^{trop}
\end{tikzcd}$$

Proof: Exactly the same in Theorem 3 (ii).