Part III - Toric Geometry

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Last updated: October 2020

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1 Introduction

In general, algebraic geometry deals with this duality between

$$\{\text{polynomial equations}\} \longleftrightarrow \{\text{varieties (schemes)}\}$$
$$f \longleftrightarrow \mathbb{V}(f).$$

Many times a statement that is hard to prove on one side will be manageable on the other side, and we try to understand things from the knowledge of both sides.

Let's consider the ring $S = \mathbb{C}[x_1, \ldots, x_n]$, and $\mathfrak{I}$ an ideal of S, (finitely) generated by $f_1, \ldots, f_s$. In what is to come, we use multi-index notation: for $\alpha \in \mathbb{N}^n$, write $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$.

There are three flavors of polynomial theories depending on the types of f_i:

- **Monomials**: each f_i has only one term (i.e. $f_i = c_i x^\alpha$).

  A monomial ideal is an ideal $\mathfrak{I} \subset S$ (finitely) generated by monomials. If $\mathfrak{I}$ is monomial, then $\mathbb{V}(\mathfrak{I})$ is a union of coordinate subspaces of $\mathbb{C}^n$.

  Examples:

  1. $x_1 = 0$ in S.
  2. $x_1 x_2 = 0$ in $\mathbb{C}[x_1, x_2]$. The cross is called "reducible" as a union of two (irreducible) lines.
  3. $x_1^2 = 0$. Then $\mathbb{V}(x_1^2)$ is the x_2 axis. But this is not "reduced" in the sense that x_1 is nilpotent.

- Let's come back to binomials in a second.

- **Trinomials**: each f_i has at most 3 non-zero monomial terms. E.g. $x_1 + x_2 + x_3$ is a trinomial, but $x_1 + x_2^2 + x_3^4 + x_4^8$ isn't.

  Lemma. *Any affine variety is a vanishing locus of a trinomial ideal.*

  More precisely, we are saying $S/\mathfrak{I}$ as a ring is isomorphic to $\mathbb{C}[x_1 \ldots, x_N]/\mathfrak{a}$ for some (usually larger) N, where $\mathfrak{a}$ is generated by trinomials. Although this is a nice result, it is pretty useless, and nothing is important here.

  Proof. (sketch) For any polynomial $a_1 x^{\alpha_1} + \cdots + a_l x^{\alpha_l}$ in $\mathfrak{I}$, we introduce $l - 3$ new variables $\{z_i\}_{i=1}^{l-3}$. Then we can replace $a_1 x^{\alpha_1} + \cdots + a_l x^{\alpha_l} = 0$ by

  $$a_1 x^{\alpha_1} + a_2 x^{\alpha_2} + z_1 = -z_1 + z_2 + a_3 x^{\alpha_3} = \cdots = -z_{l-3} + a_{l-1} x^{\alpha_{l-1}} + a_l x^{\alpha_l} = 0.$$

  Repeat this for every equation and we are done. Note this is a much larger ambient space. □

- **Binomials**: let $\mathfrak{I}$ be generated by binomials.

  Assume here $\mathfrak{I}$ is prime (then $\mathbb{V}(\mathfrak{I})$ will be both irreducible and reduced. Call this "integral").

  Proposition. *Let* $\mathbb{C} \supset X = \mathbb{V}(\mathfrak{I})$ *for the above* $\mathfrak{I}$*, with* $\dim X = m$*. Then* X *contains (a copy of)* $(\mathbb{C}^*)^m$ *as a Zariski dense open subset. Moreover, the multiplication map (group operation):*

  $$(\mathbb{C}^*)^m \times (\mathbb{C}^*)^m \to (\mathbb{C}^*)^m$$

  extends to a map

  $$X \times (\mathbb{C}^*)^m \to X.$$

Examples:

1. $\mathbb{C}^n$ with $\mathfrak{I} = (0)$. This obviously contains $(\mathbb{C}^*)^n$ as a dense open. The map is defined by

$$\mathbb{C}^n \times (\mathbb{C}^*)^n \to \mathbb{C}^n$$
$$(z_1, \ldots, z_n) \times (t_1, \ldots, t_n) \mapsto (z_1 t_1, \ldots, z_n t_n).$$

2. Take the hyperplane $\mathbb{V}(x_1 - x_2)$ in $\mathbb{C}^3$, which is isomorphic to $\mathbb{C}^2$ (and is 2-dimensional). It contains $(\mathbb{C}^*)^2 = \{(t, t, t') \mid t, t' \in \mathbb{C}^*\}$. It extends to the whole $\mathbb{V}(x_1 - x_2)$ as 0 times anything is 0.
3. Take $\mathbb{V}(x_1^2 - x_2)$ in $\mathbb{C}^3$, which is again 2-dimensional. It contains $(\mathbb{C}^*)^2 = \{(t, t^2, t') \mid t, t' \in \mathbb{C}^*\}$.

Key fact: monomial is a homomorphism $(\mathbb{C}^*)^n \xrightarrow{u} \mathbb{C}^*$. So the ideal $\mathfrak{I}$ encodes (as we require binomials equal 0) the intersection of translates of the kernels of the homomorphisms u. Thus there is a hidden algebraic torus structure here.

Definition. An *algebraic torus* over $\mathbb{C}$ of rank 1 is a group variety $\mathbb{G}_m = \mathbb{C}^*$. A rank n algebraic torus is $(\mathbb{G}_m)^n = (\mathbb{C}^*)^n$.

By group variety (aka algebraic group), we are saying:

- There are usual group axioms on $\mathbb{G}_m$.
- It's a variety: $(\mathbb{C}^*)^n = \mathbb{V}(x_1 x_2 \cdots x_{n+1} - 1)$ (RHS mapping to $(x_1, \ldots, x_n)$ gives the required identity).
- Both: the multiplication and its inverse are both morphisms of varieties.

lecture 2

If X is an affine variety,
$$X = \mathbb{V}(\mathfrak{I})$$
for some ideal $\mathfrak{I} \subset \mathbb{C}[x_1, \ldots, x_n]$. But by doing so, we are embedding X into the ambient space $\mathbb{A}^n_{\mathbb{C}}$. $S/\mathfrak{I}$ has many presentation of a quotient of an polynomial ring.

Notation: $\mathrm{Spec}(S/\mathfrak{I}) = \mathbb{V}(\mathfrak{I})$, and $\mathrm{Spec}(R) = \mathbb{V}(\mathfrak{I})$ where $R = S/\mathfrak{I}$.

Definition. An *(affine) toric variety* X is an irreducible (affine) variety over $\mathbb{C}$ of dimension l, containing a copy of $(\mathbb{C}^*)^l$ as a dense open subset, such that the multiplication map $m : (\mathbb{C}^*)^l \times (\mathbb{C}^*)^l \to (\mathbb{C}^*)^l$ extends to a group action $X \times (\mathbb{C}^*)^l \to X$.

Remark.

1. Since X is irreducible, Zariski open & non-empty imply dense.
2. We can drop affine, and by variety, we mean a finite-type $\mathbb{C}$-scheme.
3. The embedding of $(\mathbb{C}^*)^l$ is part of the data. Any $p_1 \neq p_2$ on $\mathbb{P}^1$ can be moved to 0 and ∞ by a Mobius map, and thus $\mathbb{P}^1 \setminus \{p_1, p_2\} \cong \mathbb{C}^*$.

Example.

1. $(\mathbb{C}^*)^l$ and $\mathbb{C}^l$.
2. $(\mathbb{C}^*)^{l_1} \times (\mathbb{C}^*)^{l_2}$.

3. $\mathbb{P}^l$. Pick homogeneous coordinates $x_0, \ldots, x_l$. Then identify

$$(\mathbb{C}^*)^l = \{[1 : t_1 : \cdots : t_l] \mid t_i \in \mathbb{C}^*\} \hookrightarrow \mathbb{P}^l.$$

   To check density and openness, note:

$$\mathbb{P}^l \supset \mathbb{C}^l = \{[x_0 : \cdots : x_l] \mid x_0 \neq 0\} \supset (\mathbb{C}^*)^l = \{\text{all } x_i \neq 0\}.$$

4. $\mathbb{P}^{l_1} \times \cdots \times \mathbb{P}^{l_k}$.

Remark. There are infinitely many toric varieties in every dimension $l \geq 2$. What is more interesting is that there exist toric varieties X where

- X is singular.
- X is compact but not projective.
- $\operatorname{Pic}(X)$ has torsion.

All such cases can be constructed using combinatorics.

Two lattices we will use frequently:

- $N = \operatorname{Hom}_{\text{alg grp}}(\mathbb{C}^*, (\mathbb{C}^*)^l) \cong \mathbb{Z}^l$, called the *cocharacter lattice*, where the isomorphism is given by (RHS to LHS):
$$(\lambda_1, \ldots, \lambda_l) \mapsto \Big(t \mapsto (t^{\lambda_1}, \ldots, t^{\lambda_l})\Big).$$
- $M = \operatorname{Hom}_{\text{alg grp}}((\mathbb{C}^*)^l, \mathbb{C}^*) \cong \mathbb{Z}^l$, called the *character lattice*, where the isomorphism is given by :
$$(u_1, \ldots, u_l) \mapsto \Big((t_1, \ldots, t_l) \mapsto t_1^{u_1} \cdots t_l^{u_l}\Big).$$

Let X be a toric variety with torus $\mathbb{G}_m^l$ and lattices N and M. Given $\lambda \in N$ in the cocharacter lattice, i.e. a morphism

$$\lambda(t) : \mathbb{C}^* \to (\mathbb{C}^*)^l \subset X,$$

we can ask: does $\lim_{t \to \infty} \lambda(t)$ exist? If so, what is it?

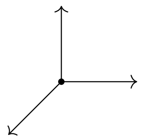
Example. $\mathbb{P}^2$ with $(\mathbb{C}^*)^2 = \{[1 : t_1 : t_2]\}$, and $N = \mathbb{Z}^2$.

Limit always exists, as $X = \mathbb{P}^2$ is proper. But what is the limit? Let $\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2$. Let's consider the case when $t \to 0$.

1. $\lambda_1, \lambda_2 > 0$. Then
$$\begin{aligned} \lambda : \mathbb{C}^* &\to (\mathbb{C}^*)^2 \\ t &\mapsto [1 : t^{\lambda_1} : t^{\lambda_2}] \to [1 : 0 : 0]. \end{aligned}$$
2. $\lambda_1 = 0$ and $\lambda_2 > 0$.
$$t \mapsto [1 : 1 : t^{\lambda_2}] \to [1 : 1 : 0].$$
3. $\lambda_1 > \lambda_2$ and $\lambda_2 < 0$.
$$[1 : t^{\lambda_1} : t^{\lambda_2}] = [t^{-\lambda_2} : t^{\lambda_1 - \lambda_2} : 1] \to [0 : 0 : 1].$$

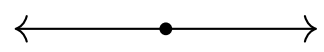
   etc.

There are a total of 7 cases, making a *fan* in $N \otimes \mathbb{R} = N_{\mathbb{R}} = \mathbb{R}^2$ of the following form (3 areas, 3 axises, and origin):

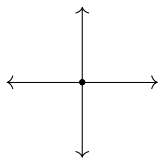


More pictures:

$\mathbb{P}^1$:

 in $\mathbb{R}$

$\mathbb{P}^1 \times \mathbb{P}^1$:



Every toric variety X determines such a fan $\Sigma \subset N_{\mathbb{R}}$.

By fan, we mean a union of *cones* in $N_{\mathbb{R}}$, where a cone is of the form $\operatorname{cone}(S) = \{v \in N_{\mathbb{R}} \mid \sum_{i=1}^{r} v_i \lambda_i, v_i \geq 0\}$ for a finite set $N \supset S = \{\lambda_1, \ldots, \lambda_r\}$. We will make this precise next time.

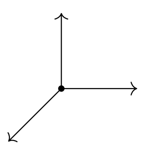
2 Fans, cones, and toric varieties over $\mathbb{C}$

lecture 3 Let X be a toric variety of dimension n, i.e.

$$X \supset (\mathbb{C}^*)^n \cong \mathbb{G}_m^n.$$

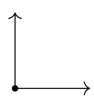
The cocharacter lattice $N = \mathrm{Hom}_{\text{alg grp}}(\mathbb{C}^*, (\mathbb{C}^*)^l) \cong \mathbb{Z}^l$.

Last time: for $\mathbb{P}^2$ we saw that, by examining each $\lambda \in N$ and the values of $\lim_{t \to 0} \lambda(t) \in X$, we obtained a diagram:



Fact: this construction works for any toric variety X, and will give rise to Σ_X a collection of subsets of N.

Example. $X = \mathbb{C}$. Then $\Sigma_X = \Sigma_{\mathbb{C}^n}$ is $\mathbb{R}^n_{\geq 0} \subset \mathbb{R}^n$. If $n = 2$, we get 4 subsets (1 area, 2 axes and origin):



and outside this first quadrant, the limit does not exist.

Natural questions to ask: what property does Σ_X have? Does this construction $X \rightsquigarrow \Sigma_X$ have an inverse construction?

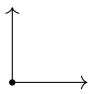
2.1 Convex geometry

Let N, M be as before, the cocharacter and character lattices. By definition, there is a pairing $\langle \cdot, \cdot \rangle : N \times M \to \mathbb{Z}$.

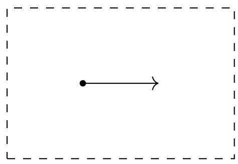
Definition. A cone σ in $N_{\mathbb{R}}$ is the non-negative span of a finite set $S \subset N$.

Example.

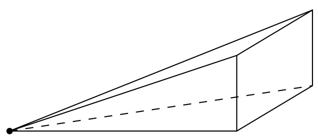
- $\sigma = \mathbb{R}^2_{\geq 0}$



- $\mathbb{R}^k_{\geq 0} \subset \mathbb{R}^n$

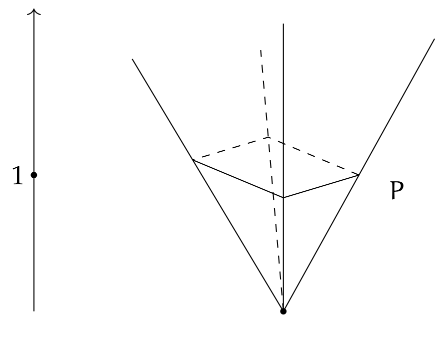


- The cone in $\mathbb{R}^3$ generated by $(e_1,\ e_2,\ e_1 + e_3,\ e_2 + e_3)$ (only non-negative span, so we need the last vector here):



which is the cone over a square (not to scale with the vectors provided...but roughly the same).

The general construction is that, take $P \subset \mathbb{R}^{n-1}$ a polytope (convex hull of a finite set), and define a cone over $P \subset \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$ at height 1.



Definition. Let $\sigma \subset N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ be a cone. Define the *dual cone* $\sigma^{\vee}$ as

$$\sigma^{\vee} = \{m \in M_{\mathbb{R}} \mid \langle v, m \rangle \geq 0 \quad \forall v \in \sigma\}.$$

The dual monoid (or semigroup) of σ is then $S_\sigma = \sigma^{\vee} \cap M$. (Exercise: show the monoid S_σ is finitely generated.)

(Note: think of this as a positivity condition. We are asking functions that evaluate to a non-negative value on all points of σ).

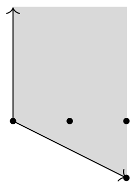
Now we can take $\mathbb{C}[S_\sigma]$ to be the "group" algebra, and take $U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$.

Lemma. U_σ *is an affine toric variety.*

Proof. We know $S_\sigma \hookrightarrow M$, so the group algebras satisfy $\mathbb{C}[S_\sigma] \hookrightarrow \mathbb{C}[M]$. But $\mathbb{C}[M]$ is the ring of Laurent polynomials in n variables (where $M \cong \mathbb{Z}^n$), and thus $\mathrm{Spec}\,\mathbb{C}[M] = (\mathbb{C}^*)^n$. So we have the toric structure $(\mathbb{C}^*)^n \hookrightarrow U_\sigma$, which also extends. □

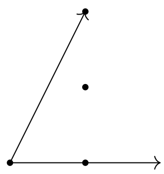
Example. In this example, let's take σ to be the cone generated by $(e_2, 2e_1 - e_2)$ in $\mathbb{R}^2$.

σ :



To calculate $\sigma^\vee$, first see that the $\mathbb{R}_{\geq 0}$ span is the following area:

$\sigma^\vee$:



And note that, to write as a $\mathbb{Z}_{\geq 0}$ span, we actually need three vectors (e_1^*, $e_1^* + e_2^*$, $e_1^* + 2e_2^*$). Then it's straightforward to see the only relation is 2 times the second equals 1+3. Put in the definition of functions on M, this translate to:

$$\mathbb{C}[S_\sigma] = \mathbb{C}[u, v, w]/(u^2 - vw).$$

One thing to note: we needed three vectors to generate $\sigma^\vee$, instead of the expected 2. This has something to do with the variety not being smooth. We will see this later.

2.2 Gluing affines

In general, we can glue (toric) affines to get general (toric) varieties. We will glue along some "face".

Definition. A *fan* in $N_\mathbb{R}$ is a collection Σ of cones that is closed under taking faces of cones and intersection, and the intersection of two cones in a face of each (picture is better than words here).

Given two cones σ, τ and inclusion $\tau \hookrightarrow \sigma$, there is an induced map $\sigma^\vee \hookrightarrow \tau^\vee$. In particular, if τ is origin, then $\tau^\vee = M_\mathbb{R}$.

lecture 4 Last time we saw this construction $\sigma \rightsquigarrow \sigma^\vee \xrightarrow{\cap M} S_\sigma \rightsquigarrow \mathbb{C}[S_\sigma] \xrightarrow{\mathrm{Spec}(-)} U_\sigma \supset (\mathbb{C}^*)^n$. Call this construction $(\dagger)$.

For a more detailed tour, see Fulton's Polyhedral geometry, section 1.2.

Definition. Let $\sigma \subset N_\mathbb{R}$ be a cone. A *supporting hyperplane* for σ is a hyperplane

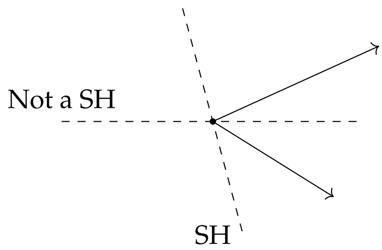
$$H_m = \{v \in N_\mathbb{R} \mid \langle v, m \rangle = 0\}$$

for some fixed $m \in M$, such that σ is contained in the associated halfspace

$$H_m^+ = \{v \in N_\mathbb{R} \mid \langle v, m \rangle \geq 0\}.$$

A *face* of σ is the intersection $\sigma \cap H_m$ for H_m a supporting hyperplane.

Example. Consider the cone below:



In particular, there are 4 faces. Two solid lines, origin, and the whole cone (because we could take $m = 0$).

Let $\tau \hookrightarrow \sigma$ be a face. This determines a morphism of toric varieties $U_\tau \hookrightarrow U_\sigma$. How do we describe this?

Let $m \in M$ be the vector that determines the supporting hyperplane for τ. So $\tau = \sigma \cap \{\langle m, v\rangle = 0\}$. Look back at the construction †. At the level of $\mathbb{C}[S_\sigma] \hookrightarrow \mathbb{C}[S_\tau]$, note m is in $\sigma^\vee$ (because σ is contained in the halfspace by definition of supporting hyperplane) and thus is in S_σ. Thus we have the equality

$$S_\tau = S_\sigma + \mathbb{Z}_{\leq 0} \cdot (m).$$

When we translate this statement to $\mathbb{C}[S_\sigma]$ where addition is multiplication, what we get is that we are adding in $1/m$ (corresponding to $-m \in S_\tau$) to $\mathbb{C}[S_\tau]$. So the inclusion $\mathbb{C}[S_\sigma] \hookrightarrow \mathbb{C}[S_\tau]$ can be identified as the localization at $1/m$ (think of $\mathbb{C}^* \hookrightarrow \mathbb{C}$ as in $\mathbb{C}[x] \hookrightarrow \mathbb{C}[x, x^{-1}] = \mathbb{C}[x]_{(x)}$). This identifies U_τ as an open set of U_σ (where $m \neq 0$).

(Try yourself: $\{0\} \hookrightarrow \mathbb{R}_{\geq 0}$ defines the map $\mathbb{C}^* \hookrightarrow \mathbb{C}$ described above.)

Example. A trivial example. If we take $\tau = \{0\}$ and $\sigma = \mathbb{R}^2_{\geq 0}$, and any supporting hyperplane that goes through the origin (e.g. $m = (1, 1)$), then $S_\tau = M$, and $S_\sigma = \mathbb{Z}_{\geq 0}$. We are inverting x and y here.

Sometimes we would use the notation χ^m to emphasize the function nature of $m \in M$.

Recall a fan Σ in $N_\mathbb{R}$ is a collection of cones such that

- every face of a cone is in Σ,
- if $\sigma_1, \sigma_2 \in \Sigma$, then $\sigma_1 \cap \sigma_2$ is a face of each, and is an element of Σ.

Now we can construct a toric variety for a given fan ($\Sigma \rightsquigarrow X_\Sigma$): given $\sigma \in \Sigma$, associate the affine toric variety U_σ. If σ_1 and σ_2 intersect in a cone τ, then we glue U_{σ_1} and U_{σ_2} along the open subset U_τ.

Notation: if X is a variety, denote by X^{an} the associated complex analytic topological space.

Example. $\mathbb{A}^1$ with two origins. We see X is not separated scheme is equivalent to X^{an} being not Hausdorff.

Lemma. *For any fan Σ, the toric variety X_Σ is a separated scheme, or equivalently X_Σ^{an} over $\mathbb{C}$ is Hausdorff.*

Proof. We will prove the Hausdorff statement, which is equivalent to the diagonal

$$X_\Sigma^{an} \to X_\Sigma^{an} \times X_\Sigma^{an}$$

is closed.

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Before we continue the proof from last time, let's revisit this general idea of gluing. Let Σ be a fan. Each $\sigma \in \Sigma$ (a face) determines an affine toric variety $U_\sigma = \operatorname{Spec}(\mathbb{C}[S_\sigma])$. Face inclusion $\tau \hookrightarrow \sigma$ determines open immersion:

$$U_\tau \hookrightarrow U_\sigma.$$

The toric variety X_Σ is determined by identifying U_τ inside both U_{σ_1} and U_{σ_2} when τ is a face of both.

Last time we also mentioned the analytic space X^{an}. If U is affine, say $U = \operatorname{Spec} A$ for some $\mathbb{C}$-algebra $\mathbb{C}[x_1, \ldots, x_n]/\mathfrak{I}$. So $U \hookrightarrow \mathbb{C}^n$ as $\mathbb{V}(\mathfrak{I})$.

But $\mathbb{C}^n$ has at least two valid topologies.

- Zariski. This is too coarse. Open sets are way too large.
- Euclidean topology where $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and give U the subspace topology. Then
  - This topology is independent of choices of ambient space (say make n a little larger).
  - Work with gluing, so we can get a space X^{an} this way.

We continue the proof from last time, where we need to show the diagonal is closed.

Proof. We have an open cover of $X_\Sigma \times X_\Sigma$ by open sets of the form $U_{\sigma_1} \times U_{\sigma_2}$ for σ_1 and σ_2 in Σ. It suffices to show that if $\tau = \sigma_1 \cap \sigma_2$, then $U_\tau \hookrightarrow U_{\sigma_1} \times U_{\sigma_2}$ is closed. (i.e. being closed can be checked affine-locally)

By standard argument in algebraic geometry, the (fiber) product of affines corresponds to the tensor product of their coordinate rings. So it suffices to check the map

$$\mathbb{C}[S_{\sigma_1}] \otimes_{\mathbb{C}} \mathbb{C}[S_{\sigma_2}] \to \mathbb{C}[S_\tau]$$

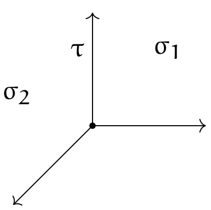
is surjective. This is because we can then identify $\mathbb{C}[S_\tau]$ as the quotient of $\mathbb{C}[S_{\sigma_1}] \otimes_{\mathbb{C}} \mathbb{C}[S_{\sigma_2}]$ by some ideal $\mathfrak{I}$, but then function in $\mathfrak{I}$ determines U_τ locally inside $U_{\sigma_1} \times U_{\sigma_2}$, thus locally closed.

To see why this map is surjective, notice that (check this! This is called separation lemma):

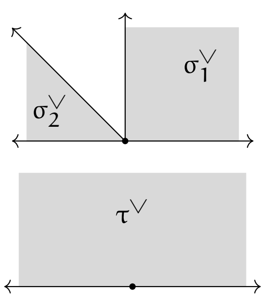
$$S_\tau = S_{\sigma_1} + S_{\sigma_2}.$$

So surjectivity follows. $\square$

As another simple example, think of the fan of $\mathbb{P}^2$:



A little calculation shows:



Hope it's then clear that any point in S_τ can be written as a sum of points from S_{σ_1} and S_{σ_2}. A proof is not hard, but needs the identity $S_\tau = S_\sigma + \mathbb{Z}_{\leq 0} \cdot m$.

3 Properties of X_Σ

3.1 Smoothness

In fact, we can only consider U_σ, as smoothness at a point is a local condition.

Let A be a $\mathbb{C}$-algebra. Then a maximal ideal $\mathfrak{m} \subset A$ determines a point in $U = \operatorname{Spec}(A)$. Say $\dim U = n$. Then U is smooth at the point if

$$\dim_{\mathbb{C}}(\mathfrak{m}/\mathfrak{m}^2) = n.$$

How can we extend this idea to toric varieties? A point, by Nullstellensatz, is a map

$$\mathbb{C}[x_1, \ldots, x_n] \twoheadrightarrow A \to A/\mathfrak{m} = \mathbb{C}.$$

If S_σ is a monoid, and $A = \mathbb{C}[S_\sigma]$. Then similarly, a monoid homomorphism $S_\sigma \to \mathbb{C}$ (where $\mathbb{C}$ is a multiplicative monoid) determines a map $\mathbb{C}[S_\sigma] \to \mathbb{C}$, and thus a point of U_σ.

Example. If $S_\sigma = \mathbb{N}^2$, to produce a homomorphism $\mathbb{C}[S_\sigma] = \mathbb{C}[x, y] \to \mathbb{C}$, we need to decide where x and y are sent to. This is exactly the same as the information in a monoid homomorphism $\mathbb{N}^2 \to \mathbb{C}$.

In particular, there are way ways to identity a point on an affine toric variety:

1. A maximal ideal of $\mathbb{C}[S_\sigma]$.
2. A surjective map of $\mathbb{C}$-algebras: $\mathbb{C}[S_\sigma] \twoheadrightarrow \mathbb{C}$.
3. A monoid map: $S_\sigma \to \mathbb{C}$, where $\mathbb{C}$ is a multiplicative monoid.

We will use these three interchangeably.

Proposition. *The affine toric variety U_σ is smooth iff the minimal generators of σ form a subset of a $\mathbb{Z}$-basis for N union 0.*

Example.

- If $\sigma = (e_1)$ in $\mathbb{R}$, then $U_\sigma = \mathbb{A}^1$. $\mathbb{C}[S_\sigma] = \mathbb{C}[x]$.
- If $\sigma = (e_1, e_2)$ in $\mathbb{R}^2$, then $U_\sigma = \mathbb{A}^2$.
- If $\sigma = 0$ in $\mathbb{R}^2$, then $U_\sigma = \mathbb{G}_m^2$.
- If $\sigma = (e_1)$ in $\mathbb{R}^2$, then $U_\sigma = \mathbb{A}^1 \times \mathbb{G}_m$. In general, situations like this will give $U_\sigma = \mathbb{A}^k \times \mathbb{G}_m^{n-k}$.

Proof. We prove the statement under a mild assumption that $\sigma^\perp = \{0\}$, i.e. σ is of full dimension n. (Exercise: remove this assumption)

There exists a distinguished "point" X_σ, by the map

$$\begin{aligned} S_\sigma &\to \mathbb{C} \\ u &\mapsto \begin{cases} 1 & u = 0, \\ 0 & u \neq 0. \end{cases} \end{aligned}$$

lecture 6 Now U_σ is smooth, then it must be smooth at this particular point X_σ. In particular, let $\mathfrak{m}$ be the associated maximal ideal of the point, then $\dim \mathfrak{m}/\mathfrak{m}^2 = n$ the dimension of U_σ.

Let's inspect $\mathfrak{m}/\mathfrak{m}^2$. For $\mathfrak{m}$, it at least contains $S_\sigma \setminus \{0\}$, because any $u \in S_\sigma \setminus \{0\}$ is mapped to 0, and thus is in the ker of the $\mathbb{C}$-algebra map. For $\mathfrak{m}^2$, it contains all sums of two elements from S_σ. So in particular, $\mathfrak{m}/\mathfrak{m}^2$ contains elements in $S_\sigma \setminus \{0\}$ that cannot be written as a sum of two other elements.

Therefore, any first non-zero element of S_σ along a ray is an element of $\mathfrak{m}/\mathfrak{m}^2$. But $\sigma^\vee$ has at most n rays. So $\dim \sigma^\vee = n$ and σ is as required. □

Corollary. *If Σ is a fan, then X_Σ smooth iff X_σ is smooth for each cone $\sigma \in \Sigma$ (unimodular).*

3.2 Normality

Definition. An affine scheme $X = \operatorname{Spec}(A)$ for A an integral domain, is normal iff A is integrally closed in its fraction field.

Remark. I wrote something about normality a while ago at HERE (click). It contains some more intuitions about how to think of all these.

- For curves, normal is the same thing as smooth (regular).
- If X is normal, then X is regular in codim 1, i.e. singular locus has codim at least 2.
- Serre's criterion for normality states: X is normal iff R_1 + S_2 (this is deep).
- Zariski's main theorem: if X is normal, then any finite and birational map $Y \xrightarrow{f} X$ is an isomorphism.

Anyway, we won't really be using all of above.

Lemma. *Let Σ be a fan. Then X_Σ is normal.*

Proof. Let $\sigma = \langle v_1, \ldots, v_h \rangle$ for some $v_i \in N$. If $\tau = \langle v_i \rangle$, then $\operatorname{Spec}(\mathbb{C}[S_\tau]) \cong \mathbb{C} \times (\mathbb{C}^*)^{n-1}$ (the $\mathbb{C}$ component on v_i direction, and $\mathbb{C}[S_\tau] = \mathbb{C}[x_1, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}]$).

But $\mathbb{C}[S_\tau]$ is obviously integrally closed. So U_τ is normal.

Fact from commutative algebra: intersection of some integrally closed rings is again integrally closed.

Thus, also using the fact that normality is affine local, we have the result. □

Remark.

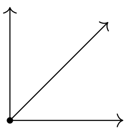
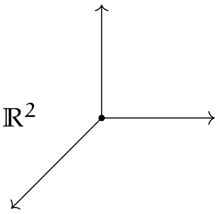
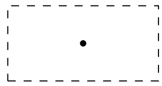
- In fact, S_2 is too weak. Toric varieties satisfy S_n for all n, and are therefore Cohen-Macaulay.
- Some authors will therefore add "normality" into the most basic definition of toric varieties (contains a copy of $(\mathbb{C}^*)^n$ and action extends).

3.3 Properness

Definition. The *support* of a fan Σ is the set $|\Sigma| = \{v \in N_\mathbb{R} \mid \exists\, \sigma \in \Sigma \text{ such that } v \in \sigma\}$.

Proposition. *A toric variety X_Σ is proper (equivalently X^{an} is compact) iff $|\Sigma| = N_\mathbb{R}$.*

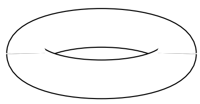
Example.

$|\Sigma| = \mathbb{R}^2_{\geq 0}$  $|\Sigma| = \mathbb{R}^2$  $|\Sigma| = 0$ 

Digression on distinguished points on toric varieties

lecture 7 Last time when we proved smoothness, we used "the" special point on our toric variety.

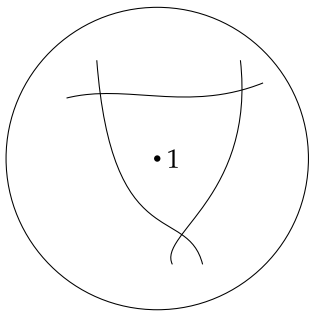
Example. If $X_\Sigma = (\mathbb{C}^*)^n$. There is obviously an identity element as a distinguished point.



Or if $X_\Sigma = \mathbb{A}^1$, which contains a $\mathbb{C}^*$, we now have two distinguished points 0 and 1.

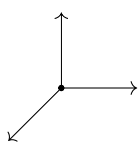
$\mathbb{C}^2 = \mathrm{Spec}(\mathbb{C}[\mathbb{N}^2])$. Both coordinate axes are a copy of $\mathbb{C}$, having their own distinguished points as toric varieties.

Observation: X_Σ has the action of $\mathbb{G}_m^n$. Each orbit of the action is isomorphic to $\mathbb{G}_m^{n-k}$ for some k. Say in the following picture of $\mathbb{P}^2$:



where each curvy line is a piece $\mathbb{V}(x), \mathbb{V}(y), \mathbb{V}(z)$, each having their own identity, and together with three more intersections. Notice there is a specialization relation: an orbit O_1 can contain O_2 in its closure.

Recall for $\mathbb{P}^2$ we have the fan:



There is an orbit-cone correspondence: each orbit of $\mathbb{G}_m^n$ corresponds to a cone of Σ, but is order reversing with respect to inclusion. We also had an order preserving correspondence that if $\tau \hookrightarrow \sigma$, then $U_\tau \hookrightarrow U_\sigma$.

How do we pick the distinguished point? Let σ be a cone, and $U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$. We wish to define a point on U_σ, and we do it as follows:

A point $x_\sigma \in U_\sigma$ is the same thing as a ring homomorphism $\mathbb{C}[S_\sigma] \twoheadrightarrow \mathbb{C}$, which is again the same as a monoid homomorphism $S_\sigma \to \mathbb{C}$. We define x_σ by:

$$\begin{aligned} S_\sigma &\to \{0,1\} \subset \mathbb{C} \\ u &\mapsto \begin{cases} 1 & u \in \sigma^\perp, \\ 0 & \text{o/w}, \end{cases} \end{aligned}$$

where the orthogonal complement $\sigma^\perp = \{m \in M_\mathbb{R} \mid \langle m, n\rangle = 0 \ \forall n \in \sigma\}$.

Sanity check: if $\sigma = 0$, then $U_\sigma = \mathbb{G}_m^n$. In this case, $\sigma^\perp = M$, and so $x_\sigma = \mathrm{id}$. Check this agrees in other examples.

If $\tau \subset \sigma$ is a face, then $U_\tau \subset U_\sigma$ open embedding, and thus $x_\tau \in U_\sigma$.

Back to properness

Example.

- $(\mathbb{P}^n_\mathbb{C})^m$ is proper.
- If X is (connected) proper, $Z \subset X$ is closed, then $X \setminus Z$ is not proper.
- $\mathbb{A}^n$ is not proper if $n \geq 1$.

In topology, (at least for metric spaces) we can test the compactness by checking sequential compactness. In algebraic geometry, we have a similar result, which we quote here:

Theorem. *(Valuative criterion of properness) If X is noetherian, finite type over $\mathbb{C}$, then X is proper iff for every morphism $f : \mathit{Spec}\,\mathbb{C}((t)) \to X$ extends uniquely to a commutative diagram:*

$$\begin{tikzcd} \mathit{Spec}\,\mathbb{C}[[t]] \arrow[r, dashed, "\exists!"] & X \\ \mathit{Spec}\,\mathbb{C}((t)) \arrow[u] \arrow[ur] & \end{tikzcd}$$

where $\mathbb{C}[[t]]$ is the formal power series ring, and $\mathbb{C}((t))$ is the formal Laurent series ring (finitely many negative terms), and the Spec map is induced from the inclusion $\mathbb{C}[[t]] \hookrightarrow \mathbb{C}((t))$.

Remark. Let's unwind this a little bit. $\mathbb{C}[t]$ is the ring of functions on the affine line $\mathbb{A}^1$. So when we invert t (corresponding to localization at the maximal ideal $(t-0)$), $\mathbb{C}[t, t^{-1}]$ corresponds to the functions on $\mathbb{A}^1 \setminus \{0\}$.

Similarly, $\mathbb{C}[[t]]$ (only two prime ideals: 0 and the maximal ideal (t)), either considered as germs of functions near origin, or the completion of the localized ring above, is the ring of functions on an infinitesimal neighborhood of $0 \in \mathbb{A}^1$. Thus $\mathbb{C}((t))$ (this is a field, so the only prime ideal is 0) is the ring of functions on a formal punctured disk.

Corollary. *If there exists a morphism $\mathbb{C}^* \xrightarrow{f} X$ such that the limit $\lim_{t\to 0} f(t)$ does not exist (ie. f does not extends to a map from $\mathbb{A}^1$), then f is not proper.*

Proof. $\mathbb{C}^*$ has ring of functions $\mathbb{C}[t, t^{-1}] \hookrightarrow \mathbb{C}((t))$. □

Lemma. *Let Σ be a fan in $N_\mathbb{R}$ and $v \in N$. By definition v defines a function $\phi_v(t) : \mathbb{C}^* \to (\mathbb{C}^*)^n \hookrightarrow X_\Sigma$. Then $\lim_{t\to 0} \phi_v(t)$ in X_Σ:*

- *does not exist if $v \notin |\Sigma|$.*

- *if σ is the smallest (wrt. inclusion) cone containing ν, then* $\lim_{t\to 0} \phi_\nu(t) = x_\sigma$.

Proof. You should really do this yourself. I will put up one later. $\square$

lecture 8 *Proof.* (to properness proposition)

Suppose $|\Sigma| \neq N_{\mathbb{R}}$ (also called not complete), choose $\nu \in N$ such that $\nu \notin \sigma$ for all $\sigma \in \Sigma$. Then such a ν gives a map $\phi_\nu(t) : \mathbb{C}^* \to X_\Sigma$. By lemma above, the limit $\lim_{t\to 0} \phi_\nu(t)$ does not exist, which would contradict the properness.

Conversely, we use the valuative criterion (abbrev. VC) to show X_Σ is proper. Notation: $K = \mathbb{C}((t))$ and $R = \mathbb{C}[[t]]$.

Fact: if X is irreducible and $U \subset X$ is open, then it suffices to check VC in the case where the image of f is inside U.

In our case, we take $U = \mathbb{G}_m^n$ the dense open torus. As in the setup of VC, let $f : \operatorname{Spec} K \to \mathbb{G}_m^n$ be a map. This would be the same as a ring map $\mathbb{C}[M] \to K$, which again is the same in monomials as a map $M \to K^*$.

Now K^* has a natural valuation structure:

$$K^* \to \mathbb{Z}$$
$$\sum_{i=k}^{\infty} a_i t^i \mapsto k \quad (a_k \neq 0).$$

These two maps together give a map

$$M \to K^* \to Z$$

which is the same as a point $\nu \in N = \operatorname{Hom}(M, \mathbb{Z})$. But by assumption Σ is complete, i.e. $\nu \in \sigma$ for some cone σ. We will show the map f has a limit inside U_σ, thus satisfying VC.

To show f extends, we want the ring map $\mathbb{C}[S_\sigma] \to K$ factors through R.

$$\begin{tikzcd} \mathbb{C}[S_\sigma] \arrow[rr] \arrow[dr, dashed] & & K \\ & R \arrow[ur, hook] & \end{tikzcd}$$

This is the same as saying (at monomial level):

$$S_\sigma \to K^* \to \mathbb{Z}$$

has image in $\mathbb{Z}_{\geq 0}$. But this is tautology, because we defined S_σ to be functions that evaluate to be non-negative. So the claim follows.

As we've already shown the space is Hausdorff, the limit must be unique. Thus by VC, our space is proper. $\square$

Note. All projective varieties are proper, as closed subset of $\mathbb{P}^n$. Converse is false. There are proper toric varieties that are not proper.

3.4 Polytope construction

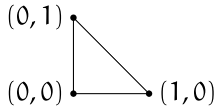
Projective toric varieties form a big class of examples of proper toric varieties. They can be obtained in the following way:

Fix a finite set of lattice point $\mathcal{A} \subset M$. Let $P_{\mathcal{A}}$ be the convex hull of $\mathcal{A}$. Let's assume that $P_{\mathcal{A}}$ is of full dimension. Then we can define a map

$$\operatorname{Spec} \mathbb{C}[M] = (\mathbb{C}^*)^{\dim M_{\mathbb{R}}} \to (\mathbb{C}^*)^{\# P_{\mathcal{A}}} \to \mathbb{P}^{\# P_{\mathcal{A}} - 1}$$

by the obvious maps (see example below), where $\# P_{\mathcal{A}}$ is the number of points inside $P_{\mathcal{A}}$. Then we define the toric variety $X_{P_{\mathcal{A}}}$ as the closure of the image.

Example. Let's take P to be the polygon below.

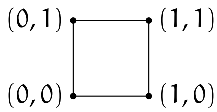


This determines the map

$$\begin{aligned} (\mathbb{C}^*)^2 \to (\mathbb{C}^*)^3 \quad &\to \mathbb{P}^2 \\ (z_1, z_2) \mapsto (1, z_1, z_2) &\mapsto [1 : z_1 : z_2]. \end{aligned}$$

The closure is then $\mathbb{P}^2$.

Another example:



This determines the map

$$\begin{aligned} (\mathbb{C}^*)^2 \to (\mathbb{C}^*)^4 \qquad\quad &\to \mathbb{P}^3 \\ (z_1, z_2) \mapsto (1, z_1, z_2, z_1 z_2) &\mapsto [1 : z_1 : z_2 : z_1 z_2]. \end{aligned}$$

Then X is given by $\mathbb{V}(XW - YZ) \subset \mathbb{P}^3$ (here by $\mathbb{V}$, we mean the Proj construction), and $X \cong \mathbb{P}^1 \times \mathbb{P}^1$. Note that X is embedded in $\mathbb{P}^3$ as the Segre embedding. We could get all Segre embeddings by these polytope construction.

4 Morphisms

lecture 9 Let T', T be algebraic tori with co/character lattices N', N, M', M respectively. Given an algebraic group homomorphism $T' \xrightarrow{\phi} T$ (by polynomials), we get the induced maps:

$$\text{Pullback (of monomials/functions):} \quad \phi^* : M \to M',$$

and

$$\text{Pushforward (of sub-objects/curves):} \quad \phi_* : N' \to N.$$

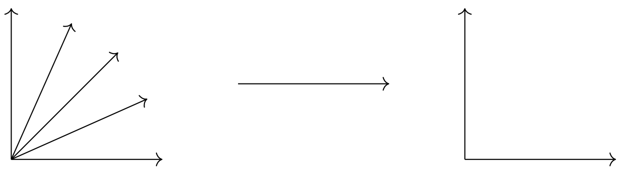
Definition. A toric morphism of toric varieties is a morphism $\phi : X_{\Sigma'} \to X_{\Sigma}$ such that $\phi|_{T'}$ is a homomorphism into T, i.e. $\phi|_{T'}$ is a homomorphism of tori, where T', T are the tori inside the two varieties. (cf. action is equivariant.)

In particular, if $\phi : X_{\Sigma'} \to X_\Sigma$ is toric, then there exists an induced map $\phi^* : N'_{\mathbb{R}} \to N_{\mathbb{R}}$. We can ask which maps on N's would induce morphisms on varieties.

Proposition. *If $\Sigma' \subset N'_{\mathbb{R}}$ and $\Sigma \subset N_{\mathbb{R}}$ are fans, and $f : N' \to N$ is a linear map of integral points such that for all $\sigma' \in \Sigma'$, $f(\sigma')$ is contained in some cone $\sigma \in \Sigma$, then there exists a well-defined induced map $X_{\Sigma'} \to X_\Sigma$.*

Example.

- Consider the map between fans of $\mathbb{C}$ (gen. by $(1, 0) \in N_{\mathbb{R}} = \mathbb{R}$) to that of $\mathbb{C}^2$ (gen. by $(1, 0), (0, 1) \in N'_{\mathbb{R}} = \mathbb{R}^2$) by the diagonal morphism $r \mapsto (r, r)$. Then the image of the cone is fully contained in the cone of $\mathbb{C}^2$. This induces a map $\mathbb{C} \to \mathbb{C}^2$ by $z \mapsto (z, z)$.
- Consider a fan in $\mathbb{R}^2$ with many rays subdividing the first quadrant, mapping to the fan of $\mathbb{C}^2$ by $(x, y) \mapsto (x, y)$. This will also induce a map between the toric varieties.



- (Non-example) The inverse map in the above example. This will map the unique 2-dim cone to something not in a cone.

Proof. (Sketch) Step 1: affine case. If Σ' and Σ both consist of a single cone and its faces, then a map $\sigma' \to \sigma$ induces $S_\sigma \to S_{\sigma'}$, and so $\mathbb{C}[S_\sigma] \to \mathbb{C}[S_{\sigma'}]$ and $U_{\sigma'} \to U_\sigma$.

Step 2: If $x \in U_{\sigma'} \cap U_{\tau'}$, then the prescription in step 1 is independent of choices of cones in which we do this (check yourself). $\square$

Example.

- $\Sigma_{\mathbb{P}^1} \to \Sigma_{\mathbb{P}^1 \times \mathbb{P}^1}$ as diagonal.
- *Change of lattice.* Consider the lattice map of multiplication by 2 on $\mathbb{Z}$, $[2]a = 2a$. This corresponds to the squaring map $\mathbb{C} \to \mathbb{C}$.

  Similarly if $(l_1, \dots, l_n) \in \mathbb{Z}^n_{\geq 0}$, then the diagonal matrix $\begin{bmatrix} l_1 & & \\ & \ddots & \\ & & l_n \end{bmatrix} : \mathbb{Z}^n \to \mathbb{Z}^n$. For any fan in $\mathbb{R}^n$, we get an induced endomorphism

$$\begin{tikzcd} X_\Sigma \arrow[r] & X_\Sigma \\ T \arrow[u, hook] \arrow[r] & T, \arrow[u, hook] \end{tikzcd}$$

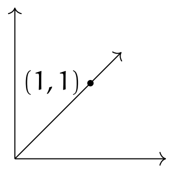
  where the T is the torus inside X_Σ, and $T \to T$ is the l_i-th power on the i-th coordinate.

  An example of when this is useful is the line bundle pullback map by Frobenius.

4.1 Blowups

Blowup at a point on a plane

We work in dimension 2. Let Σ be the first-quadrant fan ($X = \mathbb{C}^2$) and denote $\mathrm{Bl}(\sigma)$ to be:



Obviously there is a map $\mathrm{Bl}(\mathbb{C}^2) \to \mathbb{C}^2$ between the corresponding toric varieties.

In particular, the construction of $\mathrm{Bl}(\mathbb{C}^2)$ involves two charts corresponding to two cones in the fan, each of which a copy of $\mathbb{C}^2$, glued along the ray in $N_{\mathbb{R}}$, which is $\mathbb{C} \times \mathbb{C}^*$. In coordinates, $U_{\sigma_1} = \operatorname{Spec} \mathbb{C}[X, X^{-1}Y]$, $U_{\sigma_2} = \operatorname{Spec} \mathbb{C}[Y, Y^{-1}X]$, glued together $\operatorname{Spec} \mathbb{C}[X, Y, X^{-1}, Y^{-1}]$. Explicitly, $\mathrm{Bl}_0(\mathbb{C}^2) = \mathbb{V}(XT_1 - YT_2) \subset \mathbb{C}^2 \times \mathbb{P}^1$. We will say more about this in the next lecture.

lecture 10 *Remark.* The blowup we constructed is a quasi-projective variety. It is neither affine (contains $\mathbb{P}^1$) nor projective (not proper). In fact, there is a (birational) morphism

$$\pi : \mathrm{Bl}_0(\mathbb{C}^2) \to \mathbb{C}^2$$

that restricts to an isomorphism between

$$\mathrm{Bl}_0(\mathbb{C}^2) \setminus \pi^{-1}(0) \to \mathbb{C}^2 \setminus 0.$$

So we could take a sequence diverging to ∞ in $\mathbb{C}^2 \setminus 0$ and pullback to the blowup, and therefore the blowup is not proper.

Blowup morphism π has several properties:

- It is a *proper* morphism: the associated morphism of topological spaces $\pi^{an} : (\mathrm{Bl}_0\mathbb{C}^2)^{an} \to (\mathbb{C}^2)^{an}$ is proper.
- It is a projective morphism.
- It's an example of a non-flat morphism (side remark here: pullback of cohomology classes/cycles for non-flat morphisms aren't just the class of the preimage; instead we should take self intersections).

  If $X \to Y$ is flat, then the dimension of the fibers $p^{-1}(y)$ should be independent of the choices of $y \in Y$, which fails here if we take origin and any other point.
- It is birational, and an isomorphism away from $\pi^{-1}(0)$.

Higher dimension story

Let's first consider blowup of $\mathbb{C}^n$ at origin. The fan of $\mathbb{C}^n$ consist of a single n-dimensional cone $\sigma = \mathbb{R}^n_{\geq 0}$.

Define $\mathrm{Bl}(\sigma)$ as follows: let e_i be the standard unit vectors, and $e_0 = \sum_{i=1}^n e_i$. Then cones of $\mathrm{Bl}(\sigma)$ are precisely those generated by a subset of $\{e_0, \ldots, e_n\}$ not containing $\{e_1, \ldots, e_n\}$.

Exercise: construct $\mathrm{Bl}_0(\mathbb{C}^n) = X_{\mathrm{Bl}(\sigma)}$, and the morphism

$$\mathrm{Bl}_0(\mathbb{C}^n) \to \mathbb{C}^n.$$

4.2 Resolution of singularities

Recall we have this valuative criterion for properness: a map $\pi : X \to Y$ is proper iff for every diagram

$$\begin{tikzcd}
\mathrm{Spec}(\mathbb{C}((t))) \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\
\mathrm{Spec}(\mathbb{C}[[t]]) \arrow[r] \arrow[ur, dashed] & Y,
\end{tikzcd}$$

there exists a unique arrow $\mathrm{Spec}(\mathbb{C}[[t]]) \to X$ such that the diagram commutes.

Lemma. *A toric morphism $\pi : X_{\Sigma'} \to X_\Sigma$ induced by a map $\Sigma' \to \Sigma$ is proper iff $\pi^{-1}(|\Sigma|) = |\Sigma'|$.*

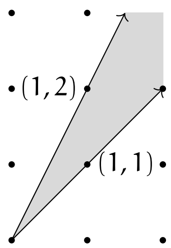
Proof. Apply VC locally on $X_{\Sigma'}$. $\square$

Remark. Blowups of cones $\mathbb{R}^n_{\geq 0}$ satisfy this support condition, and are therefore proper.

Let Σ be a fan, then every proper birational toric morphism $X_{\Sigma'} \to X_\Sigma$ is determined by a refinement of cones $\Sigma' \to \Sigma$ (and an isomorphism of lattices $N' \to N$). If in addition, Σ' is smooth, then such a morphism is referred to a *resolution of singularities*.

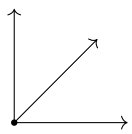
Remark. Resolutions exist in characteristic zero by works of Hironaka. For char p, it's an open question in general.

Example. Toric surfaces singularities. Let σ be the following region in $N_\mathbb{R} = \mathbb{R}^2$:



One way to see U_σ isn't smooth is that $\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2 \neq 1$.

Let $\tilde{\sigma}$ be the variety corresponding to



Then $X_{\tilde{\sigma}}$ is smooth (both pieces are smooth). This is an example of resolution of singularities.

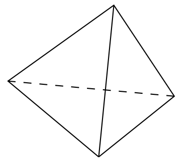
Similarly, $\sigma = \langle (1,0), (1,3) \rangle$ is not smooth.

Theorem. *Every toric variety X_Σ has a toric resolution of singularities.*

lecture 11 There is an important class of examples we forgot to mention.

Example.

- *Projective space.* Let Σ be the fans with rays generated by $e_1, \dots, e_n$, and $e_0 = -\sum_{i=1}^n e_i$. The k-dimensional cones are the positive spans of size k subsets of $\{e_0, \dots, e_n\}$ for $1 \leq k \leq n$. If we intersect Σ in $\mathbb{R}^3$ with a sphere centered at 0, we get the following picture:



where each face corresponds to a 3-dim cone, i.e. a copy of $\mathbb{C}^3$. These affines are then glued together to give $\mathbb{P}^n$.

- *Weighted projective space.* Recall that we could define $\mathbb{P}^n$ as the quotient of $\mathbb{C}^{n+1}$ by a $\mathbb{C}^*$ action: $\lambda \cdot (z_0, \dots, z_n) = (\lambda z_0, \dots, \lambda z_n)$. Similarly, for fixed integers $(d_0, \dots, d_n)$, we define another action

$$\lambda \cdot (z_0, \dots, z_n) = (\lambda^{d_0} z_0, \dots, \lambda^{d_n} z_n),$$

and define the weighted projective space as:

$$\mathbb{P}(d_0, \dots, d_n) := (\mathbb{C}^{n+1} \setminus 0)/\sim.$$

Proposition. $\mathbb{P}(d_0, \dots, d_n)$ *is a toric varieties.*

Proof. $(\mathbb{C}^*)^{n+1} \subset \mathbb{C}^{n+1}$ projects to a dense torus and acts on $\mathbb{P}(d_0, \dots, d_n)$. □

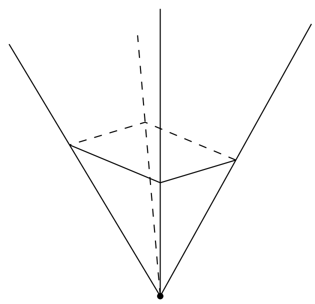
This is also an example of the Proj$(-)$ construction. As a toric variety, it can also be constructed by the following fans:

Let $\Sigma \subset N_{\mathbb{R}}$ be the fan of $\mathbb{P}^n$. We may consider this as a fan in a (refined) lattice generated by $(\frac{1}{d_0}e_0, \dots, \frac{1}{d_n}e_n)$, denoted by N'. This fan is *simplicial*, but not smooth in general (in particular, every toric surface is simplicial). See example sheet for details.

> A cone is simplicial if its generators are linear independent over $\mathbb{R}$. It is smooth/regular if its generators are part of a $\mathbb{Z}$-basis of N. A fan is simplicial (resp. smooth) if each cone is simplicial (resp. smooth).

Terminology: Singularities appearing in simplicial toric varieties are called *quotient singularities*, and the variety is an *orbifold*.

Example. The cone over a square in $N_{\mathbb{R}} = \mathbb{R}^3$ certainly needs more than three (and thus not linear independent) basis vectors.



It is not simplicial, and not smooth.

> To compute this, note that for a polyhedral cone $\sigma = H^+_{m_1} \cap \cdots \cap H^+_{m_n}$, $\sigma^\vee = \langle m_1, \dots, m_n \rangle$. This gives the same semigroup for the affine $\mathbb{V}(xy - zw)$, which relates back to the polytope construction in section 3.4.

Back to resolution for toric surfaces. Since being smooth is a local property, we only need to worry about affine toric surfaces.

The ultimate goal: for a cone σ in $\mathbb{R}^2$, there exists a refinement $\tilde{\sigma} \to \sigma$ of the cone, such that $\tilde{X}_\sigma$ is smooth (see example from last time).

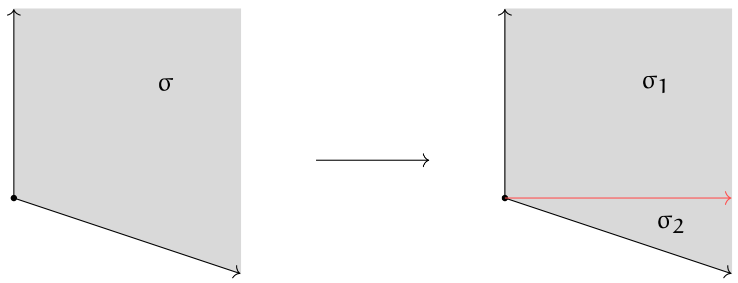
Proof. We want to put a cone σ in a "standard form". To do this, we need to do a basis change. For any $A \in SL_n(\mathbb{Z})$, and σ a cone in $\mathbb{R}^n$, the image $A\sigma$ under A gives an isomorphic toric variety as the original σ (we will only deal with $n = 2$ here).

Let σ be a cone generated by primitive vectors v_1, v_2 (first integral points on the corresponding rays).

- Since v_1 is primitve, we could take this as an element of the new basis; i.e. after a changing of basis, $v_1 \mapsto (0, 1)$, and wlog $v_2 \mapsto (m, x)$ for some $m > 0$.
- Then apply the map $\begin{bmatrix} 1 & 0 \\ c & 0 \end{bmatrix}$, we can move (m, x) to any $(m, x + cm)$. In particular, there is one such that $-m < x + cm \leq 0$.
- Since v_1, v_2 are primitive, m and $x + cm$ must be coprime.

Thus we obtained the normal form of a cone: $\langle (0, 1), (m, -k) \rangle$ for $m > 0$, $0 \leq k < m$ and $(m, k) = 1$.

Now for any cone we started with, turn it into the standard form $\langle (0, 1), (m, -k) \rangle$. We can subdivide this cone by inserting a new ray generated by $(1, 0)$:

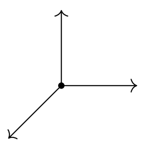


Now σ_1 is obviously smooth (with generators e_1, e_2), and we can turn $\sigma_2 = \langle (1, 0), (m, -k)$ into a normal form $\langle (0, 1), (m', -k')$. But we are rotating 90 degrees here, and thus $m' = k < m$. By a simple induction, we would eventually get $\sigma_i = \langle (1, 0), (0, 1)$, which is smooth. □

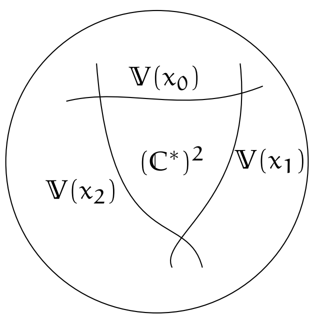
5 Orbit-cone correspondence

5.1 Orbit corresponding to a cone

lecture 12 To start off, recall we have the fan for $\mathbb{P}^2$:



and we draw it as something like this:



There are 7 orbits for the $(\mathbb{C}^*)^2$ actions (recall $(\lambda, \mu) \cdot [x_0 : x_1 : x_2] = [x_0 : \lambda x_1 : \mu x_2]$):

- There is one corresponding to the copy of $(\mathbb{C}^*)^2 = \{x_i \neq 0 \ \forall i\}$.
- There are three corresponding to the "interiors" of each coordinate line, identified as $\{x_i = 0, x_j \neq 0, x_k \neq 0\}$.
- There are three corresponding to the coordinate points $\{x_i = x_j = 0, x_k \neq 0\}$.

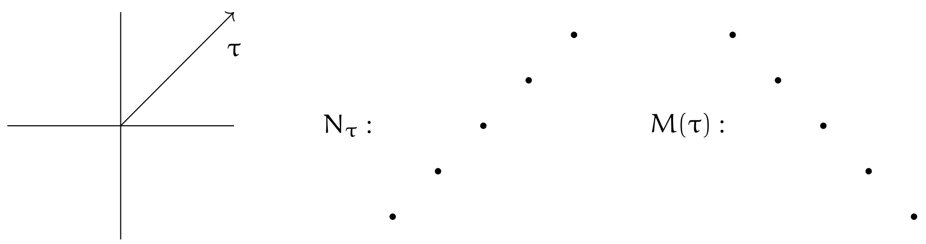
And we remarked a while ago, that there is a natural bijection between the orbits of the torus action and the cones in the fan.

Goal: generalize this to any X_Σ for a fan Σ.

Let $\tau \in \Sigma$ be a cone. There are some associated lattices:

$$\begin{aligned} N_\tau &= (\tau \cap N)^{gp} \quad \text{the group generated by } \tau \cap N, \\ N(\tau) &= N/N_\tau, \\ M(\tau) &= \tau^\perp \cap M. \end{aligned}$$

Example.



These define tori. Specifically, the *torus associated to* $N(\tau)$ is

$$T_{N(\tau)} = \mathrm{Spec}(\mathbb{C}[M(\tau)]) = N(\tau) \otimes_{\mathbb{Z}} \mathbb{C}^*.$$

Here $N(\tau) = \mathbb{Z}^k$ is just a new lattice, and $T_{N(\tau)} = (\mathbb{C}^*)^k$. In particular, the projection $N \twoheadrightarrow N(\tau)$ induces an action T_N (the torus of X_Σ, which is $N \otimes \mathbb{C}^*$) on $T_{N(\tau)}$ by:

$$T_N \times T_{N(\tau)} \to T_{N(\tau)} \times T_{N(\tau)} \xrightarrow{\text{torus multiplication}} T_{N(\tau)}.$$

We will embed $T_{N(\tau)}$ into X_Σ. To do this, note

$$\begin{aligned} U_\tau &= \mathrm{Spec}(\mathbb{C}[\tau^\vee \cap M]) = \mathrm{Hom}_{\mathrm{monoid}}(\tau^\vee \cap M, \mathbb{C}) \\ T_{N(\tau)} &= \mathrm{Spec}(\mathbb{C}[M(\tau)]) \quad\ = \mathrm{Hom}_{\mathrm{gp}}(\tau^\perp \cap M, \mathbb{C}^*). \end{aligned}$$

In plain English, we are trying to extend functions on integral points of $\tau^\perp$ (something evaluates to be 0) to functions on integral points of $\tau^\vee$ (something evaluates to be non-negative). And there is a natural candidate, *extension by 0*:

We embed $T_{N(\tau)}$ into U_σ by:

$$\begin{aligned} \mathrm{Hom}_{\mathrm{gp}}(\tau^\perp \cap M, \mathbb{C}^*) \quad &\to \quad \mathrm{Hom}_{\mathrm{monoid}}(\tau^\vee \cap M, \mathbb{C}), \\ \left(f : \tau^\perp \cap M \to \mathbb{C}^*\right) \quad &\mapsto \quad x \mapsto \begin{cases} f(x) & \text{if } x \in \tau^\perp \cap M, \\ 0 & \text{o/w}. \end{cases} \end{aligned}$$

This is well defined since τ is fixed. For notation, we alternatively use O_τ the orbit when we think of the torus $T_{N(\tau)}$ inside X_Σ.

Alternative perspective

What is happening here is that there is a perfect pairing

$$\langle\, , \rangle : \tau^\perp \cap M \times N(\sigma) \to \mathbb{Z}$$

induced by the natural pairing on $M \times N$. And this will induce an isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*) \cong T_{N(\sigma)}.$$

For a cone τ, we have a distinguished point $x_\tau \in U_\tau \subset X_\Sigma$. This immediately gives us an orbit:

$$O'_\tau = T_N \cdot x_\tau \subset X_\tau.$$

And this is precisely $T_{N(\tau)} = O_\tau$ we just constructed, via the following isomorphism:

$$O'_\tau = \{\gamma : S_\tau \to \mathbb{C} \mid \gamma(m) \neq 0 \iff m \in \sigma^\perp \cap M\} \cong \mathrm{Hom}(\sigma^\perp \cap M, \mathbb{C}^*) = O_\tau.$$

5.2 Orbit closures

Given $\tau \in \Sigma$, we know $O_\tau \subset U_\tau \subset X_\Sigma$. But the closure, while generally larger than O_τ itself, is guaranteed to always be a toric variety. In particular, we are adding in all the orbits of cones of which τ is a face. This is a case of a more frequently used name called *stratification*.

Definition. Given $\tau \in \Sigma$, the *star fan* $\Sigma(\tau)$ is defined to be (as a set, no fan structure yet):

$$\Sigma(\tau) = \{\sigma \in \Sigma \mid \tau \subset \sigma \text{ as a face}\}.$$

For each $\sigma \in \Sigma(\tau)$, let $\overline{\sigma}$ be the image cone in $N(\tau) \otimes \mathbb{R}$ under the quotient map $N_\mathbb{R} \to N(\tau)_\mathbb{R}$:

$$\overline{\sigma} = (\sigma + N(\tau) \otimes \mathbb{R})/N(\tau) \otimes \mathbb{R}.$$

Then $\overline{\sigma}$'s have a fan structure.

For each $\sigma \in \Sigma(\tau)$, there exists an affine toric partial compactification of orbit O_τ:

$$U_\sigma(\tau) := \mathrm{Spec}(\mathbb{C}[\tau^\perp \cap \sigma^\vee \cap M]).$$

These glue over $\sigma \in \Sigma(\tau)$ to give a toric variety $X_{\Sigma(\tau)}$ as a compactification of (both classical and Zariski) O_τ, and contains it as a dense open.

Theorem. *(Orbit-cone correspondence) Let X_Σ be a toric varieties of the fan $\Sigma \in N_\mathbb{R}$. Then*

1. *There is a bijective correspondence*

$$\begin{aligned} \{\textit{cones } \sigma \in \Sigma\} &\longleftrightarrow \{T_N \textit{ orbits of } X_\Sigma\} \\ \sigma &\longmapsto O_\sigma, \\ \textit{minimal cone } \sigma \in \Sigma \textit{ with } O \subset U_\sigma &\longleftarrow O. \end{aligned}$$

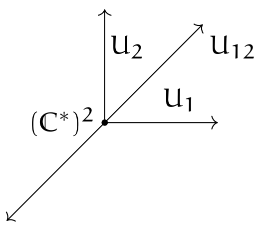
2. *If* $\dim N_{\mathbb{R}} = n$, *then* $\dim O_\sigma = n - \dim \sigma$.
3. *The affines* U_σ *is the union of orbits*
$$U_\sigma = \bigcup_{\tau \text{ a face of } \sigma} O_\tau.$$
4. *The orbit closure is a union of orbits:*
$$\overline{O_\tau} = \bigcup_{\tau \text{ is a face of } \sigma} O_\sigma.$$

lecture 13 As before, we fix a fan $\Sigma \in N_{\mathbb{R}}$. For every $\tau \in \Sigma$, we have an orbit $O_\tau \hookrightarrow X_\Sigma$.

- This is a *locally closed subscheme*, meaning that it is a closed subscheme of an open subscheme.
- In particular, the closure $V(\tau) := X_{\Sigma(\tau)} \hookrightarrow X_\Sigma$ is a closed embedding.

5.3 Examples

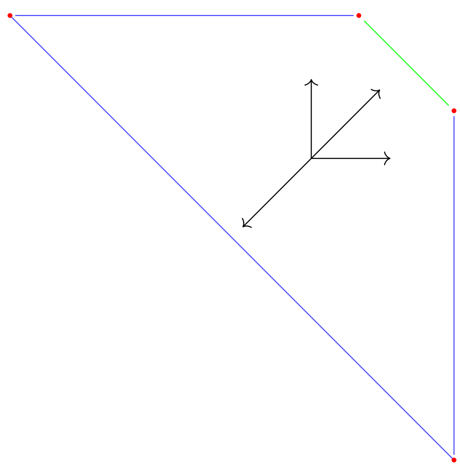
Consider the blowup of $\mathbb{P}^2$ at a point. We've seen the fan should be



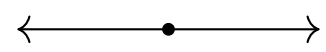
This has 4 copies of $\mathbb{C}^2$ glued together. In particular, U_1, U_2 are copies of $\mathbb{C}^2$, and U_{12} is $\mathbb{C} \times \mathbb{C}^*$ ($\mathbb{C}[S_{U_{12}}] \cong \mathbb{C}[x, y, y^{-1}]$). Origin corresponds to the big torus $(\mathbb{C}^*)^2$.

- *Open subschemes*: all U_τ for $\tau \in \Sigma$ together with the inclusion by inserting a single Laurent polynomial.
- *Closed subschemes*: all $V(\tau)$ for $\tau \in \Sigma$. In particular, for $\tau = \langle(1,1)\rangle$, we know $N_\tau = \langle(1,1)\rangle$, and $N(\tau) = \mathbb{R}^2/(1,1)$ (imagine this as the line at infinity perpendicular to the ray τ, sort of like a "wall". See picture below).

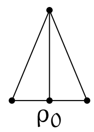
  Inside this (1-dimensional) wall lives the star fan. $\Sigma(\tau)$ has three elements. The cone τ itself is mapped to origin of the wall, and cones of either U_1 or U_2 is mapped to a ray of either direction of the wall. In particular, X_{Σ_τ} is $\mathbb{P}^1$.



Here each colored line or dot represents a lattice $N(\sigma)$. And inside the green line, which is $N(\tau)$, we have $\mathbb{P}^1$.



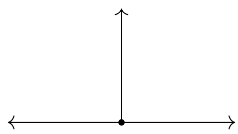
Another example: let Σ be the cone over this



Consider the ray through ρ_0 (a little abuse of notation, we just call this ρ_0). Then $N(\rho_0) = \mathbb{R}^2$. What is the star fan structure?

- ρ_0 itself is mapped to origin.
- 2-dimensional cones that contain ρ_0 as a face will map to rays. These give 3 rays, generated by $(-1, 0), (1, 0), (0, 1)$ respectively.
- 3-dimensional cones will map to 2-dimensional cones in $N(\rho_0)$. Namely the 2 images are $\langle(-1, 0), (0, 1)\rangle$ and $\langle(1, 0), (0, 1)\rangle$.

All in all, the star fan $\Sigma(\rho_0)$ is



which is $\mathbb{C} \times \mathbb{P}^1$, and it is the closure of $\mathbb{C} \times \mathbb{C}^*$. In particular, we are blowing up $\mathbb{C}^3$ along a coordinate line. You could work out other cases of blowing up coordinate subspaces of $\mathbb{C}^n$ analogously.

Compare this example with the blowup of $\mathbb{C}^2$ or $\mathbb{C}^3$ at a point.

6 Divisors

Suppose $X = \operatorname{Spec}(A)$ is smooth and affine, and A is the ring of regular functions on X. A divisor is the data of the vanishing locus $\mathbb{V}(f)$ for $f \in A$ along with the order of vanishing at each point of $\mathbb{V}(f)$.

E.g. $X = \mathbb{C}, A = \mathbb{C}[t]$, then t^2 has order of vanishing 2 at origin.

More generally, one can consider a formal sum $\sum_{i=1}^{k} n_i [D_i]$ where $D_i = \mathbb{V}(f_i)$ and $n_i \in \mathbb{Z}$.

Fact. If X is smooth, every codimension 1 subvariety is locally the vanishing locus of a single regular function. Highbrow version is to say, "Weil divisors are Cartier on a smooth scheme".

6.1 Definitions

lecture 14 *Note*: every thing in this section is (once you get used to it) intuitively easy, but generally very hard to compute. Don't be fooled by this innocence.

Recall we want to consider codimension 1 subvarieties. But there are too many, and we need a measure to control them.

Motivating example: there exists a group $\mathbb{Z}$ that naturally encodes the degrees of hypersurfaces (codim 1 subvarieties) of $\mathbb{P}^n$. In particular, degree 1 means linear hyperplanes; degree 2 means quadrics; degree 3 means cubics, etc.

The degree of a divisor (hypersurface in this case) doesn't see smoothness, normality, or any other intrinsic geometric properties of a divisor. Degree 3 hypersurfaces can be a union of three lines, elliptic curves, etc.

Valuation rings and divisors

From now on, X is normal, irreducible variety (or integral scheme). Let $D \subset X$ be an irreducible subvariety of codimension 1. Consider the ring

$$\mathcal{O}_{X,D} = \{f \in \mathbb{C}(X) \text{ the function field of } X \mid \exists U \subset X, U \cap D \neq \emptyset \text{ st } f \text{ is regular on } D\}.$$

More concretely, take any affine $\mathrm{Spec}(A) \subset X$ such that $\mathrm{Spec}(A) \cap D \neq \emptyset$. Then $D|_{\mathrm{Spec}(A)}$ corresponds to a prime ideal P of A, and $\mathcal{O}_{X,D}$ is the localization of A at P (functions defined over the generic point of D).

$\mathcal{O}_{X,D}$ is a local ring, and since P is of height 1, it is a valuation ring, i.e. there is a valuation map

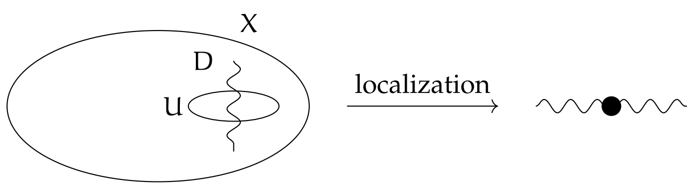
$$\nu_D : \mathcal{O}_{X,D} \to \mathbb{Z}$$

measuring order of vanishing of any f along D.

Example. $X = \mathbb{A}^1$, $D = p = \{pt\}$. Then $\mathbb{C}(X)$ is the rational functions over one variable, elements of the form $f(x)/g(x)$. In practice, to compute the order of vanishing of f/g at p, we expand f/g at p as a Laurent series, and take the exponent of the leading term.

E.g., at $D = 0$, $1/x^2$ has order -2. x^3 has order 3.

In higher dimension, localizing turns the generic point into a closed point. The picture in mind should roughly be like this:



where D is turned into a maximal ideal (closed point), and we can also pass any function f to localization to get a function on the right.

Given a rational function $f \in \mathbb{C}(X)$, there is an associated object:

$$\mathrm{div}(f) = \sum_{D \text{ irreducible divisor of } X} \mathrm{ord}_D(f)[D].$$

Basic fact: this sum is finite.

Definition. A *prime divisor* in X is an irreducible/reduced codimension 1 subvariety. A *Weil divisor* is an element of $\mathrm{Div}(X) = \bigoplus_{D \text{ prime divisor}} \mathbb{Z}D$.

Given a rational function $f \in \mathbb{C}(X)$, $\mathrm{div}(f) \in \mathrm{Div}(X)$. Such divisors are called *principal*.

Definition. The *class group* is defined to be

$$\mathrm{Cl}(X) := \mathrm{Div}(X) \,/\, \{D \mid D \text{ principal}\}.$$

Cartier divisor

If $X = \mathrm{Spec}(A)$ is affine. A divisor $D \in \mathrm{Div}(X)$ is principal if we can write $D = (f)$ for f in the fraction field of A. A Weil divisor D is Cartier if it's locally principal, i.e. there exists an open covering $X = \bigcup U_i$ such that $D|_{U_i}$ is principal.

Generally on a variety X, if every Weil divisor if Cartier, then X is called factorial.

Note:

1. Smooth implies factorial.
2. In the definition of class group, if we quotient out all Cartier divisors by the principal ones, we would get the *Picard group*.

> There is no reason why we wouldn't consider the set of all codimension 2 subvarieties, or codimension k subvarieties. We will come back to these, known as *Chow groups*, later on in the course.

6.2 Weil divisors on toric varieties

Two things to note:

1. On a toric variety, given $m \in M$, then $\chi^m \in \mathbb{C}[M] \subset \mathbb{C}(X)$ is a regular function on the variety.
2. Each ray ρ in the fan Σ of X gives rises to a codimension 1 torus orbit closure

$$V(\rho) =: D_\rho \in \mathrm{Div}(X),$$

called a *boundary divisor*.

Proposition. *If X is toric, then $Cl(X)$ has the presentation*

$$Cl(X) = \bigoplus_{\rho \text{ rays}} \mathbb{Z}D_\rho / \{div(\chi^m) \mid m \in M\}.$$

This is saying that every divisor class if equivalent to a finite sum of boundary divisors.

Lemma. *Given $m \in M$, then*

$$div(\chi^m) = \sum_{\rho \text{ rays}} \langle v_\rho, m\rangle D_\rho,$$

where v_ρ is the first lattice point on the ray ρ.

Proof. (lemma) Exercise.

lecture 15 Let W be an irreducible variety. We say W is affine if W can be identified with $\mathrm{Spec}(\Gamma(W, \mathcal{O}_W))$ called the affinization. In particular, if W is proper, then the affinization is $\mathrm{Spec}(\mathbb{C})$, which is a point.

But we do have (the function field) $\mathbb{C}(W)$, the stalk of $\mathcal{O}_W$ at the generic point; or equivalently the function field of $\mathcal{O}_W(U)$ for any affine $U \subset W$. This field is in general very big. In order to organize this, we take rational functions with prescribed order of poles along a fixed Weil divisor:

$$\mathcal{O}_W(D) = \{f \in \mathbb{C}(W) \mid (f) + D \geq 0\}.$$

This is an $\mathcal{O}_W$-module.

The transition from D to $\mathcal{O}_W(D)$ leads to the isomorphism between the Picard group defined via line bundles and via Cartier divisors.

Back to toric varieties. We want to prove the previous proposition:

Given $m \in M$, then $\mathrm{div}(\chi)^m$ is a principal divisor supported on the torus-invariant boundary divisor of X_Σ. This gives a map $M \to \mathrm{Div}_T(X_\Sigma)$ to T-invariant Weil divisors of X_Σ. Then there is an exact sequence computing $\mathrm{Cl}(X_\Sigma)$ given by:

$$M \to \mathrm{Div}_T(X_\Sigma) \to \mathrm{Cl}(X_\Sigma) \to 0.$$

> Geometrically, we are pushing any divisor to the boundary to get an T-invariant boundary divisor. But this requires limit of subvarieties, so we will present a different proof.

Proof. We need two lemmas:

Lemma. Given prime (irreducible divisors) $D_1, \ldots, D_N$ in X, and $U = X \setminus \{D_1, \ldots, D_N\}$, there is an exact sequence:

$$\bigoplus_{j=1,\ldots,N} \mathbb{Z}D_j \to Cl(X) \to Cl(U) \to 0.$$

This is true for any normal variety.

Lemma. If $X = Spec(R)$ for a UFD R, then $Cl(X) = 0$. This essentially boils down to the fact that every codim 1 prime ideal is principal.

Then:

- Take $X = X_\Sigma$ a toric variety, and $\{D_j\} = \{D_\rho\}_{\rho \in \Sigma(1)}$. Since $U = X \setminus \{D_\rho\}$ is the torus

  $$T = \mathrm{Spec}(\mathbb{C}[M]),$$

  and the Laurant polynomial ring is a UFD, we see $\mathrm{Cl}(T) = 0$. Therefore by lemma 1, we have

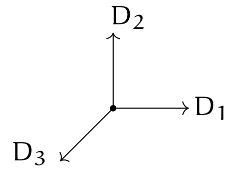
  $$\bigoplus \mathbb{Z}D_\rho \twoheadrightarrow \mathrm{Cl}(X_\Sigma).$$

- Composition $M \to \mathrm{Div}_T(X_\Sigma) \to \mathrm{Cl}(X_\Sigma)$ giving 0 is trivial.
- Consider the kernel of the map $\mathrm{Div}_T(X_\Sigma) \to \mathrm{Cl}(X_\Sigma)$. Given $D \in \mathrm{Div}_T(X_\Sigma)$. By definition, D is supported on boundary divisors, and $D|_T = \emptyset$. If $D = \mathrm{div}(f)$ is principal for some $f \in \mathbb{C}(X_\Sigma)^*$, then consider f as as element in $\mathbb{C}(T)^*$. Since $\mathrm{div}(f)$ gives 0, f is in fact, in $\mathbb{C}[T]^* = \mathbb{C}[M]^*$. So $f = c \cdot \chi^m$ for some $c \in \mathbb{C}^*$. But $\mathrm{div}(c \cdot \chi^m) = \mathrm{div}(\chi^m)$. So M surjects into the kernel of $\mathrm{Div}_T(X_\Sigma) \to \mathrm{Cl}(X_\Sigma)$. □

Remark. If the rays of Σ span $N_\mathbb{R}$, then $M \to \mathrm{Div}_T(X_\Sigma)$ is injective as well (a condition called *having no torus factors*).

Example.

- Let Σ be the fan of $\mathbb{P}^2$.



Then the three rays correspond to the three boundary divisors D_1, D_2, D_3. Since $\mathbb{P}^2$ is smooth, we have:

$$\mathrm{Cl}(\mathbb{P}^2) = \mathrm{Pic}(\mathbb{P}^2) = \mathbb{Z}D_1 \oplus \mathbb{Z}D_2 \oplus \mathbb{Z}D_3/(D_1 - D_3 = 0, D_2 - D_3 = 0) = \mathbb{Z}[H].$$

We usually write [H] for (*the* divisor class of) a hyperplane (codim 1) in $\mathbb{P}^n$.

Notice that to get the relations, it suffices to check $\mathrm{div}(\chi^{(0,1)})$ and $\mathrm{div}(\chi^{(1,0)})$ (as these form a basis of M). Using the lemmas from last time, we see

$$\mathrm{div}(\chi^{(0,1)}) = \langle(1,0),(0,1)\rangle D_1 + \langle(0,1),(0,1)\rangle D_2 + \langle(-1,-1),(0,1)\rangle D_3 = D_2 - D_3,$$

and similarly $\mathrm{div}(\chi^{(1,0)}) = D_1 - D_3$.

- Let Σ be the fan of $\mathbb{P}^1 \times \mathbb{P}^1$, consisting of four rays generated by $(1,0), (0,1), (-1,0), (0,-1)$ respectively. Then:
$$\mathrm{Cl} = \mathbb{Z}^4/(D_1 - D_3, D_2 - D_4) \cong \mathbb{Z}^2.$$
- More generally, $\mathrm{Cl}(\mathbb{P}^n) = \mathbb{Z}$, and $\mathrm{Cl}(X_{\Sigma_1} \times X_{\Sigma_2}) = \mathrm{Cl}(X_{\Sigma_1}) \oplus \mathrm{Cl}(X_{\Sigma_2})$.

6.3 Cartier divisors on toric varieties

lecture 16 **Example.** Take $X = \mathbb{V}(xy - zw) \subset \mathbb{A}^4$ the (toric variety corresponding to) cone over a square. (Note: we also have $\mathbb{V}(X_0X_1 - X_2X_3) \cong \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ the cone over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$. This $\mathbb{V}$ is a Proj construction. Alternatively, the first one has class group $\mathbb{Z}$ where the second one has class group $\mathbb{Z}^2$)

It is singular, and there is a Weil divisor on X that is not Cartier (locally principal), namely $\{x = z = 0\}$ (this is obviously dimension 2, or codim 1). Near origin, we cannot write this using just one equation.

Proposition. *If σ is a cone, and U_σ the affine toric, then any T-invariant Cartier divisor D is given by $D = \mathrm{div}(\chi^m)$ for some $m \in M$, and consequently $\mathrm{Pic}(U_\sigma) = 0$.*

Remark. Recall we calculated that the cone over a square has non-trivial class group.

Proof. Let $R = \mathbb{C}[S_\sigma]$. Any T-invariant Cartier divisor is, in particular, a T Weil divisor, and so we write $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$. Consider:

$$\big(H^0(U_\sigma, \mathcal{O}_{U_\sigma}(D)) := \big)\{f \in FF(R) \mid \mathrm{div}(f) + D \geq 0\} \cup \{0\} \subset FF(R).$$

Namely the meromorphic functions that has poles at worst prescribed by D. This is a fractional ideal I_D in $FF(R)$.

In general, if D is a Weil divisor on a normal variety X, then $\mathcal{O}_X(D)$ is a well-defined coherent sheaf of $\mathcal{O}_X$-modules. In particular, if $X = \operatorname{Spec}(R)$ (R noetherian), then the global sections $\Gamma(X, \mathcal{O}_X(D))$ is a finitely generated R-module.

To prove the last assertion, write $D = \sum a_i D_i$. We can find $g \in R \setminus 0$ that vanishes on each D_i. Then $\nu_{D_i}(g) > 0$ for all i, and we can find m with $m\nu_{D_i}(g) > a_i$. Thus $m\operatorname{div}(g) - D \geq 0$. Now for any global section f, $\operatorname{div}(f) + D \geq 0$, so that

$$\operatorname{div}(g^m f) = m\operatorname{div}(g) + \operatorname{div}(f) = m\operatorname{div}(g) - D + \operatorname{div}(f) + D \geq 0.$$

So $g^m\Gamma(X, \mathcal{O}_X(D)) \subset R$ is finitely generated. This also show I_D is a fractional ideal.

Note that $\mathcal{O}_X(D)$ is a line bundle (locally free of rank 1) iff D is Cartier.

Now D is T-invariant, so I_D is T-invariant, and we can decompose I_D as

$$I_D = \bigoplus_{\chi^m \in I_D} \mathbb{C} \cdot \chi^m.$$

We want to show I_D is principal.

D is Cartier, so we can investigate what I_D looks like near the distinguished point $x_\sigma \in U_\sigma$. Let $\mathfrak{m}$ be the maximal ideal of x_σ. D Cartier means that around this point D is principal. In other words,

$$I_D/\mathfrak{m}I_D \text{ must be 1 dimensional vector space over } R/\mathfrak{m}.$$

Assume it is generated by the image of $\chi^{m_0} \in I_D$. Now graded Nakayama says we can lift this back to I_D to get a generator for I_D. Thus I_D is principal. $\square$

Global T-invariant Cartier divisor

Warning: notations are not consistent with those in lectures.

Given a Cartier divisor D, we've shown $D|_{U_\sigma}$ is principal, i.e. of the form $\operatorname{div}(\chi^{m_\sigma})$ for each $\sigma \in \Sigma$, subject to the compatibility condition that $\chi^{m_{\sigma_1}}$ and $\chi^{m_{\sigma_2}}$ should agree when restricting to the open set corresponding to σ_{12}.

Let's be a little more precise here. For two choices of m_σ, say m_σ and m'_σ, when do they determine the same principal divisor locally?

We have $\operatorname{div}(\chi^{m_\sigma}) = \operatorname{div}(\chi^{m'_\sigma}) \Leftrightarrow \langle \nu_\rho, m_\sigma \rangle = \langle \nu_\rho m'_\sigma \rangle \Leftrightarrow \langle \nu_\rho, m_\sigma - m'_\sigma \rangle = 0\ \forall \rho \in \sigma^{(1)} \Leftrightarrow \langle u, m_\sigma - m'_\sigma \rangle = 0\ \forall u \in \sigma \Leftrightarrow m_\sigma - m'_\sigma \in \sigma^\perp \cap M$. So m is unique in $M/M(\sigma)$ where $M(\sigma) := \sigma^\perp \cap M$, and this quotient identifies all T-Cartier divisors.

For each $\sigma \in \Sigma$, we pick a $m_\sigma \in M/M(\sigma)$ such that $\langle m_\sigma, \nu_\rho \rangle = a_\rho$ (there could be a minus sign in other references) Then a T-Cartier divisor is the same as the data

$$\{(U_\sigma, \chi^{m_\sigma})\}_{\sigma \in \Sigma}.$$

They are compatible in the sense that if τ is a face of σ, then $m_\sigma \equiv m_\tau \mod M(\tau)$. It's rather heavy description though.

The data of each m_σ have a nice combinatorial description.

Definition. A *piecewise linear function* (abbrev. PL) on Σ is a continuous function:

$$\phi : |\Sigma| \to \mathbb{R}$$

such that $\phi|_\sigma$ is linear for each σ, and $\phi(|\Sigma| \cap N) \subset \mathbb{Z}$.

To see how $D = \sum a_\rho D_\rho$ gives a PL function, let m_σ be as above, with $\langle v_\rho, m_\sigma \rangle = a_\rho$, then on each cone σ, we define the linear function to be $u \mapsto \langle m_\sigma, u \rangle$. Compatibility guarantees this function is well-defined on the overlaps. The converse is also true:

Theorem. *The map from* T*-Cartiers to PL functions described above is an isomorphism (with addition of functions as group law).*

For the inverse map, suppose we have constructed a ϕ_D from a Cartier D. Then we can recover D by:

$$D = \sum_\rho \phi_D(v_\rho) D_\rho.$$

For a quick recap:

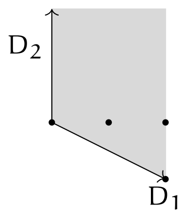
- WEIL: given by integers attached to each boundary divisor.
- CARTIER: given by PL functions linear in each cone.

Proposition. *If* X_Σ *is smooth, then every* T*-Weil divisor is* T*-Cartier.*

Proof. Although this statement is true in general, we have a nice combinatorial proof here. Using the notation as above, we can set $\phi(v_\rho) = a_\rho$. Since X_Σ is smooth, we are setting values on generators here. So we are done. $\square$

Remark. In singular cases, e.g.

σ :



Say we are given a Weil divisor $D = aD_1 + bD_2$. We can, of course, set $\phi(2, 1) = a$, and $\phi(0, 1) = b$. But then, ϕ is not defined on $(1, 0)$. Of course we could try to work out the value of $\phi(1, 0) = (a - b)/2$ and hope it is integral, but it's not guaranteed, i.e. not all Weil are Cartier.

Remark. In fact, for a toric variety X_Σ coming from a fan, that X_Σ is smooth, that $\mathrm{Pic}(X_\Sigma) = \mathrm{Cl}(X_\Sigma)$, and that every Weil divisor is Cartier, are three equivalent conditions. Furthermore, that X_Σ is simplicial, that $\mathrm{Pic}(X_\Sigma)$ has finite index in $\mathrm{Cl}(X_\Sigma)$, and that every Weil divisor has an (integer) multiple which is Cartier, are three equivalent conditions.

Example.

- The fan for $\mathbb{C}^1$ is a single ray generated by $(1, 0)$. For any divisor $D = nD_\rho$, define the PL function by $\phi(x) = nx$ for $x \geq 0$.
- If Σ is the fan for $\mathbb{P}^1$, i.e. generated by two rays $(1, 0)$ (with boundary divisor D_1) and $(-1, 0)$ (with boundary divisor D_2). Then any divisor $D = aD_1 + bD_2$ defines a function:

$$\phi_D(x) = \begin{cases} ax & \text{if } x \geq 0, \\ bx & \text{if } x \leq 0. \end{cases}$$

6.4 Introduction to intersection theory

lecture 17 *Basic idea*: if X is an n-dimensional variety, we want to build some groups $A_k(X)$ called *Chow groups* consisting of all $\mathbb{Z}$-linear combinations of k-dimensional subvarieties of X, up to a notion of *rational equivalence*, such that $\mathrm{Cl}(X) = A_{k-1}(X)$.

In good situation (usually when X is smooth), we expect an intersection product:

$$A_k(X) \times A_l(X) \to A_{l+k-n}(X)$$

such that if $Z_1 \in A_k(X), Z_2 \in A_l(X)$ are two subvarieties, and $Z_1 \cap Z_2 = Z_1 \times_Z Z_2$ is reduced and is in A_{l+k-n}, then $[Z_1 \cap Z_2] = [Z_1] \cdot [Z_2]$. (The funny dimension should actually happen in codim, where $A^k \times A^l \to A^{k+l}$)

In particular, for variety X, if C is a complete curve (meaning proper 1-dim subvariety), and D is a Cartier divisor ($(n-1)$-dim), then we expect an intersection pair $C \cdot D \in A_0(X)$. If we assume $A_0(X)$ is trivial (all points are "equivalent"), then this will just give an integer (alternatively, we are counting the number of points in $C \cap D$ with some multiplicity), satisfying:

- if E is another Cartier divisor, then $C \cdot (D + E) = C \cdot D + C \cdot E$.
- if $D \sim E$, then $C \cdot D = C \cdot E$.

Remark. Such a theory exists. For D Cartier, C smooth curve, $C \cdot D$ is constructed as follows: take $\mathcal{O}_X(D)$ to be the invertible sheaf of meromorphic functions with poles bounded by D (where $D \leftrightarrow \mathcal{O}_X(D)$ determines each other up to divisor equivalence), and take $\iota : C \hookrightarrow E$, then

$$C \cdot D := \deg(\iota^* \mathcal{O}_X(D)).$$

To define degree, pick a Cartier divisor $E = \sum n_i p_i$ (codim 1 on a curve are just points) such that $\iota^* \mathcal{O}_X(D) = O_C(E)$, then $\deg(\iota^* \mathcal{O}_X(D)) := \sum n_i$.

On $\mathbb{P}^1$, all line bundles are of the form $\mathcal{O}_{\mathbb{P}^1}(d)$ for some $d \in \mathbb{Z}$. If $d \geq 0$, we can think of $\mathcal{O}(d)$ (or global sections) as degree d homogeneous polynomials on $\mathbb{P}^1$. Also $\deg(\mathcal{O}(d)) = d$.

Example. In $\mathbb{P}^2$, if C is a degree c curve, and D is a degree d curve (in particular a divisor), then $C \cdot D = cd$, also known as Bezout's theorem.

Back to toric varieties. For C a T-invariant boundary curve, D a T-Cartier, we want to compute $C \cdot D$ in terms of combinatorics.

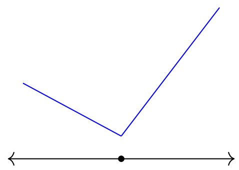
Fix $\Sigma \subset N_{\mathbb{R}}$ a fan, and τ an $(n-1)$-dim cone (so $C = V(\tau)$ is of dimension 1, i.e. a curve; moreover $\mathbb{P}^1$). Note completeness of C implies $\tau = \sigma \cap \sigma'$ for two top dimensional cones.

Let D be a T-Cartier divisor, and suppose $D|_{U_\sigma} = \mathrm{div}(\chi^{m_\sigma})$, $D|_{U'_\sigma} = \mathrm{div}(\chi^{m_{\sigma'}})$. Choose $u \in \sigma' \cap N$ such that the image of u in $N(\tau)_{\mathbb{R}}$ generates the lattice $N(\tau)$. Then we have:

Proposition. *The intersection product is given by:*

$$C \cdot D = \langle m_\sigma - m_{\sigma'}, u \rangle \in \mathbb{Z}.$$

Remark. Given a T-Cartier divisor on $\mathbb{P}^1$, we obtain a PL function $f : \mathbb{R} \to \mathbb{R}$ that looks like:



The associated line bundle $\mathcal{O}_{\mathbb{P}^1}(f)$ has degree the sum of outwards slopes at origin.

Proof. We reduce to the case $X_\Sigma = U_\sigma \cup U_{\sigma'}$ and our curve $C = V(\tau)$. We may adjust the T-Cartier divisor by a principal divisor to assume that $D|_{U_\sigma}$ is empty and $D|_{U_{\sigma'}} = \operatorname{div}(\chi^{m_\sigma - m_{\sigma'}})$.

Now observe that m_σ and $m_{\sigma'}$ must agree on τ, i.e. $m_\sigma - m_{\sigma'} \in \tau^\perp \cap M$. Thus $m_\sigma - m_{\sigma'}$ is a rational function on $V(\tau)$. Now restricting D to $V(\tau) \cong \mathbb{P}^1$:

1. On the open set $U_\sigma \cap V(\tau)$, D restricts to empty.

2. On $U_{\sigma'} \cap V(\tau)$, D restricts to $\operatorname{div}(\chi^{m_\sigma - m_{\sigma'}})$.

A direct computation gives the number is $\deg(D|_C) = \langle m_\sigma - m_{\sigma'}, u\rangle$. □

7 Line bundles

7.1 Polytopes and global sections of divisors

lecture 18 Let X_Σ be a toric variety and D a T-Cartier divisor. Associated to D is the space

$$\mathcal{O}_{X_\Sigma}(D) \rightsquigarrow H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)).$$

Here $\mathcal{O}_{X_\Sigma}(D)$ is a line bundle.

The space of global sections $H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))$ could be as small as the 0 space (meaning no non-trivial global sections; an example would be $\mathcal{O}_{\mathbb{P}^n}(-1)$), or as large as infinite dimensional (e.g. $\mathcal{O}_{\mathbb{A}^1}$ on $\mathbb{A}^1$, then $H^0 = \mathbb{C}[x]$, which is an infinite dimensional $\mathbb{C}$ vector space). However if X is proper, then $H^0(X, \mathcal{O}(D))$ is finite dimensional.

Example. If we consider $\mathcal{O}_{\mathbb{P}^n}(d)$ on $\mathbb{P}^n$, then $H^0 = \{\text{degree } d \text{ homogeneous polynomials in } n+1 \text{ variables}\}$. Here $\mathcal{O}(d) = \mathcal{O}(d \cdot H)$ for any hyperplane (class) H.

Let $\phi : |\Sigma| \to \mathbb{R}$ be a PL function associated to a divisor $D = \sum a_\rho D_\rho$. These data determine a polytope (because it is the intersection of finitely many halfspaces):

$$M_{\mathbb{R}} \supset P_D := \{m \in M_{\mathbb{R}} \mid \langle m, v_\rho\rangle \le a_\rho \ \forall \rho \in \Sigma^{(1)}\}.$$

Let's investigate this a little bit more.

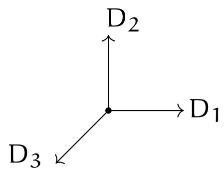
Lemma. *The (functions associated to the) lattice points in $P_D \cap M$ form a ($\mathbb{C}$-vector space)-basis for $H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))$.*

Proof. First note that since D is T-invariant, any $f \in H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))$ satisfies $\operatorname{div}(f) + D \ge 0$, and thus $\operatorname{div}(f)|_T \ge 0$. So $f \in \mathbb{C}[M]$. By the decomposition from last time, we can write

$$H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = \bigoplus_{\operatorname{div}(\chi^m)+D\ge 0} \mathbb{C}\cdot\chi^m.$$

Locally over any U_σ, asking χ^m has poles bounded by $D|_{U_\sigma}$ is the same as saying $P_D|_{U_\sigma} \cap M$ form a basis for global sections (over U_σ) of $D|_{U_\sigma}$. Together we have the required result. □

Example. Consider $X_\Sigma = \mathbb{P}^2$ with $v_1 = (1,0)$ and $D_1 = V(\langle v_1 \rangle)$.



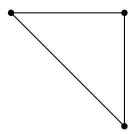
The PL function corresponding to D_1 is:

$$\phi_{D_1}((x,y)) = \begin{cases} x, & \text{if } x \geq 0 \text{ and } y \geq 0 \\ x - y, & \text{if } (x,y) \text{ is in the lower right area} \\ 0, & \text{otherwise.} \end{cases}$$

Some linear algebra calculation gives that P_{D_1} is the region bounded by

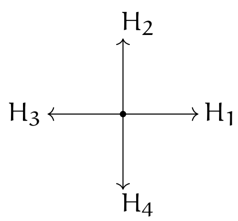
$$\begin{aligned} x &\leq 1, \\ y &\leq 0, \\ -x - y &\leq 0. \end{aligned}$$

So the polytope associated to P_{D_1} is:



⚠ Recall we have the polytope construction, which in this case would give us back the toric variety $\mathbb{P}^2$.

Example. On $\mathbb{P}^1 \times \mathbb{P}^1$:



Similarly take $D = H_1$. Then P_D is cut out by:

$$\begin{aligned} x &\leq 1, \\ -x &\leq 0, \\ y &\leq 0, \\ -y &\leq 0. \end{aligned}$$

Thus P_D is the unit length interval. In particular, if we take the cone over a interval, we won't recover $\mathbb{P}^1 \times \mathbb{P}^1$.

Remark. The key difference between the two divisors is that the first divisor is ample, but the second isn't. We will discuss more in the next section.

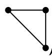
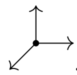
Recall how to construct a toric variety from a polytope: given $P \subset M_{\mathbb{R}}$ a lattice polytope, we take

$$\mathrm{im}\,\overline{\{(\mathbb{C}^*)^{\dim M} \to (\mathbb{C}^*)^{\#P} \to \mathbb{P}^{\#P-1}\}}.$$

Fact: if P is of full dimension, then X_P is isomorphic to X_{Σ_P}, where Σ_P is the *normal fan* of P.

Example. Intuitively in a picture, the normal fan is constructed as follows:

Given a polytope P, and a facet F, we can associate the outward normal ν_F. For any face $Q \subset P$, we may associate $\sigma_Q = \langle \nu_F \mid F \text{ contains } Q \rangle$. Together these form the fan Σ_P.

Back to $\mathbb{P}^2$. For $D = D_1 + 0 \cdot D_2 + 0 \cdot D_3$, the lattice polytope is , and the normal fan is .

Definition. A T-Cartier divisor D is *(combinatorially) ample* if

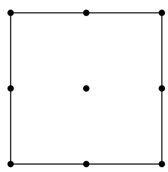
$$D \rightsquigarrow \phi_D \rightsquigarrow P_D \rightsquigarrow \Sigma_{P_D} = \Sigma_X.$$

Example. Notice that if $\phi : |\Sigma| \to \mathbb{R}$ is a PL for Σ, and $\Sigma' \to \Sigma$ is a refinement, then ϕ is linear on each of the cones of Σ' as well. However, P_D only depends on D. In particular, we have at least two cases where ampleness could fail:

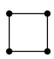
- P_D lowered dimension.
- missed some subdivision.

7.2 Motivation: projective morphisms

lecture 19 **Example.** Suppose we are given a lattice polytope in $M_{\mathbb{R}} = \mathbb{R}^2$ as follows:



This gives a morphism $(\mathbb{C}^*)^2 \to (\mathbb{C}^*)^9$ by $(z_1, z_2) \mapsto (1, z_1, z_1^2, z_2, z_1 z_2, z_1^2 z_2, z_2^2, z_1 z_2^2, z_1^2 z_2^2)$. In $\mathbb{P}^8$, we have a torus with coordinate $(1, t_1, \ldots, t_8)$. So $(\mathbb{C}^*)^2 \to (\mathbb{C}^*)^9 \to \mathbb{P}^8$ and we take the closure. The variety is $X_{\mathbb{P}} \cong \mathbb{P}^1 \to \mathbb{P}^1 \hookrightarrow \mathbb{P}^8$.

However, a different polytope  gives $X_{\mathbb{P}} \cong \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. So different polytopes with the same normal fan arise as different projective embeddings of a toric variety. (Alternatively, think of a line bundle L and $L^{\otimes 2}$ and their associated polytopes)

Suppose we have an embedding $X_\Sigma \hookrightarrow \mathbb{P}^n$. There is a god-given line bundle $\mathcal{O}_{\mathbb{P}^n}(1) := \mathcal{O}(H_\infty)$ (for a hyperplane H_∞) on the right, whose global sections are homogeneous linear forms. We can restrict $\mathcal{O}_{\mathbb{P}^n}(1)$ to X_Σ to a line bundle $\mathcal{O}_\Sigma(1)$ (just think of restricting functions).

- In the language of line bundles, we are pulling back the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^n$ to X_Σ.
- In the language of divisors, we are restricting the divisor to $H_\infty \cap X_\Sigma$, and take the line bundle associated to this.

Note: If D on X arises as a pullback of a hyperplane along a map $\begin{tikzcd} X \arrow[r, hook] & \mathbb{P}^n \\ D \arrow[u] \arrow[r] & H \arrow[u, hook] \end{tikzcd}$, then we call D *very ample*. A divisor D is *ample* if $n \cdot D$ is very ample for some $n \in \mathbb{Z}_{>0}$.

Unofficially, if P_D is the polytope of D, then $n \cdot P_D$ is the polytope of $n \cdot D$. Then ample means that some dilation of P_D is a lattice polytope with normal fan the same of X. Check the example at the beginning of this subsection.

But why do we require an embedding? In particular, everything still makes sense if we simply have a map $X_\Sigma \to \mathbb{P}^n$.

Definition. If L is a line bundle on X such that for all $x \in X$, there exists some global section $s \in H^0(X, L)$ with $s(x) \neq 0$, then L is called *basepoint free* or *globally generated*. Alternatively L is globally generated if it arises by pulling back $\mathcal{O}_{\mathbb{P}^n}(1)$ along a morphism $X \to \mathbb{P}^n$.

Note: tautologically, for any $f : X \to \mathbb{P}^n$ and $x \in X$, $f(x) \in \mathbb{P}^n$ means that some coordinate of $f(x)$ is not zero, say the i-th. Then $z_i \in H^0(\mathbb{P}^n, \mathcal{O}(1))$ satisfies $z_i(f(x)) \neq 0$.

Obviously not all line bundles satisfy this property. Some line bundles have no global sections at all.

Remark. There exist smooth toric varieties that are proper ($|\Sigma|$ is complete) but *no* line bundle on them is ample, i.e. they are non-projective.

7.3 Positivity vs convexity

We assume fans are complete in this section.

Definition. Let $\phi : |\Sigma| \to \mathbb{R}$ be a PL function linear on each cone. Then ϕ is *convex* (or lower convex) if for all $v, w \in \Sigma$ and $t \in [0, 1]$,

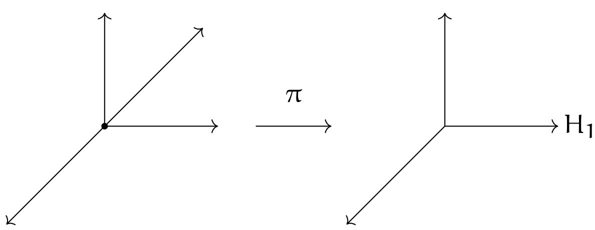
$$\phi(tv + (1-t)w) \leq t\phi(v) + (1-t)\phi(w).$$

A convex function ϕ above is called *strictly convex* if for any two maximal cones σ, σ', the linear functions $\phi|_\sigma$ and $\phi|_{\sigma'}$ are distinct.

Proposition. *Let* D *be a* T*-Cartier divisor on* X_Σ *with PL function* $\phi_D : |\Sigma| \to \mathbb{R}$*. Then:*

- D *is globally generated iff* ϕ_D *is convex.*
- D *is ample iff* ϕ_D *is strictly convex.*

Example. Consider the map from $\mathrm{Bl}_0\mathbb{P}^2 \to \mathbb{P}^2$:



and the line bundle $O(1) = O(H_1)$ on $\mathbb{P}^2$. The PL function ϕ has values 1, 0, 0 along the three rays. Then:

- $\pi^*\phi$ is a PL function on the fan of $\mathrm{Bl}_0\mathbb{P}^2$.
- It is convex but not strictly convex. Obviously the blowup is not an embedding into $\mathbb{P}^2$. But the existence of π shows the line bundle must be globally generated.

lecture 20 Recall that an invertible sheaf L is call globally generated if $L \cong f^*\mathcal{O}_{\mathbb{P}^n}(1)$ for a morphism $f : X \to \mathbb{P}^n$.

Example. Tautologically $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ (which is also very ample) and $(X, \mathcal{O}_X)$ are globally generated. Also $(\mathbb{P}^n \times \mathbb{P}^m, \pi_1^*\mathcal{O}_{\mathbb{P}^n}(1))$ where π_1 is the first projection, is globally generated; but it is not ample, as any maps induced won't even see the second factor.

Proof. We prove the first assertion. The second one can be found in Fulton's book.

First note that on X_Σ, $\mathcal{O}_X(D)$ is generated by global sections iff for each (maximal) cone $\sigma \in \Sigma$, the local principal functions m_σ satisfy

$$m_\sigma \in P_D.$$

This is true because this condition guarantees the function to be a local section, but the compatibility of m_σ also shows these local sections glue to give a global section.

Now ϕ_D is convex iff $\phi_D(u) \leq \langle m_\sigma, u\rangle$ for all $u \in |\Sigma|$ and any (maximal) cone σ, since ϕ is the PL term, and pairing with m_σ is the linear term.

Thus ϕ convex iff $m_\sigma \in P_D$. □

Example. Here are two examples of convex but not strictly convex PL functions:

1. On any Σ, take ϕ by dotting with m. This is globally linear.
2. The blowup map of $\mathbb{P}^2$, and by pulling back the PL function corresponding to the Cartier divisor H_1. Notice that the subdivided cones have identical linear functions, and therefore not strictly convex.

   In other words, we require the domain of linearity of any local function to NOT include any other cone.

7.4 Non-projective proper varieties

Note if ϕ is convex, P_D the associated polytope, and $\{m_1, \ldots, m_k\} \in P_D \cap M$, then these k-sections give a map $X_\Sigma \to \mathbb{P}^{k-1}$.

Theorem. *There exists a complete fan Σ in $\mathbb{R}^3$ such that Σ does not admit any strictly convex PL function.*

Corollary. *There exist non-projective proper algebraic varieties.*

The first (quite convoluted) construction of a non-projective proper variety was due to Hironaka. But here the toric construction is actually quite easy.

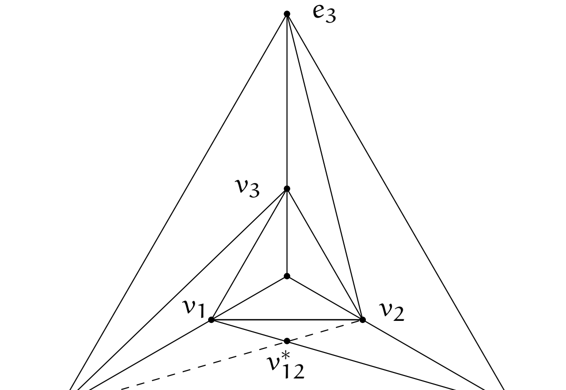
We list two more results here that we will not prove. In some sense, they are improvements of the previous theorem.

Theorem. *(Payne-Fujino) There exists a smooth toric 3-fold X_Σ that is proper, but any morphism $X_\Sigma \to \mathbb{P}^n$ for any n is constant.*

Theorem. *(Toric Chow lemma) For all toric variety X_Σ, there exists a subdivision $\widetilde{\Sigma} \to \Sigma$ such that $X_{\widetilde{\Sigma}}$ is quasi-projective. Moreover, if X_Σ is proper, then $X_{\widetilde{\Sigma}}$ is projective.*

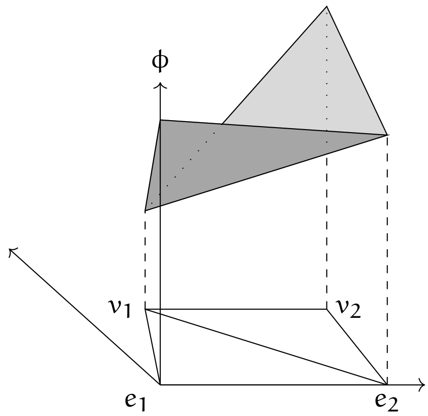
We give the construction to the very first theorem here.

Proof. Consider the cone over the following diagram:



We build a fan Σ by replacing the first orthant $\langle e_1, e_2, e_3 \rangle$ in $\Sigma_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}$ by the cone above (here we take $(\mathbb{P}^1)^3$ to make the fan complete).

Consider a PL function $\phi : |\Sigma| \to \mathbb{R}$. Assume it's strictly convex. While v_1, e_2 lie in a 2-dimensional cone, e_1, v_2 do not. The PL function restricted on the 2-dimensional trapezium e_1, v_1, v_2, e_2 looks roughly like this:



Notice that the convexity condition requires the two sheets to go down and then up again.

As a consequence,

$$\phi(v_1) + \phi(e_2) < \phi(e_1) + \phi(v_2).$$

We may adjust ϕ by any global linear function, and therefore we can assume $\phi(e_i) = 0$ (three values determine such a global linear function). Then $\phi(v_1) < \phi(v_2)$. By symmetry, we get

$$\phi(v_1) < \phi(v_2) < \phi(v_3) < \phi(v_1).$$

So we have a contradiction. □

Remark. To see how we can subdivide to give a projective variety (by toric Chow), just connect the missing three lines.

8 Further topics

8.1 Semi-stable reduction

lecture 21 **Conjecture**. (Ode, "Strong factorization", toric case) Let X, Y be two toric smooth compactifications of a torus T, then there exists a factorization of the rational map $\begin{array}{ccc} X & \dashrightarrow & Y \\ \uparrow & & \uparrow \\ T & \xrightarrow{\text{id}} & T \end{array}$ of the form:

$$\begin{tikzcd} & Z \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ X \arrow[rr, dashed] & & Y \end{tikzcd}$$

where ϕ and ψ are a sequence of blowups of a torus invariant subvariety.

Remark. The result for toric surfaces are known (easy). General case is open.

Theorem. *(Abramovich-Karu-Matsuki-Włodarczyk) (Weak factorization) For X, Y as above, there is a weak factorization:*

$$\begin{tikzcd} & X_1 \arrow[dl] \arrow[dr] & & X_2 \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & X_k \arrow[dl] \arrow[dr] & \\ X & & X_1' & & X_2' & & \cdots & & X_k' = Y, \end{tikzcd}$$

where each arrow is a sequence of blowups along smooth centers.

Remark. AKMW used this toric weak factorization to prove the weak factorization for arbitrary smooth varieties X, Y sharing a common dense open $U \subset X$, $U \subset Y$, using a method called *toroidalization*. We might talk about this in the last lecture.

Morphisms

Let $\phi : X \to Y$ be an equivariant toric morphism. Some of the niceness properties of ϕ that we can ask for are:

- Flatness ($\implies$ equidimensionality: $\dim \phi^{-1}(y)$ should be constant for all $y \in Y$).
- (All fibers are smooth) $\rightsquigarrow$ impractical.
- Reducedness of the fibers.

Definition. A morphism $\phi : X \to Y$ is called *weakly semistable* if it's flat with reduced fibers.

In general, ϕ toric doesn't imply weakly semistable. Nonetheless, if $\phi : X \to Y$ is toric and semistable, and $\psi : Z \to Y$ is toric, then $X \times_Y Z$ is toric.

Theorem. *(Abramovich-Karu, '00, weakly semistable reduction) If $\phi : X \to Y$ is toric and surjective with fan morhpism $\Sigma_X \to \Sigma_Y$, then there exist:*

1. *subdivisions of Σ_X and Σ_Y, and*
2. *lattice refinement of lattice of Y,*

such that the resulting map $\widetilde{\Sigma}_X \to \widetilde{\Sigma}_Y$ induces $\widetilde{X} \to \widetilde{Y}$ weakly semistable.

Remark. Toroidalization once again yields a weakly semistable reduction result for arbitrary surjective maps of varieties.

We discuss flat reduction and reduced reduction separately.

Flatness and fiber dimension

Fact (Chevalley): If $F : X \to Y$ is a proper morphism of varieties, then the function:

$$\begin{aligned} Y &\to \mathbb{Z}_{\geq 0} \\ y &\mapsto \dim f^{-1}(y) \end{aligned}$$

is upper semicontinuous in Zariski topology.

What this is saying is that this function should be constant, except along a Zariski closed subset where it should go up, inside which there could be anothe closed subset where the function could go up again.

Example. $\mathrm{Bl}_{\mathrm{pt}}\mathbb{P}^2 \to \mathbb{P}^2$.

Lemma. *Let $\phi : X \to Y$ be a proper (this can be relaxed) map between toric varieties. Then ϕ is equidimensional iff, in the induced map $\phi : \Sigma_X \to \Sigma_Y$, the image of every cone in Σ_X is a cone in Σ_Y,*

Proof. Since this is a local (at cone level) statement, we could reduce to case where:

- Σ_Y is a single cone τ;
- we focus on the map $X \to Y$ on an open set $U_\sigma \subset X$;
- σ and τ are full dimensional.

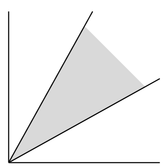
Notice that the generic fiber of ϕ has dimension $\mathrm{rk}(N_\sigma) - \mathrm{rk}(N_\tau)$. To see this, just pick the identity in the torus of U_τ, and then the fiber is the closure of the kernel of the homomorphism

$$T_\sigma \to T_\tau.$$

Now let x_τ be the distinguished point of U_τ. We claim that (exercise: unwind and check this statement): $\phi^{-1}(x_\tau)$ is a union of orbits of X, namely those orbits $V(\sigma')$ where σ' is a face of σ and the image of σ' meets the interior of τ.

Now $\dim(V(\sigma')) = \mathrm{rk}(N_\sigma) - \dim(\sigma')$. For equidimensionality to hold, the image of σ' must, therefore, have dimension $\dim(\tau)$. This holds for every face σ' mapping to τ. Therefore we conclude $\phi(\sigma)$ has to be a cone.

A non-example:



Here the image is the shaded area, and the two faces would fail.

lecture 22 *Remark.* Miracle flatness theorem (over $\mathbb{C}$): for a morphism of schemes $f : X \to Y$ where X is Cohen-Macaulay and Y is smooth, then f is flat iff f is equidimensional.

Corollary. *If Σ_Y is a smooth fan, then $f : X \to Y$ is flat iff the image of each cone is a cone.*

Proposition. *(Flat reduction) If $\Sigma_1 \to \Sigma_2$ is any surjective morphism of fans, then there exist subdivisions $\widetilde{\Sigma}_1 \to \widetilde{\Sigma}_2$ of Σ_1 and Σ_2, such that the induced map $\widetilde{\Sigma}_1 \to \widetilde{\Sigma}_2$ is a flat morphism of fans.*

Note that subdivisions correspond to proper birational toric morphisms.

We introduce a new technique: let Σ be a fan and $\phi : |\Sigma| \to \mathbb{R}$ be a convex PL function with finitely many domains of linearity, and *not necessarily linear in each cone*. Then the domains of linearity form a subdivision of the fan Σ. Subdivisions associated to such PL functions are:

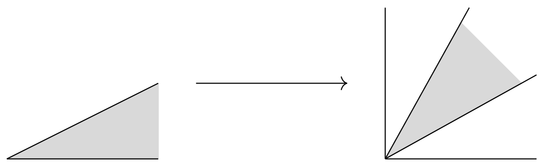
- Regular subdivision (in the sense of convex geometry).
- Correspond to projective toric birational maps $X_{\widetilde{\Sigma}} \to X_\Sigma$.

Example. All blowups.

Proof. The proof consists of three steps:

1. Assume Σ_2 is a single (maximal) cone τ.

   Given any cone $\sigma \in \Sigma_1$, its image under the map $f : \Sigma_1 \to \Sigma_2$ is a strictly convex subcone of τ. For example, it could be:

   Then there exist $l_{\sigma,1}, \dots, l_{\sigma,n}$ linear functions that cut out the image:

   $$f(\sigma) = \bigcap_{k=1}^{n} \{l_{\sigma,k} \geq 0\}.$$

   Now consider the PL function on τ given by:

   $$\psi_\tau := \sum_{k=1}^{n} |l_{\sigma,k}|.$$

   Then the associated subdivision $\widetilde{\tau} \to \tau$ satisfies that the image of σ in $\widetilde{\tau}$ is a *union* of cones. This property is also stable under further subdivision of $\widetilde{\tau}$.

2. Observe that ψ_τ can be extended to a PL function on the whole of $N_\mathbb{R}$. Let $\tau_1, \dots, \tau_r$ be cones of Σ_2, each with its own ψ_{τ_1}. Then define

   $$\psi = \sum_{i=1}^{r} \psi_{\tau_i}.$$

   Then $\widetilde{\Sigma}_2 \to \Sigma_2$ defined by ψ satisfies the above property: for each cone $\sigma \in \Sigma_1$, the image is a union of cones in $\widetilde{\Sigma}_2$.

   ⚠ $\Sigma_1 \to \widetilde{\Sigma}_2$ is not a morphism of fans, as the image of a cone is not contained in some cone.

3. Now we modify Σ_1, in the sense that we are doing the pushout of

$$\begin{tikzcd} \widetilde{\Sigma}_1 \arrow[r] \arrow[d] & \widetilde{\Sigma}_2 \arrow[d] \\ \sigma_1 \arrow[r] & \Sigma_2. \end{tikzcd}$$

To do this, we have a composition (strictly speaking, not a morphism of fans; but nonetheless a set function) $\phi : \Sigma_1 \to \widetilde{\Sigma}_2 \xrightarrow{\psi} \mathbb{R}$. Then domains of linearity of ϕ subdivide Σ_1 to give a morphism of fans $\widetilde{\Sigma}_1 \to \widetilde{\Sigma}_2$ with the required conditions. $\square$

Note that the way the proof went suggests:

Corollary. *For the subdivisions described above, we could:*

- *take subdivisions to be projective (regular).*
- *make Σ_2 a smooth fan.*

Reducedness criterion

Proposition. *Let $f : X_{\Sigma_1} \to X_{\Sigma_2}$ be an equidimensional toric morphism with associated fan morphism f_Σ. Then the fibers of f are reduced iff for every cone $\sigma \in \Sigma_1$ with image $\tau \in \Sigma_2$, the image of the lattice is the lattice in the image cone:*

$$f_\Sigma(N_\sigma) = N_\tau.$$

(Recall that $N_\tau = (\tau \cap N)^{gp}$)

A non-example: $\mathbb{P}^1 \to \mathbb{P}^1$ squaring map has $f^{-1}(0)$ not reduced.

Remark. It's possible to perform a change of lattice on Σ_2 to satisfy this condition. Some words to keep in mind for this: cyclic covering, ramified base change, or root stack construction.

8.2 Chow groups and ring structure

This is the theory for higher (co)dimensional subvarieties that extends class group and picard group which are in codim 1.

We define the k-cycles:

$$Z_k(X) := \bigoplus_{\substack{\text{irreducibile } W \subset X \\ \dim W = k}} \mathbb{Z}\langle W \rangle.$$

Define a *rational equivalence*: two subvarieties W_1, W_2 of dimension k are rational equivalent if there exists a subvariety W_{12} of dimension $k+1$ and a rational function on it $f \in \mathbb{C}(W_{12})$ such that $\mathrm{div}(f) = W_1 - W_2$. Then define the k-th Chow groups as

$$A_k(X) := Z_k(X)/\{\text{rational equivalence}\}.$$

lecture 23 Quick summary: for an algebraic variety X of dimension n, we assume the existence of the groups $A_k(X)$. ⚠ if $X = E$ an elliptic curve, then $A_0(E)$ is uncountable over $\mathbb{C}$, while $H_0(E) = \mathbb{Z}$.

Chow groups have some basic functorial properties:

- Proper pushforward: if $f : X \to Y$ is a proper morphism, then there exists a pushforward map $f_* : A_k(X) \to A_k(Y)$. Given an irreducible k-dim subvariety $W \subset X$, f is proper implies $f(W)$ is closed, i.e. a subvariety of Y. So we can define $f_*([W]) := w[f(W)]$ for some $w \in \mathbb{Z}$ to account for the degree:

$$w = \begin{cases} 0 & \text{if } f(W) \text{ has } \dim < k. \\ \deg(\mathbb{C}(W)/\mathbb{C}(f(W)) & \text{o/w.} \end{cases}$$

Example. $\mathbb{P}^1 \to \mathbb{P}^1$ the squaring map. Then the pushforward map between $A^1 = \mathbb{Z}$ is the multiplication by 2 map, corresponding to the field extension by adjoining square roots.

- Flat pullback: (for today) write $A^k(X) = A_{n-k}(X)$. If $f : X \to Y$ is flat of fiber dimension r, then there exists a flat pullback map $A^k(Y) \to A^k(X)$ defined by $f^*([W]) = [f^{-1}(W)]$. Notice that $f^{-1}(W)$ has dimension $r + (n-k) = (n+r) - k$, so is of codimension k in X.

For both proper push and flat pull, commutativity with rational equivalence needs to be checked. The standard reference is Fulton's Intersection Theory.

If X is smooth, then $A^*(X) := \bigoplus_k A^k(X)$ has a graded ring structure: given subvarieties $W \in A^k(X)$ and $W' \in A^{k'}(X)$, if $W \cap W' \subset X$ is reduced (easy) and of expected codimension $k + k'$ (usually codim is too low: any self intersection), then $[W] \cdot [W'] := [W \cap W']$.

If dimension of the intersection is not right, one uses *excess intersection formula*. Notice that we've seen an example of this: if C, D are two curves, we can treat one of them as a divisor and use the linear equivalence to "perturb" it to get a transverse intersection.

Example. If $X = \mathbb{A}^n$, then $A_0(X) = 0$ and $A_n(X) = \mathbb{Z}$. Given any 0-cycle (i.e. point) P on X, pick any line though P, and a linear function f on this line that vanishes only at P. Then $\operatorname{div}(f) = P - 0$. So $P \sim 0$.

As stated, in codimension 1, $A^1(X) = \operatorname{Cl}(X)$.

Theorem. *If X is toric, then $A^k(X)$ is generated by classes $[V(\sigma)]$ for those cones σ with $\dim(\sigma) = k$.*

Note we've already proved this for $k = 1$. The T-invariant boundary divisors generate the class group. The general result follows by similar arguments with one extra piece called *excision sequence*:

Let $Z \subset X$ be a closed subvariety, and $U = X \setminus Z$. Then $i : Z \subset X$ is a closed immersion (proper) and $j : U \subset Z$ is an open immersion (flat). Then:

$$A_k(Z) \xrightarrow{i_*} A_k(X) \xrightarrow{j^*} A_k(U) \to 0$$

is exact for all k. (There are more on the left here)

Proof. (to theorem, sketch) We filter X into pieces

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$$

where X_i is the union of $V(\sigma)$ for those σ of dimension at least $n - i$. Then $X_i \setminus X_{i-1}$ is the disjoint union of algebraic tori. By above excision sequence, it suffices to understand $A_k((\mathbb{C}^*)^l)$ for all l, but excision could be applied again here. $\square$

Remark. If X is smooth, then every invariant subvariety $V(\sigma)$ of a dimension k cone σ is the intersection $\bigcap_{i=1}^k V(\rho_i)$ where ρ_i are the rays of σ. In other words, if X is smooth, then boundary divisors generate each group multiplicative (and the ring as an algebra).

Theorem. *Let X_Σ be a smooth and projective toric variety. Then there is an isomorphism of graded rings:*

$$A^*(X) = \frac{\mathbb{Z}[D_\rho \mid \rho \textit{ is a ray of } \Sigma]}{\substack{\textit{(1) Linear equivalence on } \langle D_\rho \rangle \textit{ (class group relation),} \\ \textit{(2) } \{\prod_{i=1}^q D_{\rho_i} = 0 \textit{ if } \rho_1, \ldots, \rho_q \textit{ don't form a cone of } \Sigma\}}}$$

("(Quotient) Stanley-Reisner presentation")

Example. If $X_\Sigma = \mathbb{P}^n$, then $A^*(X)$ is the quotient of the polynomial ring $\mathbb{Z}[D_1, \dots, D_{n+1}]$ by the relations:

- $D_i = D_j$ for all i, j; for convenience, call this class H.
- $H^{n+1} = 0$, as the only rays that don't form a cone is everything.

Thus $A^*(\mathbb{P}^n) = \mathbb{Z}[H]/(H^{n+1})$, which is the same as the cohomology ring.

Remark. If X_Σ is smooth projective, then $A^*(X_\Sigma) = H^{2*}(X_\Sigma; \mathbb{Z})$ as rings.

8.3 Logarithmic geometry

lecture 24 This area was founded by the papers of Fontaine-Illusie, and Kato. The motivating examples are toric varieties.

Let X be a toric variety. Given $U \subset X$ an open affine subset, there is a set of functions defined by

$$\mathcal{M}_X(U) := \{f \in \mathbb{C}^* \oplus M \mid f \text{ is regular on } U\}.$$

In other words, these are functions that are invertible outside the boundary divisor:

$$\mathcal{M}_X(U) = \{f \in \mathcal{O}_X(U) \mid f|_{U \setminus \partial X} \text{ is invertible}\}.$$

We have actually been doing this implicitly: if $U_\sigma \subset X_\Sigma$ is an affine open, then $U_\sigma \rightsquigarrow S_\sigma = \sigma^\vee \cap M$, and tensor this with $\mathbb{C}^*$ (think of this as coefficients of the monomials), we get $\mathcal{M}_X(U)$. Notice that $\mathcal{M}_X(U)$ is a monoid, and $\mathcal{M}_X$ is a sheaf of (multiplicative) monoids.

Definition. A *logarithmic scheme* is a pair $(X, \mathcal{M}_X)$, where X is a scheme, and $\mathcal{M}_X$ is a sheaf of monoids, together with a sheaf map:

$$\alpha : \mathcal{M}_X \to \mathcal{O}_X,$$

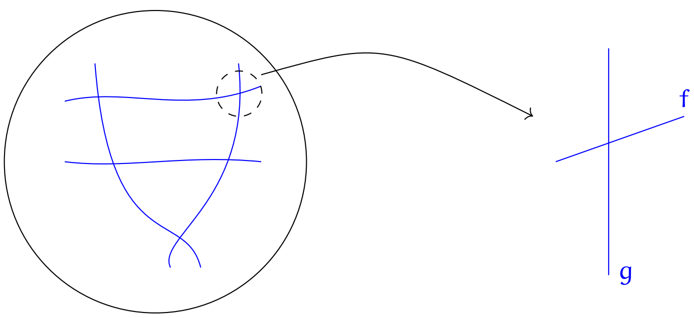
such that $\alpha^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$ is an isomorphism.

Example.

- All toric varieties are log schemes, where the key fact is that we have the boundary divisor to give us the monoid sheaf.
- Given a normal variety X and $D \subset X$ a Cartier divisor, we can similarly define

  $$\mathcal{M}_X(U) = \{f \in \mathcal{O}_X(U) \mid f|_{U \setminus D} \text{ is invertible}\}.$$

  In a picture:



Here the blue curves form a divisor on X, and suppose the intersection point is locally defined by $fg = 0$ (note that the assumption that we can write it of the form is called *normal crossings*, and we assume the divisor is a normal crossing here; although generally I'm not entirely sure we can drop this when defining divisorial log structure).

Then any powers of f and g are (locally) invertible away from the blue crossing, so the powers of both give

$$\mathbb{C}^* \otimes \mathbb{N}^2 \subset \mathcal{O}_X(U).$$

All log schemes $(X, \mathcal{M}_X)$ come with a fan, called the *tropicalization of* X. To build it, note $\mathcal{M}_X$ contains $\mathcal{O}_X^*$ as a subsheaf. We can build

$$\overline{\mathcal{M}}_X := \mathcal{M}_X / O_X^*$$

to keep track of the exponents of monoid functions. In good cases (divisorial log structure for a normal crossings divisor, described above), the sheaf $\overline{\mathcal{M}}_X$ is finitely generated (e.g. the local picture above has stalk $\mathbb{N}^2$).

If at any point p, the stalk $\overline{\mathcal{M}}_{X,p}$ is finitely generated, then we can associate to this point a cone σ_p whose monoid is exactly this stalk. These glue to form a fan Σ_X.

Since toric construction is local, this procedure generalizes to (X, Σ_X).

We again have some nice correspondence between the combinatorial side and the geometry side of things:

Example.

- A PL function $\phi : \Sigma \to \mathbb{R}$ gives an associated line bundle $\mathcal{O}_X(\phi)$.
- A subdivision $\widetilde{\Sigma} \to \Sigma$ induces an (affine) proper birational map $\widetilde{X} \to X$.

One problem though: for X toric, $\Sigma_X \leftrightarrow X$ is a perfect bijection. But this is false for log schemes. What additional (combinatorial) data on Σ and some restrictions on X are required for a similar reconstruction result?

If X is a Calabi-Yau variety (or an abelian variety), this is the SYZ mirror symmetry / Gross-Siebert program.

–End of lecture notes–

Appendices

The appendix contains some basic introductions to a few more topics. There won't be any (long) proof involved.

A Log geometry

We treat log geometry more axiomatically and systematically here. References can be found HERE (Ogus) or HERE (Gross).

Basic definitions

Definition. A *monoid* is a set with a binary operation that is associative and commutative, and there is an identity element in the set. We will usually use the multiplicative notation, and so the identity is denoted by 1.

The groupification P^{gp} of a monoid P is $P \times P/\sim$ where $(x, y) \sim (m, n)$ iff $axn = aym$ for some $a \in P$.

Definition. A *prelog structure* on a scheme X is the data of an étale sheaf of monoids M_X on X and a morphism $\alpha_X : M_X \to \mathcal{O}_X$ of sheaves of monoids. A prelog structure is a *log structure* if the induced morphism $\alpha_X^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$ is an isomorphism. A *log scheme* is a scheme with a log structure.

Every prelog structure has an associated log structure (logification...?), the pushout of $\mathcal{O}_X^* \leftarrow \alpha_X^{-1}(\mathcal{O}_X^*) \to M$.

Definition. A *log morphism* between two log schemes X and Y is a scheme morphism $f : X \to Y$, equipped with a lift of the natural map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ as a map of sheaves of monoids $f^{-1}M_Y \to M_X$. A log morphism induces a map at the level of *ghost sheaf* $\overline{M}_X := M_X/\alpha_X^{-1}(\mathcal{O}_X^*)$:

$$f^\flat : f^{-1}\overline{M}_Y \to \overline{M}_X.$$

Example. Let's consider the toric variety $\mathbb{P}^2$ with three boundary divisor given by $x_i = 0$, with the dense open torus $(\mathbb{C}^*)^2$. We can associate to it the divisorial log structure, locally generated by $\mathcal{O}^*$ and some subset of $\{x_0, x_1, x_2\}$. The ghost sheaf has stalks:

- 0 at a generic point in the open dense torus;
- $\mathbb{N}$ at the relative interior of each $x_i = 0$;
- $\mathbb{N}^2$ at the intersection of any two of the boundary divisors.

To relate to toric varieties, recall for each cone σ we have the monoid S_σ, and $U_\sigma = \mathbb{C}[S_\sigma]$ is a scheme. The inclusion $S_\sigma \to \mathbb{C}[S_\sigma]$ induces a sheaf morphism from the constant sheaf of value S_σ to $\mathcal{O}_X$. This gives a prelog structure, and we can take the associated log structure. Call this the *standard log structure* on $\mathbb{C}[P]$, which can be done whenever P is a *fs* (fine and saturated) monoid.

Definition. A *fs* monoid is a monoid P such that:

- P^{gp} is finitely generated.

- $P \to P^{gp}$ is injective.
- If $nx \in P$ for some $n \in \mathbb{N}_{\geq 0}$ and $x \in P^{gp}$, then $x \in P$.

Up to torsion, a fs monoid is the same thing as a toric monoid.

Definition. A *fs log scheme* is a log scheme X such that for every point $x \in X$, there exists a neighborhood U_x, a fs monoid P_x, a scheme map $f_x : U_x \to \operatorname{Spec} \mathbb{C}[P_x]$, such that the log structure on U_x induced by X is the pullback of the standard log structure on $\operatorname{Spec} \mathbb{C}[P_x]$

Example. (Divisorial log structure) Let X be a regular scheme, and D a reduced normal crossing divisor. Then the sheaf M_X of functions invertible outside D is a fs log structure. For example, take D to be the toric boundary of an (affine or glued) toric variety, then this divisorial log structure coincides with the standard log structure described previously. In this case, the ghost sheaf is $\mathbb{N}^r$ at the intersection of r components of D.

One thing we can do is to redo algebraic geometry putting log everywhere: log smooth, log étale, log differential, log stacks... See references for precise definitions. In particular, log smooth does not imply flat, contrary to the usual case.

Remark. Up to torsion, a log smooth morphism is étale locally the pullback of a **dominant morphism of toric varieties**.

Log moduli

This (very short) section might assume some familiarity of $\overline{M}_{g,n}$.

Since the boundary $\Delta = \overline{M}_{g,n} \setminus M_{g,n}$ forms a normal crossing divisor, $\overline{M}_{g,n}$ carries a natural log structure $M_{\overline{M}_{g,n}}$ of this NC divisor.

It's probably surprising to see that log smoothness includes some degenerate objects, including nodal singularities, which coincide with objects in the boundary of $\overline{M}_{g,n}$. In fact there is an isomorphism between the two.

Let's have a few more definitions here, for the sake of stating the result:

Definition. A *prestable* log curve is a flat proper log smooth morphism between fs log schemes $\pi : C \to W$ whose fibers are reduced and connected curves. It is *stable* if its underlying curve is stable.

Theorem. *The moduli space of stable log curves is $\overline{M}_{g,n}$ as a fs log stack for the divisorial log structure given by boundary divisors.*

Remark. There is a notion of *basic log structure* hidden here. Intuitively, this is a "universal" structure of which any map $X \to \underline{X}$ can be lifted to $X \to X^{bas}$. In particular, basic log structure has a universal property that rules out infinite automorphisms, and turns the (Artin) stack of all stable log maps with fixed target space B into a Deligne-Mumford stack. More information can be found on Kato's paper for references.

Similar results can be said for stable maps. In particular, we have this ultimate theorem:

Theorem. *If $X \to B$ is proper and log smooth, then there is a natural perfect obstruction theory on $\mathcal{M}(X/B, \beta)$ of stable log maps of type β, which is a fs log stack of finite type, defining a virtual fundamental class.*

Remark. Note that it's not true in general that the boundary of $\overline{M}_{g,n}(X, \beta)$ forms a normal crossings divisor (maybe not a divisor at all, usually too high dimension; it is true for some nice cases, e.g. $\overline{M}_{0,n}(\mathbb{P}^r, d)$), it still carries a log structure with respect to which it is log-smooth.

B Cox ring and "quotients"

We know $\mathbb{P}^n$ can be defined easily as $(\mathbb{A}^{n+1} \setminus \{0\})/\sim$, with a graded coordinate ring $\mathbb{C}[x_1, \dots, x_n]$ by degree. The *Cox homogeneous coordinate ring* does a similar job for toric varieties.

Let X be the toric variety corresponding to a fan Σ. Recall that each ray of Σ defines a boundary divisor, and they together generate the class group. We have a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & \mathrm{Div}_T(X) \arrow[r] \arrow[d] & \mathrm{Pic}(X) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M \arrow[r] & \mathbb{Z}^s \arrow[r, "f"] & A^1(X) \arrow[r] & 0 \end{tikzcd}$$

where $s = |\Sigma^{(1)}|$, the rows are exact and the vertical arrows are inclusions.

For each ray $\rho \in \Sigma^{(1)}$, we introduce a variable x_ρ, and consider the polynomial ring

$$S := \mathbb{C}[x_\rho : \rho \in \Sigma^{(1)}].$$

Note that a monomial $\sum_\rho x_\rho^{a_\rho}$ determines a divisor $D = \sum_\rho a_\rho D_\rho$. We also write this monomial as x^D. We grade the ring S as follows:

$$\text{the } \textit{degree} \text{ of a monomial is } f(D) = [D] \in A^1(X).$$

Using the exact sequences above, two monomials $\sum_\rho x_\rho^{a_\rho}$ and $\sum_\rho x_\rho^{b_\rho}$ have the same degree iff $a_\rho - b_\rho = \langle m, v_i \rangle$, where $m \in M$ and v_i are the primitive vectors corresponding to the ray ρ_i. Thus we can define the degree α vector space

$$S_\alpha = \bigoplus_{\deg(x^D) = \alpha} \mathbb{C} \cdot x^D,$$

and then S decomposes into

$$S = \bigoplus_{\alpha \in A^1(X)} S_\alpha.$$

Example. If $X = \mathbb{P}^n$, then $A^1(X) = \mathbb{Z}[H]$, and the map f sends each hyperplane H_i to the divisor class $[H]$. Thus $S_n = \bigoplus_{\sum \rho_i = n} \mathbb{C} \cdot (x_0^{\rho_0} \cdots x_n^{\rho_n})$, and $S = \mathbb{C}[x_0, \dots, x_n]$ with usual grading.

Example. For the weighted projective space $\mathbb{P}(p_0, \dots, p_n)$, $S = \mathbb{C}[x_1, \dots, x_n]$ with $\deg(x_i) = p_i$.

Recall from algebraic geometry that $H^0(\mathbb{P}^n, \mathcal{O}(n)) = S_n$. This generalizes.

Proposition. *If $\alpha = [D] \in A^1(X)$, then there is an isomorphism*

$$\phi_D : S_\alpha \cong H^0(X, \mathcal{O}_X(D)).$$

Moreover, ϕ_D form a group in the sense that there is a commutative diagram:

$$\begin{tikzcd} S_\alpha \otimes S_\beta \arrow[r] \arrow[d] & S_{\alpha+\beta} \arrow[d] \\ H^0(X, \mathcal{O}_X(D)) \otimes H^0(X, \mathcal{O}_X(E)) \arrow[r] & H^0(X, \mathcal{O}_X(D+E)). \end{tikzcd}$$

Now notice that we only used information on the rays of a fan, so the reverse process won't actually work here. We need to specify which (maximal) cones are actually in the fan for the reverse process to work. This is related to an ideal of S.

For a cone $\sigma \in \Sigma$, let $\hat{\sigma}$ be the divisor $\sum_{\rho \notin \sigma(1)} D_\rho$, and let the corresponding monomial be $x^{\hat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho$. Then define the *irrelevant ideal* to be

$$B := \langle x^{\hat{\sigma}} : \sigma \in \Sigma \rangle \subset S.$$

Notice that it suffices to go over the maximal cones $\sigma \in \Sigma$. Therefore, if two fans Σ and Σ' with the same ray structure and Cox ring, then $\Sigma = \Sigma'$ iff $B = B'$.

Example. Consider the fans of $\mathbb{P}^2$ and $\mathbb{P}^2 \setminus \{pt\}$. Notice that for $\mathbb{P}^2$ the irrelevant ideal is the usual $B = \langle x_0, x_1, x_2 \rangle \subset S$.

In particular, if we define

$$Z := \mathbb{V}(B) \subset \mathbb{C}^s$$

(which can be shown to have codim at least 2) then, just as in $\mathbb{P}^n$, we could take quotient of $\mathbb{C}^s - Z$ to reconstruct our toric variety X.

Homogeneous coordinate and quotients

Let's start by applying the functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ to the exact sequence at the beginning of the section:

$$0 \to G := \mathrm{Hom}_{\mathbb{Z}}(A^1(X), \mathbb{C}^*) \to (\mathbb{C}^*)^s \to T^n.$$

Since $(\mathbb{C}^*)^s$ acts on $\mathbb{C}^s$, and G is a subgroup, G also acts on $\mathbb{C}^s$ naturally by:

$$g \cdot t := (g([D_\rho])t_\rho)$$

where $g \in \mathrm{Hom}_{\mathbb{Z}}(A^1(X), \mathbb{C}^*)$, $t = (t_\rho) \in \mathbb{C}^s$, and D_ρ is the divisor corresponding to the ray x_ρ. Then:

Theorem. *Let notations be as above, then:*

1. *The set* $\mathbb{C}^s \setminus Z$ *is* G*-invariant.*
2. X *is (naturally) isomorphic to the categorical quotient* $(\mathbb{C}^s \setminus Z)/G$.
3. X *is the geometric quotient iff* X *is simplicial.*

In particular, for any cone $\sigma \in \Sigma$, the action of G on $\{(x_i) \in \mathbb{C}^s : x_i = 0 \text{ for } v_i \in \sigma\}$ with Z removed, then this is the closed orbit $V(\sigma)$ corresponding to σ. From here, we could have toric Nullstellensatz, graded modules, etc. See Cox's very readable original paper "The Homogeneous Coordinate Ring of a Toric Variety".

C Tropical geometry

Although there is a solid background of what tropical geometry is, involving tropical semi ring and etc, here (personally) tropical data is just a collection of discrete data in a very organized way.

Background

We work over what is called a tropical semiring $\mathbb{R}^{\text{trop}} := \mathbb{R} \cup \{+\infty\}$ with addition and multiplication given by:

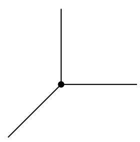
$$a \oplus b = \min(a, b)$$
$$a \odot b = a + b.$$

Note there is no additive inverse. Also note that a monomial in $\mathbb{R}^{\text{trop}}[x_1, \ldots, x_n]$ of the form $a_I x^I$ actually means $a_I + \sum i_k x_k$, and thus a polynomial is a global minimum of all of these linear functions. If we use the same notation of lattices M, N as in torics, then alternatively, a polynomial is a map $f : M_{\mathbb{R}} \to \mathbb{R}$ where, for $S \subset N = \mathbb{Z}^n$ a finite set,

$$f(x) = \sum_{I \in S} a_I x^I := \min\{a_I + I \cdot x\}.$$

We define the locus $V(f)$ to be the points in $M_{\mathbb{R}}$ where f is not linear.

Example. Consider $f = 1 \oplus (0 \odot x_1) \oplus (0 \odot x_2) = \min\{1, x_1, x_2\}$. Note 0 is the multiplicative identity here. Then $V(f)$ looks like:



with origin being $(1, 1)$.

Tropicalization of subvarieties of tori

The tropical semiring serves as a target space for a valuation. In addition to the common valuations, e.g. $\mathbb{Q}$ or $\mathbb{C}$ with $\nu(a) = 0$ for all but 0, or the p-adic, one more important valuation is the following.

We define the field $K\{t\}$ of *Puiseux series with real powers locally converging at zero* by:

$$K\{t\} = \{\phi : U \to \mathbb{R} : \phi(t) = \sum_{j \in I} a_j t^j, a_j \in \mathbb{C}^*, t \in U\}$$

where $0 \in U \subset \mathbb{R}$ is some open neighborhood, and $I \subset \mathbb{R}$ a totally ordered set. This is an algebraically closed field with a non-Archimedean valuation defined by

$$\text{val}(\sum_{j \in I} a_j t^j) = \min(I).$$

Nevertheless, for every polynomial $f \in K[x_1, \ldots, x_n]$, we can define its tropicalization by replacing the Puiseux series coefficients from K with their valuations in $\mathbb{R}^{\text{trop}}$. In fact, this works for any $f \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Example. Consider $f = 0 \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, and we want to consider $\mathrm{trop}(T^n)$ for $T = K^*$. Then $\mathrm{trop}(f)$ is smooth everywhere, and we get the $V(\mathrm{trop}(f)) = \mathbb{R}^n$. So we say $\mathrm{trop}(T^n) = \mathbb{R}^n$.

Tropicalization of toric varieties

Intuitively, we only need to glue the pieces together in the correct way.

Example. $\mathbb{A}^1$ consists of a T^1 and a point, thus tropicalizing it gives an $\mathbb{R}^1$ and a point, i.e. $\mathbb{R}^{\mathrm{trop}}$. $\mathbb{P}^1$ consists of a T^1 with two extra points: the T^1 with each point should be a copy of $\mathbb{R}^{\mathrm{trop}}$, thus $\mathrm{trop}(\mathbb{P}^1)$ is the quotient of two copies of $\mathbb{R}^{\mathrm{trop}}$ over the common open set $\mathbb{R}$. It's homeomorphic to $[0, 1]$ with standard topology.

Definition. Let Σ be the fan of a toric variety X. Recall we defined $N(\sigma) = N_{\mathbb{R}}/\mathrm{span}(\sigma)$. Then as a set, X^{trop} is the disjoint union:

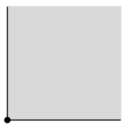
$$X^{\mathrm{trop}} = \coprod_{\sigma \in \Sigma} N(\sigma).$$

We associate to each cone σ the space $U_\sigma^{\mathrm{trop}} := \mathrm{Hom}(\sigma^\vee \cap M, \mathbb{R}^{\mathrm{trop}})$, and thus

$$U_\sigma^{\mathrm{trop}} = \coprod_{\tau \preceq \sigma} N(\tau).$$

Each patch has the pointwise-convergence topology induced from $(\mathbb{R}^{\mathrm{trop}})^k$, and they are glued together along common faces to give X^{trop}.

Example. Let Σ be the fan defining $\mathbb{P}^2$. Then $\mathrm{trop}(\mathbb{P}^2)$ has three copies of $(\mathbb{R}^{\mathrm{trop}})^2$ glued together along their faces. In pictures, notice $(\mathbb{R}^{\mathrm{trop}})^2$ looks something like this:



where the solid stuff are at infinity. So three copies of them are glued together along the boundaries (identifying one edge and the area inside for each), giving us something like a triangle-shaped stuff...

People generally deal with tropical curves, and they are much easier to describe and draw. See EXAMPLES HERE (taken from LSGNT Topics in Geometry materials).

To relate back to log geometry, log curves have a well-defined tropicalization process, and, as a result of which, the moduli space will have some nice (...well up to how you define nice) properties, e.g. see HERE.