PART III TORIC GEOMETRY (LENT 2022)
EXAMPLE SHEET 1

Notes: As always, we fix a lattice N which gives rise to a dual lattice $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ and an associated algebraic torus $T = \operatorname{Spec} \mathbb{C}[M]$. Unless stated otherwise, "cone" means "strictly convex rational polyhedral cone" and "toric variety" means "normal toric variety". There are loads more exercises to be found in Fulton and Cox–Little–Schenck, and I encourage you to try whichever ones appeal to you.

Theory

(1) Given a cone $\sigma \subseteq N_{\mathbb{R}}$ prove that the double dual recovers the original cone:
$$(\sigma^{\vee})^{\vee} = \sigma.$$
This justifies the use of the word "dual".

(2) Given a cone $\sigma \subseteq N_{\mathbb{R}}$ prove that σ is full-dimensional[1] if and only if $\sigma^{\vee}$ is strictly convex.

(3) Let Σ and Σ' be fans in vector spaces $N_{\mathbb{R}}$ and $N'_{\mathbb{R}}$. Work out for yourself the correct definition of the product fan $\Sigma \times \Sigma'$ in $N_{\mathbb{R}} \oplus N_{\mathbb{R}}$. Show that there is a natural isomorphism:
$$X_{\Sigma \times \Sigma'} \cong X_{\Sigma} \times X_{\Sigma'}.$$[2]
"The construction of a toric variety from a fan commutes with products."

(4) In lectures, we claimed that the toric variety X_{σ} is smooth if and only if σ is generated by a subset of a $\mathbb{Z}$-basis for N. Complete the proof of this statement. Give an example to show that if σ is generated by a subset of a $\mathbb{Q}$-basis then X_{σ} need not be smooth.

(5) Let X be a *not-necessarily normal* toric variety with dense torus T. Recall that we partitioned the lattice $N = \mathrm{Hom}_{\mathrm{AlgGp}}(\mathbb{C}^{\star}, T)$ of T based on the limits of one-parameter subgroups of T inside X. If X were normal, this would give the fan of X and therefore determine X uniquely. Give examples to show that, without the normality assumption, this data does not uniquely determine X.

[1] The *span* of σ is the smallest vector subspace $V \subseteq N_{\mathbb{R}}$ containing σ. It can be defined as $V = \sigma + (-\sigma)$. By definition σ is full-dimensional if and only if $V = N_{\mathbb{R}}$.

[2] As always, product of schemes means fibre product over the ground field: $X_{\Sigma} \times_{\operatorname{Spec} \mathbb{C}} X'_{\Sigma}$. For obvious reasons, we usually omit the $\operatorname{Spec} \mathbb{C}$ from the notation.

(6) Let $S \subseteq M$ be an affine semigroup. The *saturation* of S is defined to be:

$$S^{\mathrm{sat}} = \{m \in M : cm \in S \text{ for some } c \in \mathbb{Z}_{\geq 1}\}.$$

Clearly S^{sat} is saturated, and S is saturated if and only if $S = S^{\mathrm{sat}}$. Consider the inclusion

$$\mathbb{C}[S] \subseteq \mathbb{C}[S^{\mathrm{sat}}].$$

Look up "integral closure" of an integral domain, and prove that $\mathbb{C}[S^{\mathrm{sat}}]$ is the integral closure of $\mathbb{C}[S]$. The dual morphism is known as the normalisation of $\operatorname{Spec}\mathbb{C}[S]$. In each of the following examples, write down equations in affine space for both $\operatorname{Spec}\mathbb{C}[S]$ and its normalisation $\operatorname{Spec}\mathbb{C}[S^{\mathrm{sat}}]$, and study the morphism between them:

(a) $S = 2\mathbb{N} + 3\mathbb{N} \subseteq \mathbb{Z}$,
(b) $S = (1,1)\mathbb{N} + (1,0)\mathbb{N} + (0,2)\mathbb{N} \subseteq \mathbb{Z}^2$.

Practice

(1) Consider the following 3 cones in $\mathbb{R}^2$:
- $\sigma_1 = \mathrm{Cone}\langle(1,0),(1,1)\rangle$
- $\sigma_2 = \mathrm{Cone}\langle(1,1),(0,1)\rangle$
- $\tau = \mathrm{Cone}\langle(1,1)\rangle$

These cones (together with their faces) assemble to form a fan Σ. Express the resulting toric variety X_Σ as a gluing of two affine spaces. Using this description, express X_Σ as a closed subvariety of $\mathbb{C}^2 \times \mathbb{P}^1$. Study the composite morphism $X_\Sigma \to \mathbb{C}^2 \times \mathbb{P}^1 \to \mathbb{C}^2$. What do its fibres look like?

(2) Give an explicit construction of the toric variety $\mathbb{P}^2 \times \mathbb{P}^1$ from a fan – you should describe each affine toric open subset X_σ, the gluing morphisms, and why the resulting toric variety is $\mathbb{P}^2 \times \mathbb{P}^1$.

(3) Consider the fan Σ in $\mathbb{R}^2$ whose rays are generated by $(1,0)$, $(0,1)$ and $(-1,-1)$ and which *does not* have any 2-dimensional cones. Describe the toric variety X_Σ explicitly. Calculate the Euler characteristic[3] of the topological space X_Σ^{an}.[4]

(4) Give an example of a toric variety X of dimension 3 (*a toric threefold*) and a point $x \in X$ such that the tangent space T_xX at this point has dimension 4. Are there any restrictions on the dimension of the tangent space of a point on a toric threefold?

(5) (Hirzebruch surfaces.) Let Σ_r be the fan in $\mathbb{R}^2$ whose rays are given by

$$(1,0),(0,1),(-1,r),(0,-1)$$

[3] If you are confused by this, first calculate the Euler characteristic of $\mathbb{P}^2$ by giving it a cell decomposition.

[4] X_Σ^{an} means X_Σ with the "usual" i.e. Euclidean topology, as opposed to the Zariski topology. The Euclidean topology is defined locally be embedding each affine open set in $\mathbb{C}^n$ and restricting the Euclidean topology on $\mathbb{C}^n$. These agree on overlaps, hence glue to give a topology on X_Σ^{an}.

for r a non-negative integer, and whose two-dimensional cones are spanned by adjacent pairs of vectors, i.e:

$$\begin{aligned}&\mathrm{Cone}\langle(1,0),(0,1)\rangle,\\&\mathrm{Cone}\langle(0,1),(-1,r)\rangle,\\&\mathrm{Cone}\langle(-1,r),(0,-1)\rangle,\\&\mathrm{Cone}\langle(0,-1),(1,0)\rangle.\end{aligned}$$

Show that $X_r = X_{\Sigma_r}$ admits a morphism to $\mathbb{P}^1$ such that every fibre is isomorphic to $\mathbb{P}^1$ (so X_r is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$).

Soft bonus question: can you see a "shadow" of this morphism in a morphism from the fan of X_r to the fan of $\mathbb{P}^1$? (We haven't defined morphisms of fans yet, but we will. In the meantime, see if you can work out what the definition should be.)

(6) Give an example of a morphism of smooth toric varieties $X \to \mathbb{C}^2$ such that the fibre over $(0,0)$ in $\mathbb{C}^2$ is singular.

(7) Recall that $\mathcal{O}_{\mathbb{P}^1}(-1)$ is a line bundle on $\mathbb{P}^1$ whose fibre over a point is the line in $\mathbb{C}^2$ associated to that point. Prove that the total space of this bundle $\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1))$ is a 2-dimensional toric variety[5] and draw its fan. Prove that there is an open embedding of $\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(1))$ into the Hirzebruch surface X_1.

We will see later that the X_r are all compact, so X_1 is a compactification of the total space of this bundle.

[5]Hint: try to find equations for $\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1))$ inside a simple ambient variety.