PART III TORIC GEOMETRY (LENT 2022)
EXAMPLE SHEET 3

1. The Picard group of a toric variety is always a finitely generated abelian group, and the Picard rank of a toric variety X is the rank of $\operatorname{Pic} X$. Give examples to show that the Picard rank of smooth toric surfaces is unbounded, i.e. for any integer N there is a smooth toric surface with Picard rank larger than N.

2. Recall that the Hirzebruch surface $\mathbb{F}_n$ is the complete toric surface whose fan has rays $(1, 0)$, $(0, 1)$, $(-1, n)$ and $(0, -1)$. For each of the four boundary curves C in $\mathbb{F}_n$, compute the self-intersection $C \cdot C$.

3. For $0 \leq i \leq 3$ let X_i be the blowup of $\mathbb{P}^2$ at i torus fixed points.

   (a) Calculate the pullback of each toric divisor $D \subseteq \mathbb{P}^2$ along the toric morphism $X_i \to \mathbb{P}^2$. Do this twice: once using the toric language, and once using equations for the blowup.

   (b) The *anticanonical divisor* of X_i is equal to the sum of all boundary divisors on X_i. Compare the anticanonical divisor of X_i to the pullback of the anticanonical divisor of $\mathbb{P}^2$ and compute the self-intersection of both.

4. Find a toric variety whose class group contains both torsion and non-torsion elements.

5. Let X_Σ be a toric variety and assume that Σ contains a full-dimensional cone. Prove that the Picard group of X_Σ is torsion-free.

6. ($\star$) Find an example to show that the hypothesis that the fan contains a full-dimensional cone in the previous question is necessary.

7. ($\star$) Prove that if Σ is a simplicial fan, then the map $\operatorname{Pic} X_\Sigma \to \operatorname{Cl} X_\Sigma$ is the inclusion of a finite-index subgroup.

8. ($\star$) Prove the following result. Let Σ be a fan such that all its maximal cones are full-dimensional. Then the following are equivalent:
   (a) Σ is simplicial;
   (b) Every Weil divisor on X_Σ is $\mathbb{Q}$-Cartier;
   (c) The map $\operatorname{Pic}(X_\Sigma) \otimes \mathbb{Q} \to \operatorname{Cl}(X_\Sigma) \otimes \mathbb{Q}$ is an isomorphism;
   (d) The rank of $\operatorname{Pic}(X_\Sigma)$ is $\#\Sigma(1) - n$.

9. (Deformation to the normal cone.) Let X be a smooth variety. The *deformation to the normal cone*[1] of a smooth subvariety $Z \subseteq X$ is defined to be the blowup of $Z \times \{0\}$ inside $X \times \mathbb{A}^1$. This forms a flat family over $\mathbb{A}^1$ with general fibre isomorphic to X.

   For $X = \mathbb{P}^2$ and Z a co-ordinate line, describe the deformation to the normal cone using toric geometry, paying special attention to the geometry of the central fibre. Do the same for Z a co-ordinate point.

[1] This construction is originally due to Fulton and MacPherson and applies in much greater generality than for smooth subvarieties of smooth varieties. The theory of Chow groups is based in an essential way on this construction.