PART III TORIC GEOMETRY (LENT 2022)
EXAMPLE SHEET 2

1. Consider a toric surface. Is there any a priori relationship between the number of 1-dimensional and 0-dimensional orbits? What if we restrict to proper toric surfaces? What about in higher dimensions?

2. Consider the fan morphism:

$$\begin{aligned} \mathbb{R}_{\geq 0} &\to \mathbb{R}^2_{\geq 0} \\ u &\mapsto (u, 0). \end{aligned}$$

   Describe the corresponding toric morphism $\mathbb{C} \to \mathbb{C}^2$. Is it what you expected? If so: congratulations. If not, have a think and realise why what you were expecting cannot be a toric morphism.

3. For any $r \geq 1$ construct a fan Σ and a proper map to $\mathbb{R}_{\geq 0}$ such that the associated toric morphism $X \to \mathbb{A}^1$ has every nonzero fibre isomorphic to $\mathbb{P}^1$, and has zero fibre consisting of exactly r irreducible components.

4. Draw the fans of:
   - the blow-up of $\mathbb{A}^2$ at the origin;
   - the blow-up of $\mathbb{A}^3$ at the origin;
   - the blow-up of $\mathbb{A}^3$ in the line $V(x_1, x_2)$.

   Ponder the orbit-cone correspondence in this context.

5. ($\star$) Consider the variety X obtained by taking $\mathbb{P}^n$ and blowing up a single closed dimension k coordinate plane. Carefully describe the fan for this variety (the answer will certainly depend on k). Identify all the torus orbit closures of X.

6. Consider the *Cremona transform:*

$$\begin{aligned} (\mathbb{C}^\star)^n &\to (\mathbb{C}^\star)^n \\ (t_1, \ldots, t_n) &\mapsto (t_1^{-1}, \ldots, t_n^{-1}). \end{aligned}$$

   Describe the self-map on the cocharacter lattice that induces this map. Does this map extend to the compactification $\mathbb{P}^n$? What about $(\mathbb{P}^1)^n$?

7. Let X be the blowup of $\mathbb{P}^2$ at its three torus fixed points and let $\pi : X \to \mathbb{P}^2$ be the blowup morphism. Prove that the coordinatewise reciprocal map defined in the previous question extends to an endomorphism $\phi : X \to X$.

   Now let $\ell \subseteq \mathbb{P}^2$ be a line that does not pass through the coordinate points, and therefore lifts to a unique subvariety in X. Give an explicit description of the subvariety of $\mathbb{P}^2$ obtained as $\pi \circ \phi \circ \pi^{-1}(\ell)$.

8. ($\star$) Construct a toric variety X with dense torus $T \cong (\mathbb{C}^\star)^n$ with the following two properties: (1) the Cremona transform on T extends to a regular (i.e. well-defined) self-map on X, and (2) admits a toric birational morphism $X \to \mathbb{P}^n$.
9. Let σ be the 3-dimensional cone obtained as a cone over a square. Identify[1] the associated toric variety with the affine cone over $\mathbb{P}^1 \times \mathbb{P}^1$. Give a toric resolution of singularities $\pi : X \to U_\sigma$ with the additional constraint that the morphism is an isomorphism in codimension 1. Practically, this is requiring that π is a bijection on torus orbits of dimension 2 and 3.
10. ($\star$) Prove or give a counterexample: For toric morphisms of toric varieties $X_1 \to Z$ and $X_2 \to Z$ the fibre product of schemes $X_1 \times_Z X_2$ is always a toric variety.
11. (Fulton.) Fix a basis $e_1, \dots, e_n$ for N and let σ be the cone generated by:
$$e_1, e_2, \dots, e_{n-1}, -e_1 - e_2 - \cdots - e_{n-1} + me_n.$$
Show that $X_\sigma = \mathbb{C}^n/\mu_m$ where μ_m is the group of mth roots of unity, acting diagonally on $\mathbb{C}^n$. Show that X_σ is the cone over the m-uple Veronese embedding $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^{N-1}$ where $N = \binom{m+n-1}{n-1}$ is the number of degree-m monomials in n variables.[2]
12. Give a toric morphism of toric varieties $f : X \to Y$ and a point $y \in Y$ such that the scheme theoretic fiber $X \times_Y \{y\}$ is non-reduced. ($\star$) Can you give a combinatorial condition that guarantees that all scheme theoretic fibers of f are reduced?
13. ($\star$) Given an arbitrary fan Σ in $N_\mathbb{R}$ prove that there exists a subdivision $\widetilde{\Sigma}$ of Σ such that every cone of $\widetilde{\Sigma}$ is generated by a subset of an $\mathbb{R}$-basis for $N_\mathbb{R}$ (i.e. is simplicial)[3]. Prove the ultimate statement: there is a further refinement of $\widetilde{\Sigma}$ where every cone is generated by a lattice basis.

[1] You may need to look at the Wikipedia pages for Segre embedding and for affine cone.

[2] If you're struggling, start with $n = 2$.

[3] A more pretentious way of saying this is that X_Σ has a toric resolution of singularities by a smooth orbifold, or even more pretentiously, a smooth Deligne–Mumford stack.