THE NUMBER OF TORSION DIVISORS IN A STRONGLY F-REGULAR RING IS BOUNDED BY THE RECIPROCAL OF F-SIGNATURE

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Abstract. Polstra showed that the cardinality of the torsion subgroup of the divisor class group of a local strongly F-regular ring is finite. In this thesis, we first provide an expository introduction to the field of F-singularities before improving upon Polstra's result by proving that the reciprocal of the F-signature of a local strongly F-regular ring R bounds the cardinality of the torsion subgroup of the divisor class group of R. Sections 2, 3, and 4 are intended to provide background on relevant theory, section 5 presents new proofs of known results and slight modifications of generally known results, section 6 contains the main contributions of this thesis, and section 7 describes efforts to generalize both our primary result and more broadly the main theorems pertaining to F-singularities.

Contents

1. Introduction

The purpose of this thesis is broadly to provide exposition regarding basic notions in the study of F-singularities, singularities of varieties or schemes which admit characterizations via the Frobenius endomorphism, and to prove a new result connecting F-signature to a ring's torsion divisors. Given this, the reader will not be surprised to learn that we primarily work with prime characteristic rings R for which the Frobenius endomorphism $F : R \to R$ is a finite map.

As stated above, the main result of this thesis concerns an invariant known as F-signature. This was first introduced by Smith and Van den Bergh [SV97], was formally defined by Huneke and Leuschke [HL02], and was proven to exist under general hypotheses by Tucker [Tuc12]. Because we work only with integral domains, for our purposes we define the F-signature of R to be the limit

$$s(R) := \lim_{e \to \infty} \frac{\operatorname{frk} F^e_* R}{\operatorname{rank}_R F^e_* R}.$$

Here, $\operatorname{frk} F^e_* R$ denotes the free-rank of $F^e_* R$, the maximal rank of a free-module appearing in a direct sum decomposition of $F^e_* R$. We discuss the motivation and meaning of this definition in detail later in the thesis.

The ring R is said to be strongly F-regular if for each nonzero $r \in R$ there is some $e \in \mathbb{N}$ and $\varphi \in \operatorname{Hom}_R(F^e_* R, R)$ such that $\varphi(F^e_* r) = 1$. Aberbach and Leuschke proved that a local ring of prime characteristic is strongly F-regular if and only if its F-signature is positive [AL03]. Every strongly F-regular ring is a normal domain and therefore has a well-defined divisor class group, which we call $\operatorname{Cl}(R)$. Polstra showed that if R is strongly F-regular, then the torsion subgroup of $\operatorname{Cl}(R)$ is finite [Pol20]. Together, these results lend plausibility to the following theorem, the primary original contribution of this thesis:

Theorem. *Let $(R,\mathfrak{m},k)$ be a local F-finite and strongly F-regular ring of prime characteristic. Then the cardinality of the torsion subgroup of the divisor class group of R is bounded by $1/s(R)$ where $s(R)$ is the F-signature of R.*

The author notes that $1/s(R)$ has previously been used to establish upper bounds on other related invariants, notably on the order of the étale fundamental group of a strongly F-regular ring [CST18] and on the order of an individual torsion divisor D in a strongly F-regular ring [Car17]. These results further motivate this article. We further note that the techniques employed here are largely inspired by the novel proof in [PS19, Theorem 3.8] of the classic result first proven in [HL02]: $s(R)=1$ if and only if R is regular.

Unless stated otherwise, all rings are assumed to be commutative with identity and Noetherian.

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2. The Frobenius Map

2.1. **First Definitions and Facts.** Suppose R is a ring with identity 1_R and consider a ring homomorphism $\varphi : \mathbb{Z} \to R$. Since $\mathbb{Z}$ is generated as an Abelian group by 1 and since $\varphi(1) = 1_R$ by the definition of a ring morphism, there is precisely one such morphism φ. For any element $n \in \mathbb{Z}$, we therefore write $n \in R$ to mean "the image of n under the φ" without confusion.

The kernel of any ring homomorphism is an ideal and because $\mathbb{Z}$ is a PID, $\ker\varphi = (m)$ for some $m \in \mathbb{Z}$. Without loss of generality, we may choose $m \geq 0$, and we call m the characteristic of R. This is summarized in the following definition.

Definition 2.1. Let R be a ring. We denote by $\mathsf{char}(R)$ the *characteristic* of R, and define $\mathsf{char}(R)$ to be the nonnegative integer m such that (m) is the kernel of the unique homomorphism $\mathbb{Z} \to R$.

Remark 1. For a ring R, let $\mathrm{Frac}(R) = S^{-1}R$ denote the total ring of fractions where S is the set of nonzero divisors in R, which is necessarily multiplicatively closed. Then $\mathsf{char}(R) = \mathsf{char}(\mathrm{Frac}(R))$.

Proof. Let $K = \mathrm{Frac}(R)$ and consider the map $\varphi : R \to \mathrm{Frac}(R)$ defined $\varphi(f) = \frac{f}{1}$. If $\varphi(f) = 0$, then by the definition of localization there is some $u \in S$ such that $u(f-0) = uf = 0$. Since S does not contain zero divisors, $f = 0$, and therefore φ is injective. The kernel of the composition $\mathbb{Z} \to R \to \mathrm{Frac}(R)$ is therefore equal to the kernel of $\mathbb{Z} \to R$. By uniqueness of $\mathbb{Z} \to \mathrm{Frac}(R)$, we conclude that $\mathsf{char}(R) = \mathsf{char}(\mathrm{Frac}(R))$. $\square$

We also make note of the following fact:

Remark 2. If R is a domain, then $\mathsf{char}(R)$ is either 0 or prime.

Proof. Suppose $\mathsf{char}(R) = n \cdot m$ where neither n nor m is zero or a unit. This means (nm) is strictly contained in both (n) and (m), so n and m must both be nonzero in R. However, nm generates the kernel of $\mathbb{Z} \to R$, so $nm = 0$ in R. Hence R is not a domain. $\square$

The case of prime characteristic is of particular interest to us, for a ring of prime characteristic comes equipped with an additional symmetry which we call the *Frobenius endomorphism*.

Definition 2.2. Let R be a ring of prime characteristic $p > 0$. The map $F : R \to R$ defined $r \longmapsto r^p$ is called the Frobenius endomorphism. For any $e \in \mathbb{N}$, the e-th iterate of the Frobenius morphism is written F^e, and is the map $F^e(r) = r^{p^e}$.

Proposition 2.1. *If R is a ring of prime characteristic $p > 0$, then F is an endomorphism.*

Proof. It is clear that $F(0) = 0$, $F(1) = 1$, and $F(rs) = F(r)F(s)$ for $r, s \in R$ as these properties hold in any ring. It remains to be shown that $F(r+s) = F(r) + F(s)$.

First, consider the binomial coefficient $\binom{p}{n}$ for $0 < n < p$. We may expand this as

$$\binom{p}{n} = \frac{p!}{n!(p-n)!} = \frac{p \cdot (p-1) \cdot \ldots \cdot 1}{n!(p-n)!} = p \cdot \frac{(p-1)!}{n!(p-n)!}.$$

The number $\binom{p}{n}$ is always an integer, so because p is prime, $n!(p-n)!$ must divide $(p-1)!$. Therefore, as an element of R,

$$\binom{p}{n} = p \cdot \frac{(p-1)!}{n!(p-n)!} = 0 \cdot \frac{(p-1)!}{n!(p-n)!} = 0.$$

With this, we now see that for any $r, s \in R$,

$$F(r+s) = (r+s)^p = \sum_{n=0}^{p} \binom{p}{n} r^{p-n} s^n = r^p + 0 \cdot r^{p-1} s + \cdots + 0 \cdot r s^{p-1} + s^p = r^p + s^p = F(r) + F(s),$$

and we conclude that F is an endomorphism on R. $\square$

Proposition 2.2 (Basic properties of F)**.** *Suppose R is a ring of prime characteristic.*

(a) R is reduced if and only if F is injective.

(b) Let $X = \mathrm{Spec}(R)$. Then the map $F' : X \to X$ induced by F is the identity on X viewed as a topological space.

Proof.

(a) ($\Longrightarrow$) If R has a nilpotent element r, then there is some smallest $n \in \mathbb{Z}$ such that $r^n = 0$. Choosing the smallest e such that $p^e \geq n$, we have that $F^{e-1}(r) \neq 0$

since $p^{e-1} < n$, but

$$F\left(F^{e-1}(r)\right) = F^e(r) = r^{p^e} = r^{p^e - n} r^n = 0.$$

This means $\ker F$ is nontrivial.

$(\Longleftarrow)$ If F is not injective, then $F(r) = r^p = 0$ for some $r \in R$. This means r is nilpotent, so R is not reduced.

(b) The induced map $F' : X \to X$ is defined $F'(\mathfrak{p}) = F^{-1}(\mathfrak{p})$, so we need to show $\mathfrak{p} = F^{-1}(\mathfrak{p})$ for any prime ideal $\mathfrak{p}$ of R. Since every ideal is closed under multiplication, it is clear that $F(a) = a^p \in \mathfrak{p}$ for any $a \in \mathfrak{p}$, so $\mathfrak{p} \subseteq F(a)$. If instead $a \in F^{-1}(\mathfrak{p})$, then $F(a) = a^p \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime, either $a \in \mathfrak{p}$ or $a^{p-1} \in \mathfrak{p}$. If $a \in \mathfrak{p}$ then we are done, and if $a^{p-1} \in \mathfrak{p}$ we may deduce that either $a \in \mathfrak{p}$ or $a^{p-2} \in \mathfrak{p}$. Repeating this process inductively shows that $a \in \mathfrak{p}$, giving us the second inclusion.

$\square$

There are many ways in which the Frobenius endomorphism may be utilized to study properties of prime characteristic rings and schemes. In this note, we will focus our attention solely on the following collection of endofunctors:

$$F_*^e : \mathsf{Mod}(R) \to \mathsf{Mod}(R).$$

These functors are obtained by restricting the R-action along the Frobenius endomorphism. More precisely, for a ring R of prime characteristic and M an R-module, $F_*^e(M)$ is exactly M as an Abelian group. Denoting by $F_*^e m$ the element of $F_*^e(M)$ corresponding to $m \in M$, we define the R-action on $F_*^e(M)$ by $r \cdot F_*^e m = F_*^e r^{p^e} m$.

Remark 3 (F_*^e is exact)**.** Any R-linear map $\varphi : M \to P$ of R-modules is already linear over $F^e(R)$, so $F_*^e \varphi$ is exactly the map φ. Since $F_*^e(-)$ is simply the identity on underlying Abelian groups, it is therefore exact.

We will be particularly interested in the collection $\{F_*^e R\}_{e\in\mathbb{N}}$ of R-modules resulting from applying F_*^e to R itself. Though we are primarily think of $F_*^e R$ as an R-module, it is useful to note that $R \cong F_*^e R$ as rings via the ring isomorphism $r \longmapsto F_*^e r$.

There are two primary ways to think about these R-modules, and it is useful to keep both in mind. If R is reduced, then $\mathsf{Frac}(R) = \prod_i K_i$ for some collection of fields $\{K_i\}_i$, and in particular, $\overline{\mathsf{Frac}(R)} = \prod_i \overline{K_i}$. Letting $K = \mathsf{Frac}(R)$, we define

$$R^{1/p^e} := \left\{ s \in \overline{K} \;\middle|\; s^{p^e} \in R \right\}.$$

It is easily checked that R^{1/p^e} is a ring. We also have containment $R \subseteq R^{1/p^e}$, with equality only when R contains all its p^eth roots. The map $\mathbb{Z} \to R^{1/p^e}$ factors through the inclusion $R \hookrightarrow R^{1/p^e}$, meaning $\mathsf{char}(R) = \mathsf{char}(R^{1/p^e})$. Finally, we can then view the eth iterate of the Frobenius map as a ring morphism $F^e : R^{1/p^e} \to R$ defined $F^e(r^{1/p^e}) = r$. The map F^e is injective since R is reduced and is surjective since R^{1/p^e} contains all p^eth roots of elements in R, so F^e is an isomorphism of rings. These properties are summarized in the following proposition,

Proposition 2.3. *Let R be a reduced ring of prime characteristic $p > 0$ and K be the total ring of fractions of R. Then*

(a) R^{1/p^e} is unique up to isomorphism.
(b) $R \subseteq R^{1/p^e}$
(c) R^{1/p^e} is a ring of prime characteristic $p > 0$.
(d) $F^e : R^{1/p^e} \to R$ defined by $F^e(r^{1/p^e}) = r$ is an isomorphism of rings.

From (d) we obtain the immediate corollary:

Corollary 2.4. *The map $\psi : F_*^e R \to R^{1/p^e}$ is an isomorphism of R modules.*

Proof of Corollary 2.4. As a set, $F_*^e(R) = R$, so as a map of Abelian groups ψ_e is simply the inverse of $F^e : R^{1/p^e} \to R$. It remains to show that it is R-linear.

Recall for $s \in R$ and $F^e_* r \in F^e_* R$ that $s \cdot F^e_* r = F^e_*(s^{p^e} r)$, where this final term is a product in R. Notice:

$$s\psi_e(r) = sr^{1/p^e} =^* (s^{p^e} r)^{1/p^e} = \psi_e(s^{p^e} r) = \psi_e(s \cdot r).$$

The equality marked $*$ follows from the fact that $r \longmapsto r^{1/p^e}$ is a ring map from $R \to R^{1/p^e}$. Since ψ_e respects R-action, it is R-linear and we conclude that it is an isomorphism of R-modules. $\square$

This corollary allows us to switch freely between R^{1/p^e} and F^e_*. While we primarily use $F^e_* R$ in these notes, it is useful to keep both notions in mind. The R^{1/p^e} modules are particularly useful when considering computations in which the R-action on $F^e_* R$ becomes obfuscating.

With these preliminary definitions out of the way, we now introduce a result of Kunz which demonstrates how the Frobenius map can be used to characterize rings of prime characteristic.

Theorem 2.5 ([Kun69])**.** *A ring R of prime characteristic $p > 0$ is regular if and only if $F^e : R \to F^e_* R$ is flat for some (equivalently all) $e > 0$.*

We present a proof of the forward direction the following section under the additional assumption that R is local.

2.2. **F-finite Rings.**

Definition 2.3. A ring R of prime characteristic $p > 0$ is said to be F-finite if $F^e : R \to R$ is finite for some (equivalently all) $e > 0$. Equivalently, R is F-finite if $F^e_* R$ is finitely generated as an R-module for some (equivalently all) $e > 0$.

The following proposition shows that "F-finiteness" is preserved under basic ring constructions.

Proposition 2.6. *Let R be an F-finite ring of characteristic $p > 0$. Then*

(a) If $I \subseteq R$ is an ideal then R/I is F-finite.

(b) If W is a multiplicative set then $W^{-1}R$ is F-finite.

(c) Both $R[x]$ and $R[[x]]$ are F-finite.

Note that this proposition does *not* tell us whether F-finite is a local condition – in fact, a counterexample was recently found which demonstrates a ring may be F-finite at every localization and yet fail to be F-finite itself [DI20]. Using the above proposition, we obtain the following corollary:

Corollary 2.7. *Let R be an F-finite ring. If an R-module M is finitely generated then $F_*^e M$ is finitely generated as well.*

Proof. For any integer N, the map $F_*^e(R^{\oplus N}) \to (F_*^e R)^{\oplus N}$ defined

$$F_*^e(r_1,\ldots,r_N) \longmapsto (F_*^e r_1,\ldots,F_*^e r_N)$$

is easily seen to be an isomorphism. Furthermore, since M is finitely generated, there exists a surjective map $R^{\oplus N} \to M$ for some N. The induced map $F_*^e(R^{\oplus N}) \to F_*^e M$ is still surjective by the exactness of $F_*^e(-)$, and as the module $(F_*^e R)^{\oplus N} \cong F_*^e(R^{\oplus N})$ is finitely generated by the assumption that R is F-finite, $F_*^e M$ is also finitely generated. $\square$

We now proceed to the main fact of this section. A corollary of the following proposition constitutes the forward direction of the proof of theorem 2.5.

Proposition 2.8. *If $(R,\mathfrak{m},k)$ be an F-finite local ring of prime characteristic $p > 0$ then $F_*^e R$ is finitely generated as an R-module.*

To prove this, we first require the following lemma:

Lemma 2.9. *Suppose R is a prime characteristic ring and $I \subseteq R$ is an ideal. An element $r \in R$ is a zero divisor on R/I if an only if it is a zero divisor on $F_*^e R/(I \cdot F_*^e R)$.*

Proof of Lemma 2.9. Fix an element $r \in R$, and consider the maps

$$\varphi : R/I \xrightarrow{r\cdot} R/I \qquad \text{and} \qquad \varphi_r^e : \frac{F_*^e R}{I \cdot F_*^e R} \xrightarrow{r\cdot} \frac{F_*^e R}{I \cdot F_*^e R}.$$

Suppose that r is a zero divisor on $F_*^e R/I \cdot F_*^e R$, i.e. that $r \cdot F_*^e s \in I \cdot F_*^e R$ for some $s \in R$ where $F_*^e s \notin I \cdot F_*^e R$. Then there are some $m_1, ..., m_n \in R$ and some $a_1, ..., a_n \in I$ such that

$$\begin{aligned} r \cdot F_*^e s = F_*^e \left(r^{p^e} \cdot s \right) &= a_1 \cdot F_*^e m_1 + ... + a_n \cdot F_*^e m_n \\ &= F_*^e \left(a_1^{p^e} m_1 \right) + ... + F_*^e \left(a_n^{p^e} m_n \right) \\ &= F_*^e \left(a_1^{p^e} m_1 + ... + a_n^{p^e} m_n \right). \end{aligned}$$

Since $F_*^e(-)$ is the identity on sets, $r^{p^e} \cdot s = a_1^{p^e} m_1 + ... + a_n^{p^e} m_n$. This is an element of I since $a_1, ..., a_n \in I$, so r is a zero divisor on R/I. Note that the above set of equalities suffice to show that $F_*^e s \in I \cdot F_*^e R$ and therefore that $s \in I$, so we know $s \notin I$.

Suppose now that $r \cdot s = a \in I$ for some $r, s \in R$. Then

$$r^{p^e} s^{p^e} = a^{p^e} \cdot 1$$

which implies

$$r \cdot F_*^e \left(s^{p^e} \right) = a \cdot F_*^e (1) \in I \cdot F_*^e R$$

so r is a zero divisor on $F_*^e R/(I \cdot F_*^e R)$. □

We may now proceed to the proof of proposition 2.8

Proof of Proposition 2.8. We first show that $\mathsf{depth}(F_*^e R) = \mathsf{dim}(R)$. Fix $e > 0$, and consider a $F_*^e R$-regular sequence $(x_1, ..., x_n)$. Because x_i is not a zero divisor on $\frac{F_*^e R}{(x_1, ..., x_{i-1}) \cdot F_*^e R}$, by lemma 2.9 it is not a zero divisor on $R/(x_1, ..., x_{i-1})$ either, hence $(x_1, ..., x_n)$ is a regular sequence on R. Since R is a regular local ring and therefore Cohen-Macaulay, $n = d$ and therefore $\mathsf{depth}(F_*^e R) = d$. Furthermore, $F_*^e R$ is finitely generated because R is F-finite, hence we may apply the Auslander-Buchsbaum theorem:

$$\mathrm{pd}_R(F^e_* R) + \mathrm{depth}(F^e_* R) = \dim(R),$$

which tells us $\mathrm{pd}_R(F^e_* R) = 0$. This occurs only when $F^e_* R$ is projective, and because every finitely generated projective module over a Noetherian local ring is free [Mat80], F^e_R is free. $\square$

Every free module is flat, hence proposition 2.8 implies the forward direction of theorem 2.5.

3. F-Signature

Recall that a **singular point** or a **singularity** P on a scheme X is a point at which $\mathcal{O}_{X,P}$ is not a regular ring. The point P is *not* a singular point if and only if $\mathcal{O}_{X,P}$ is regular, and in this case we unsurprisingly refer to P as a **regular** point. Kunz's theorem, theorem 2.5, provides a nice characterization of singularities in the case that $\mathcal{O}_{X,P}$ is of prime characteristic. According to Kunz, to check if $\mathcal{O}_{X,P}$ is a regular point, it suffices to find precisely one $e > 0$ such that $F^e : R \longrightarrow F^e_* R$ is flat. In the case that R is Noetherian, local, and $F^e_* R$ is finitely generated, i.e., that R is F-finite, we see that $F^e_* R$ is flat if and only if it is free. Hence, to quantify how badly a point P fails to be regular, it makes sense to measure how badly $F^e_* \mathcal{O}_{X,P}$ fails to be free for all $e \in \mathbb{N}$. This is the central idea behind F-signature. As we are interested in characterizing F-singularities, we will restrict our attention primarily to local rings. We make note of efforts which globalize the notions discussed in section 7.

We first need a notion that describes the "free-ness" of a module. If R is a commutative ring and M is a finitely generated R-module then the notion we use is the *free rank* of M, which we denote $\mathrm{frk}_R(M)$ or simply $\mathrm{frk}(M)$ if R is unambiguous, and we define $\mathrm{frk}_R(M)$ to be the maximal integer n such that there exists a surjective R-linear map $M \to R^{\oplus n}$.

Proposition 3.1. *Let R be a commutative (not necessarily Noetherian) ring and M a finitely generated R-module. Then* $\mathrm{frk}(M)$ *is the largest rank of a free module F such that $M \cong F \oplus N$, where N necessarily does not have an R summand.*

Proof. Note first that if $M \cong F \oplus N$ where F is a free module and if $N \oplus R \oplus N'$ for some N', then $M \cong F \oplus R \oplus N' \cong (F \oplus R) \oplus N'$, and $\operatorname{rank}(F \oplus R) = \operatorname{rank} F + 1$. Hence, if $\operatorname{rank} F$ is maximal then N does not have a free summand.

Suppose that F is a free module of maximal degree such that there exists a surjective morphism $\varphi : M \twoheadrightarrow F$. Note that F is necessarily finitely generated because it is isomorphic to the quotient of a finitely generated module, so it has some finite basis $x_1, ..., x_n$. For each $1 \leq i \leq n$, choose m_i such that $\varphi(m_i) = x_i$. We can then define a map $\psi : F \to M$ by $x_i \longmapsto m_i$ and extend by linearity. The composition $\varphi \circ \psi$ is then the identity on F, so φ splits and we obtain $M \cong F \oplus \ker \varphi$.

Conversely, if $M \cong F \oplus N$, then the projection map $\pi : M \cong F \oplus N \to F$ surjects onto F. If F' is a free module such that $\varphi : M \twoheadrightarrow F'$ and $\operatorname{rank} F' > \operatorname{rank} F$, then by the first implication $M \cong F' \oplus N'$ where $N' = \ker \varphi$ and

$$N \cong \frac{F \oplus N}{F} \cong \frac{F' \oplus N'}{F} \cong (F'/F) \oplus N',$$

and because $\operatorname{rank} F' > \operatorname{rank} F$ the quotient F'/F is free. Thus, N has a free summand. By contrapositive, if N does not have a free summand then $\operatorname{rank} F$ is maximal. $\square$

We will be particularly interested in the free rank of $F_*^e R$. The *e-th Frobenius splitting number* of R is defined to be the free rank of $F_*^e R$ and is denoted by $a_e(R)$. Notice that if $F_*^e R$ is free, then $a_e(R) = \operatorname{rank}(F_*^e R)$. Likewise, if $a_e(R) = \operatorname{rank}(F_*^e R)$, then decomposing $F_*^e R \cong R^{\oplus a_e(R)} \oplus N$ implies that

$$\operatorname{rank}(F_*^e R) = \operatorname{rank}(R^{\oplus a_e(R)}) + \operatorname{rank}(N) = a_e(R) + \operatorname{rank}(N) = a_e(R)$$

from which we obtain $\operatorname{rank}(N) = 0$.

Since no one element subset of N is linearly independent, for each $n \in N$ there must be some $r \in R$ such that $rn = 0$. This means N is a torsion module. The inclusion $N \hookrightarrow R^{\oplus a_e(R)} \oplus N \cong F^e_* R$ means N is isomorphic to a submodule of $F^e_* R$. However, for an element $F^e_* m \in F^e_* R$, $r \cdot F^e_* m = F^e_*(r^{p^e} m) = 0 \implies r^{p^e} m = 0$. If R is a domain and $r \neq 0$, then $m = 0$, and hence the only torsion submodule of $F^e_* R$ is the trivial submodule. This implies $N = 0$ and that $F^e_* R$ is free.

Thus, $a_e(R) = \operatorname{rank}(F^e_* R)$ if and only if $F^e_* R$ is free. For any R-module M it is the case that $\operatorname{frk}(M) \leq \operatorname{rank}(M)$, so the quotient

$$s_e(R) := \frac{a_e(R)}{\operatorname{rank} F^e_* R}$$

is always between 0 and 1 and we have $s_e(R) = 1$ exactly when $F^e_* R$ is free. If $s_e(R)$ is close to 1 for all $e \in \mathbb{N}$, then R is close to being regular. It turns out that, rather than examining each $s_e(R)$ individually, it is more useful to examine the asymptotic behavior of the quotient. We call this the F-signature of R:

$$s(R) := \lim_{e \to \infty} s_e(R) = \lim_{e \to \infty} \frac{a_e(R)}{\operatorname{rank} F^e_* R}.$$

The existence of this limit is not immediately obvious. In fact, it took 10 years after Huneke and Leuschke first defined F-signature for Tucker to prove it existed under reasonably general hypotheses.

To see why one might care about F-signature, consider again our problem of quantifying the "badness" of a singularity. Rather than considering $s_e(\mathcal{O}_{X,P})$ for each $e \in \mathbb{N}$ to decide whether P is regular or not, it suffices to check the asymptotic behavior of this ratio.

Theorem 3.2 ([HL02]). *Let $(R, \mathfrak{m}, k)$ be an F-finite local ring of prime characteristic $p > 0$. Then $s(R) = 1$ if and only if R is a regular local ring.*

Hence, F-signature provides a classification of regular points in the case that $\mathcal{O}_{X,P}$ is of characteristic p and F-finite.

Limits preserve non-strict inequalities, meaning $0 \leq s(R) \leq 1$. By our previous rational, the closer $s(R)$ is to 0 the more "severe" the singularity. Clearly an F-signature of 0 is the worst case scenario, so it is natural to ask, when do we at least have $s(R) > 0$? Happily, there is a nice answer to this question, but it requires that we introduce a new type of ring.

Definition 3.1. A ring R is said to be strongly F-regular if for each $r \in R$ not contained in any minimal prime, there exists an $e \in \mathbb{N}$ and $\varphi \in \mathrm{Hom}_R(F^e_* R, R)$ such that $\varphi(F^e_* r) = 1$.

Note: As one would expect, there are also notions of F-regular rings and weakly F-regular rings. These are typically defined using tight closure theory, a different, well-developed approach to the study of F-singularities. This is the origin of the "weak implies strong" conjecture, one of the most important open problem in the study of F-singularities. While strongly F-regular rings do admit characterizations via tight closure theory, we prefer the definition provided for our purposes.

This class of rings turns out to be precisely those whose F-signature is positive:

Theorem 3.3 ([AL03])**.** *Suppose $(R, \mathfrak{m}, k)$ is a F-finite local ring of prime characteristic p. Then R is strongly F-regular if and only if $s(R) > 0$.*

One potential proof of the "weak implies strong" conjecture would be an argument which showed that if a ring R is F-finite and weakly F-regular then $s(R) > 0$. However, F-signature is in general quite difficult to compute and this method would likely not be fruitful.

4. Divisors

Divisors, in their many forms, play an important role in algebraic geometry and make up the bulk of the geometric content of our main result. Suppose R is a Noetherian normal domain and K its field of fractions. A *Weil divisor* on $\mathrm{Spec}(R)$ is a formal sum of height one primes of R over $\mathbb{Z}$ in which only finitely many coefficients are nonzero. Alternatively, if X is a scheme/variety, then we say a Weil divisor is a formal sum of codimension 1 irreducible subschemes/subvarieties. We let $\mathrm{Div}(R)$ or $\mathrm{Div}(X)$ denote this Abelian group.

4.1. **The Divisor Class Group.**

If $\mathfrak{p}$ is a height one prime of R, then $R_\mathfrak{p}$ is a regular local ring of Krull dimension 1 and is therefore a PID [Mat80]. Denote by $\pi_\mathfrak{p}$ a generator of the maximal ideal $\mathfrak{p}R_\mathfrak{p}$ of $R_\mathfrak{p}$. Every element in $R_\mathfrak{p}$ is either a unit or contained in $\langle \pi_\mathfrak{p} \rangle$, the sole maximal ideal, so we may *uniquely* write every element of R_p as $u\pi_\mathfrak{p}^N$ for some $N \geq 0$. Similarly, noting that the fraction field of a ring is isomorphic to the fraction field of a localization of that ring, we may write every element of K uniquely as $u\pi_\mathfrak{p}^N$ for some integer N. The existence of u and N is easy to see, and for uniqueness, we simply note that if $u\pi_\mathfrak{p}^N = s\pi_\mathfrak{p}^M$ for some units u and s and some integers N and M, then taking $M \geq N$ without loss of generality we see

$$\left(su^{-1}\pi_\mathfrak{p}^{M-N} - 1\right)\pi_\mathfrak{p}^N = 0,$$

and so $su^{-1}\pi_\mathfrak{p}^{M-N} = 1$ because $R_\mathfrak{p}$ is a domain. This means $\pi_\mathfrak{p}^{M-N}$ is invertible, so $M = N$ and $su^{-1} = 1$ yielding that $s = u$.

The fact that for each height one prime $\mathfrak{p} \in \mathrm{Spec}(R)$ we may uniquely write each $f \in K$ as $u\pi_\mathfrak{p}^N$ for some unit $u \in R_\mathfrak{p}$ and integer N allows us to define the following map:

$$\nu_\mathfrak{p} : K^\times \to \mathbb{Z}, \qquad \nu_\mathfrak{p}(f) = N. \tag{1}$$

Lemma 4.1. *The map defined in expression 1 is a valuation.*

Proof. Suppose $f = u\pi_{\mathfrak{p}}^N$ and $g = s\pi_{\mathfrak{p}}^M$ for units u and s and integers N and M. Then

$$\nu_{\mathfrak{p}}(fg) = \nu\left(su\pi_{\mathfrak{p}}^{M+N}\right) = M+N = \nu_{\mathfrak{p}}(f) + \nu_{\mathfrak{p}}(g).$$

Similarly, if we suppose without loss of generality that $M \geq N$, then

$$f+g = \left(u+s\pi_{\mathfrak{p}}^{M-N}\right)\pi^N. \tag{2}$$

We see $u+s\pi_{\mathfrak{p}}^{M-N} \in R$ since $M \geq N$, so $\nu\left(u+s\pi_{\mathfrak{p}}^{M-N}\right) \geq 0$ and so by what we proved above,

$$\nu_{\mathfrak{p}}(f+g) = \nu\left(u+s\pi_{\mathfrak{p}}^{M-N}\right) + \nu\left(\pi_{\mathfrak{p}}^N\right) \geq 0+N = \min\{\nu_{\mathfrak{p}}(f), \nu_{\mathfrak{p}}(g)\}. \tag{3}$$

If $N \neq M$, in which case $M > N$, then $u+s\pi_{\mathfrak{p}}^{M-N}$ is the sum of a unit and a non-unit, and is therefore itself a unit. Equation 2 then tells us $\nu_{\mathfrak{p}}(f+g) = N$. We therefore have equality in equation 3 if $\nu_{\mathfrak{p}}(f) \neq \nu_{\mathfrak{p}}(g)$, and conclude that $\nu_{\mathfrak{p}}$ is a valuation. □

The valuation $\nu_{\mathfrak{p}}$ for each height one prime $\mathfrak{p} \in \operatorname{Spec} R$ also satisfies the following additional properties. Suppose R is a normal domain, K is its field of fractions, and that $f \in K^{\times}$. Then

(a) Let $\mathfrak{p}$ be a height one prime of R. Then $f \in R_{\mathfrak{p}}$ if and only if $\nu_{\mathfrak{p}}(f) \geq 0$.
(b) There exist only finitely many height one primes $\mathfrak{p}$ such that $\nu_{\mathfrak{p}}(f) \neq 0$.

Fact (a) follows because $f \in R_{\mathfrak{p}}$ exactly when $f = u\pi_{\mathfrak{p}}^N$ for some $N \geq 0$ and a unit $u \in R_{\mathfrak{p}}$. To see (b), first notice that because we may write $f = \frac{f_1}{f_2}$ where $f_1, f_2 \in R$ and $\nu_{\mathfrak{p}}(f) = \nu_{\mathfrak{p}}(f_1) - \nu_{\mathfrak{p}}(f_2)$, it suffices to consider the case that $f \in R$. The only prime of height 0 in a domain is the zero ideal, so the minimal primes over (f) are height one, and since every ideal has finitely many minimal primes, $f \in \mathfrak{p}$ for only finitely many height one primes $\mathfrak{p}$. Because $\nu_{\mathfrak{p}}(f) \neq 0$ if and only if $f \in \mathfrak{p}$, this gives us the result.

All together, this means that each $f \in K^{\times}$ there is a well-defined Weil divisor defined as follows:

$$\operatorname{div}(f) = \sum_{\substack{\mathfrak{p} \in \operatorname{Spec} R \\ \operatorname{ht}(\mathfrak{p})=1}} v_{\mathfrak{p}}(f) \cdot \mathfrak{p}. \tag{4}$$

We call such a Weil divisor a *principal divisor.*

Lemma 4.2. *The set of all principal divisors form a subgroup of* $\operatorname{Div}(X)$*.*

Proof. Any unit $u \in R$ satisfies $v_{\mathfrak{p}}(u) = 0$ for all height one primes $\mathfrak{p}$, so $\operatorname{div}(u) = 0$. If $f, g \in K^{\times}$, then

$$\operatorname{div}(f) + \operatorname{div}(g) = \sum_{\substack{\mathfrak{p} \in \operatorname{Spec} R \\ \operatorname{ht}(\mathfrak{p})=1}} v_{\mathfrak{p}}(f) \cdot \mathfrak{p} + \sum_{\substack{\mathfrak{p} \in \operatorname{Spec} R \\ \operatorname{ht}(\mathfrak{p})=1}} v_{\mathfrak{p}}(g) \cdot \mathfrak{p} = \sum_{\substack{\mathfrak{p} \in \operatorname{Spec} R \\ \operatorname{ht}(\mathfrak{p})=1}} v_{\mathfrak{p}}(f \cdot g) \cdot \mathfrak{p} = \operatorname{div}(f \cdot g),$$

so principal divisors are closed under addition. Finally, if $f \in K^{\times}$, then $\operatorname{div}\left(f^{-1}\right)$ is the inverse of $\operatorname{div}(f)$, hence we conclude that the set of all principal divisors forms a subgroup of $\operatorname{Div}(X)$. $\square$

Given two Weil divisors D_1 and D_2, we say D_1 and D_2 are linearly equivalent and write $D_1 \sim D_2$ if $D_1 - D_2$ is a principal divisor. We say that a Weil divisor D is *effective* if all its coefficients are nonnegative. Similarly, we say D is *anti-effective* if all its coefficients are non-positive. We write $D \geq 0$ if D is effective and for Weil divisors D_1 and D_2 we write $D_1 \geq D_2$ if $D_1 - D_2$ is effective.

Proposition 4.3. *Every Weil divisor is linearly equivalent to an effective divisor.*

Proof. Let $D = a_1\mathfrak{p}_1 + ... + a_n\mathfrak{p}_n$ be an arbitrary Weil divisor of R. Choose $0 \neq f \in \bigcap_{i=1}^{n} \mathfrak{p}_n$ (which exists because $\mathfrak{p}_i \neq 0$ for $1 \leq i \leq n$). Then $v_{\mathfrak{p}_i}(f) \geq 1$ for each $1 \leq i \leq n$ and since $f \in R$, $v_{\mathfrak{q}}(f) \geq 0$ for any other height one prime as well. Choosing

$a = \max\{a_1, ..., a_n\}$ and letting $S = \operatorname{Spec} R \setminus \{\mathfrak{p}_1, ..., \mathfrak{p}_n\}$, we have

$$\begin{aligned}
D + \operatorname{div}(f^a) &= \sum_{i=1}^{n} a_i \mathfrak{p}_i \ + \sum_{\substack{\mathfrak{q} \in \operatorname{Spec} R \\ \operatorname{ht}(\mathfrak{q})=1}} \nu_{\mathfrak{q}}(f^a)\mathfrak{q} \\
&= \sum_{i=1}^{n} a_i \mathfrak{p}_i \ + \sum_{\substack{\mathfrak{q} \in \operatorname{Spec} R \\ \operatorname{ht}(\mathfrak{q})=1}} a\nu_{\mathfrak{q}}(f)\mathfrak{q} \\
&= \sum_{i=1}^{n} (a\nu_{\mathfrak{p}_i} - a_i)\mathfrak{p}_i \ + \sum_{\mathfrak{q} \in S} \nu_{\mathfrak{q}}(f)\mathfrak{q}.
\end{aligned}$$

The final term in the equality is effective since $\nu_{\mathfrak{q}}(f) \geq 0$ for all height 1 primes $\mathfrak{q}$, and because $a\nu_{\mathfrak{p}_i}(f) - a_i \geq a - a_i \geq 0$. Since $(D + \operatorname{div}(f)) - D = \operatorname{div}(f)$, we conclude that D is linearly equivalent to an effective divisor. □

We define the *divisor class group* of $X = \operatorname{Spec} R$ to be

$$\operatorname{Cl}(X) = \operatorname{Div}(X)/\sim. \tag{5}$$

Note that $\sim$ is an equivalence relation by Lemma 4.2, so this quotient is well defined. The divisor class group provides insight into the geometric structure of X, as well as into the ring structure of R. For example, we have the following fact:

Proposition 4.4. *Let R be a normal Noetherian domain. Then R is a UFD if and only if* $\operatorname{Cl}(R) = 0$.

Before we prove this fact, we first need the following two lemmas.

Lemma 4.5. *A ring R is a UFD if and only if every height one prime is principal.*

Proof. Suppose first that R is a UFD and let $\mathfrak{p}$ be a height one prime. Because every UFD is a domain, $\mathfrak{p} \neq 0$ and therefore there is some $0 \neq x \in \mathfrak{p}$. If $x = r_1 \cdot ... \cdot r_n$ is a decomposition of x into irreducible then $r_i \in \mathfrak{p}$ for some i, and by reindexing we have $r_1 \in \mathfrak{p}$. Since r_1 is irreducible, (r_1) is prime, nonzero, and therefore of at least height one. By the fact that $(r_1) \subseteq \mathfrak{p}$ and $\mathfrak{p}$ is height one, we now $\mathfrak{p}$ is a minimal prime of (r_1), and hence $(r_1) = \mathfrak{p}$.

Now suppose that every height one prime is principal $\square$

Lemma 4.6 can be thought of as an upgrade of condition (a) met by the valuation $v_{\mathfrak{p}}$.

Lemma 4.6. *Let R be a Noetherian normal domain and $f \in K^{\times}$. Show that* $\mathrm{div}(f) \geq 0$ *if and only if $f \in R$.*

Though, for our purposes, we only discuss Weil divisors in this thesis, this correspondence sheds light into how Weil divisors fit into the broader framework of algebraic geometry.

Proof. Suppose first that $f \in R$. Then for each height one prime $\mathfrak{p}$ we see $f \in R_{\mathfrak{p}} \implies v_{\mathfrak{p}}(f) \geq 0$. Each coefficient of $\mathrm{div}(f)$ is therefore nonnegative, hence $\mathrm{div}(f) \geq 0$.

Now suppose that $\mathrm{div}(f) \geq 0$, i.e. that $v_{\mathfrak{p}}(f) \geq 0$ for each height one prime. Condition (a) met by the valuations $v_{\mathfrak{p}}$ tells us $f \in R_{\mathfrak{p}}$. By [Mat80, Theorem 38] we know that

$$R = \bigcap_{\substack{\mathfrak{p} \in \operatorname{Spec} R \\ \operatorname{ht}(\mathfrak{p}) = 1}} R_{\mathfrak{p}},$$

so we conclude $f \in R$. $\square$

We now prove the proposition.

Proof of Proposition 4.4. Suppose first that R is a UFD. Then each height one prime $\mathfrak{p}$ equals $(f_{\mathfrak{p}})$ for some $f_{\mathfrak{p}} \in R$, hence $\mathrm{div}(f_{\mathfrak{p}}) = \mathfrak{p}$. Each height one prime is a principal divisor, and therefore so are finite linear combinations of primes. Thus, each divisor $D \in \mathrm{Div}(R)$ is principal, and we have that $\mathrm{Cl}(R) = 0$.

Now suppose that $\mathrm{Cl}(R) = 0$, so in particular, for every height one prime $\mathfrak{p}$ there is some $f \in K^{\times}$ such that $\mathfrak{p} = \mathrm{div}(f)$. Because $v_{\mathfrak{p}}(f) = 1$, $f \in \mathfrak{p} \cdot R_{\mathfrak{p}}$ and is therefore an element of R. Furthermore, since $\nu_{\mathfrak{q}}(f) = 0$ for any other height one prime $\mathfrak{q}$, $\operatorname{div}(f) \geq 0$ and therefore $f \in R$ by Lemma 4.6. In fact, $f \in R \cap \mathfrak{p}A_{\mathfrak{p}} = \mathfrak{p}$. Suppose now that g is any other element in $\mathfrak{p}$, so $\nu_{\mathfrak{p}}(g) \geq 1$ and $\nu_{\mathfrak{q}}(g) \geq 0$ for any other height one prime $\mathfrak{q}$. Then for any $\mathfrak{q}$,

$$\nu_{\mathfrak{q}}(g/f) = \nu_{\mathfrak{q}}(g) - \nu_{\mathfrak{q}}(f) \geq 0,$$

hence we know $g/f \in R$, again by Lemma 4.6. We know f is not a unit since $f \in \mathfrak{p}$, so $f \mid g$. Since f divides an arbitrary element of $\mathfrak{p}$, $(f) = \mathfrak{p}$. $\square$

The divisor class group also serves to generalize other notions in geometry and number theory.

Example 4.1 ([Har77, Example 6.3.2])**.** If R is a Dedekind domain, then $\mathrm{Cl}(X)$ is the ideal class group of R, as defined in algebraic number theory. Thus Proposition (4.4) generalizes the fact that R is a UFD if and only if its ideal class group is 0.

More exotically, one can see that identifying objects up to linear equivalence unifies several different constructions in algebraic geometry. In the case that X is a regular scheme over a field k, we see that

$$\frac{\left\{\begin{matrix}\text{Weil}\\ \text{Divisors}\end{matrix}\right\}}{\begin{matrix}\text{linear}\\ \text{equivalence}\end{matrix}} \cong \frac{\left\{\begin{matrix}\text{Cartier}\\ \text{Divisors}\end{matrix}\right\}}{\begin{matrix}\text{linear}\\ \text{equivalence}\end{matrix}} \cong \frac{\left\{\begin{matrix}\text{Invertible}\\ \text{Sheaves}\end{matrix}\right\}}{\text{isomorphism}} \cong \frac{\left\{\begin{matrix}\text{Line}\\ \text{Bundles}\end{matrix}\right\}}{\text{isomorphism}},$$

as shown in Harshorne's treatment of the subject [Har13].

4.2. **Divisorial Ideals.** Given a Weil divisor D, we define the *divisorial ideal* of D to be

$$R(D) = \{f \in K^{\times} \mid \operatorname{div}(f) + D \geq 0\} \cup \{0\}.$$

In the proof of Proposition 4.3, we found an $f \in K^{\times}$ such that $\operatorname{div}(f) + D \geq 0$ in order to show that D was linearly equivalent to an effective divisor. Likewise, any effective divisor D' linearly equivalent to D will satisfy $D' = \operatorname{div}(f) + D$ for some $f \in K^{\times}$, so the divisorial ideal $R(D)$ is in bijection with those effective divisors to which D is linearly equivalent. The following proposition summarizes the first properties of divisorial ideals.

Proposition 4.7. *Let R be a Noetherian normal domain with fraction field K and suppose D, D_1 and D_2 are Weil divisors.*

(a) $R(D)$ is an R-submodule of K

(b) $R(0)$ is R

(c) $R(D_1) \subseteq R(D_2)$ if and only if $D_2 \geq D_1$. In particular, $R(D) \subseteq R$ if and only if D is effective.

(d) Given $f \in K^{\times}$ we have an R-module isomorphism

$$R(D) \xrightarrow{\cdot f} R(-\operatorname{div}(f) + D).$$

(e) $R(D)$ is finitely generated as an R-module.

Proof.

(a) Suppose $f, g \in R(D)$. Then for each height one prime $\mathfrak{p} \in \operatorname{Spec}(R)$,

$$\nu_{\mathfrak{p}}(f+g) \geq \min\{\nu_{\mathfrak{p}}(f), \nu_{\mathfrak{p}}(g)\}$$

which means $\operatorname{div}(f+g) + D \geq 0$ or equivalently $f + g \in R(D)$ Furthermore, if $r \in R$ then $\operatorname{div}(r) \geq 0$ by Lemma 4.6 and so

$$\operatorname{div}(rf) + D = \operatorname{div}(r) + \operatorname{div}(f) + D \geq \operatorname{div}(f) + D \geq 0.$$

Since $R(D)$ is closed under addition and R-action and includes 0 by definition, it is an R-submodule of $K^{\times}$.

(b) A nonzero element $f \in K^{\times}$ is in $R(0)$ if and only if $\operatorname{div}(f) \geq 0$ which occurs exactly when $f \in R$, so $R(0) = R$.

(c) Suppose first that $D_2 - D_1 \geq 0$. Then for any $f \in K^\times$ $\mathrm{div}(f) + D_2 \geq \mathrm{div}(f) + D_1$, so $R(D_1) \subseteq R(D_2)$. Now suppose that $D_2 - D_1 \ngeq 0$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the collection of height one primes which appear as nontrivial summands of D_1 and D_2 and let

$$D_1 = \sum_{i=1}^{n} a_i \mathfrak{p}_i \qquad \text{and} \qquad D_2 = \sum_{i=1}^{n} b_i \mathfrak{p}_i.$$

Note that the coefficients a_i and b_i might be zero. Since $D_2 - D_1 \ngeq 0$, there must be at least one $1 \leq i \leq n$ such that $b_i - a_i < 0$, so without loss of generality take it to be $i = 1$. Choose $f_1 \in R$ so that $\nu_{\mathfrak{p}_1}(f_1) = 1$. For each $2 \geq i \geq n$, choose $f_i \in \mathfrak{p}_i \setminus \mathfrak{p}$, so that $\nu_{\mathfrak{p}_1}(f_i) = 0$ and $\nu_{\mathfrak{p}_i}(f_i) \geq 1$. If $a_1 > 0$, then

$$\mathrm{div}\left(f_1^{-a_1} + f_2^{|a_2|} + \ldots + f_n^{|a_n|}\right) + D_1 \geq 0 \cdot \mathfrak{p}_1 + \sum_{i=2}^{n} (|a_i| + a_i)\mathfrak{p}_i \geq 0,$$

but

$$\mathrm{div}\left(f_1^{-a_1} + f_2^{|a_2|} + \ldots + f_n^{|a_n|}\right) + D_2 = (b_1 - a_1) \cdot \mathfrak{p}_1 + \sum_{i=2}^{n} (|a_i| + b_i)\mathfrak{p}_i \ngeq 0,$$

so $f_1^{-a_1} + f_2^{|a_2|} + \ldots + f_n^{|a_n|} \in R(D_1) \setminus R(D_1)$. Likewise, if $a_1 < 0$, then the element $f_1^{a_1} + f_2^{|a_2|} + \ldots + f_n^{|a_n|}$ is in $R(D_1) \setminus R(D_2)$. In either case, we have the desired result.

(d) This map is well defined, since for any $g \in R(D)$, $\mathrm{div}(f \cdot g) - \mathrm{div}(f) + D = \mathrm{div}(g) + D \geq 0$. It is additive and R-linear simply because multiplication by elements in K is itself K-linear and additive. It is injective because R is a domain: $g \cdot f = 0 \implies g = 0$ or $f = 0$, and because $f \in K^\times$, $g = 0$. Finally, it is surjective because f is a field element, so $g \cdot f = h \cdot f \implies g = h$.

(e) By Proposition 4.3 there exists some element $f \in K^\times$ such that $\mathrm{div}(f) - D \geq 0 \iff -\mathrm{div}(f) + D \leq 0 \iff \mathrm{div}(f^{-1}) + D \leq 0$. Because $\mathrm{div}(f^{-1}) + D$ is anti-effective, part (c) tells us $R(-\mathrm{div}(f) + D) \subseteq R$, and is therefore finitely generated as R is Noetherian. Part (4) then tells us that

$$R(D) \xrightarrow{f} R(-\operatorname{div}(f)+D)$$

is an isomorphism of R-modules, hence D is also finitely generated.

□

We will be particularly interested in how divisorial ideals interact with restriction along Frobenius $F^e_*(-)$. The first step towards understanding this interaction is the following proposition originally due to Serre:

Proposition 4.8. *Let R be a Noetherian normal domain. A finitely generated rank 1 R-module M satisfies Serre's condition (S_2) if and only if M is isomorphic to some divisorial ideal $R(D)$.*

In particular, this means $R(D)$ is a *reflexive module* [Har94]. A reflexive module is an R-module M such that the natural map

$$j : M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M,R),R)$$

which sends $m \in M$ to the map $\varphi \in \operatorname{Hom}_R(M,R) \longmapsto \varphi(m)$ is an isomorphism. When we say "$R(D)$ satisfies Serre's condition (S_2)," we mean that for any prime ideal $\mathfrak{p} \subseteq R$

$$\operatorname{depth} R(D)_{\mathfrak{p}} \geq \min\{n, \operatorname{ht}(\mathfrak{p})\}$$

when $n = 2$. Note here that because F^e_* commutes with $\operatorname{Hom}(-,R)$ it also commutes with the reflexification functor $\operatorname{Hom}_R(\operatorname{Hom}_R(-,R),R) = (-)^{**}$.

Recall that for a prime $P \in \operatorname{Spec}(R)$, the nth symbolic power of P is defined $P^{(n)} = P^n R_P \cap R$. Divisorial ideals can be realized as the intersections of symbolic powers of primes. For a divisor $D=N_1\mathfrak{p}_1+\ldots+N_\ell\mathfrak{p}_\ell$,

$$R(D)=R(N_1\mathfrak{p}_1)\cap\ldots\cap R(N_\ell\mathfrak{p}_\ell)=\mathfrak{p}_1^{(-N_1)}\cap\ldots\cap\mathfrak{p}_\ell^{(-N_\ell)}. \tag{6}$$

Note that if $N\geq 0$ and $P\in\operatorname{Spec}(R)$ is a prime, then

$$P^{(-N)}:=\{f\in K\ \mid\ \nu_P(f)\geq -N\}\cup\{0\}.$$

This means $P^{(-N)}$ consists only of elements in k which have at most an Nth power of π_P in their denominator. We prove the following two lemmas for use in the proof of Proposition 4.11 (c). The following lemma is well known, but it's proof is included for convenience.

Lemma 4.9. *Suppose $(R,\mathfrak{m},k)$ is a local F-finite strongly F-regular ring, D is a divisor and $\mathfrak{p}\in\operatorname{Spec}(R)$ is a fixed height one prime. Then $R(D)_\mathfrak{p}\cong\langle\pi_\mathfrak{p}^{-N}\rangle$, where N is an integer such that $N\mathfrak{p}$ is the $\mathfrak{p}$ component of D. Note that we define*

$$\langle\pi_\mathfrak{p}^{-N}\rangle:=\frac{1}{\pi_\mathfrak{p}^N}\cdot R_\mathfrak{p}\subseteq k.$$

Proof. Suppose first that $D=m\mathfrak{q}$ for some height one prime $\mathfrak{q}\in\operatorname{Spec}(R)$. Then by equation (6) and using the fact that $S^{-1}(M_1\cap M_2)=S^{-1}M_1\cap S^{-1}M_2$ for any R-modules M_1 and M_2, we have

$$R(D)_\mathfrak{p}=\mathfrak{q}_\mathfrak{p}^{(-m)}=\begin{cases}R_\mathfrak{p} & \text{if }\ \mathfrak{p}\neq\mathfrak{q}\\ \mathfrak{q}^{-m}R_\mathfrak{q} & \text{if }\ \mathfrak{p}=\mathfrak{q}\end{cases}.$$

Since $\mathfrak{q}$ is height one, notice that $\mathfrak{q}^{-m}R_\mathfrak{q}=\langle\pi_\mathfrak{q}^{-m}\rangle$. For an arbitrary divisor $D=D=N_1\mathfrak{q}_1+\ldots+N_\ell\mathfrak{q}_\ell$, we then obtain

$$R(D)_\mathfrak{p}=\big(\mathfrak{q}_1^{(-N_1)}\cap\ldots\cap\mathfrak{q}_\ell^{(-N_\ell)}\big)_\mathfrak{p}=\langle\pi_\mathfrak{p}^{-m}\rangle,$$

where m is the coefficient of the $\mathfrak{p}$ summand in D. Note that if $m = 0$, we obtain $R(D)_{\mathfrak{p}} = R_{\mathfrak{p}}$. $\square$

The following lemma is again well known, but it's proof is included for convenience.

Lemma 4.10. *Suppose $(R, \mathfrak{m})$ is a local principal ideal domain of prime characteristic $p > 0$. Denote by $\langle \pi \rangle$ the maximal ideal $\mathfrak{m}$. Then for any integers $n, m \in R$,*

$$F_*^e \langle \pi^n \rangle \otimes_R \langle \pi^m \rangle \cong F_*^e \langle \pi^{n+mp^e} \rangle$$

via the isomorphism $\varphi : F_*^e x \otimes y \longmapsto F_*^e (xy^{p^e})$.

Proof. We first establish that this map is a R-module homomorphism. It is R-multiplicative: if $r \in R$, $x \in \langle \pi^n \rangle$ and $y \in \langle \pi^m \rangle$, then

$$\begin{aligned}\varphi\big(r \cdot (F_*^e x \otimes_R y)\big) &= \varphi(F_*^e r^{p^e} x \otimes_R y) \\ &= F_*^e (r^{p^e} x y^{p^e}) \\ &= r \cdot F_*^e (xy^{p^e}) = r \cdot \varphi(F_*^e x \otimes_R y),\end{aligned}$$

and by extending additively to arbitrary tensors we have that φ is R-linear. To see that it is an isomorphism, we define a map

$$\psi : F_*^e \langle \pi^{n+mp^e} \rangle \to F_*^e \langle \pi^n \rangle \otimes_R \langle \pi^m \rangle, \; F_*^e (xy^{p^e}) \longmapsto F_*^e x \otimes_R y$$

Every element of $\langle \pi^{n+mp^e} \rangle = \langle \pi^{mp^e} \cdot \pi^n \rangle = \langle \pi^n \rangle \cdot \langle \pi^m \rangle^{p^e}$ may be realized as a product $x \cdot y^{p^e}$ where $x \in \langle \pi^n \rangle$ and $y \in \langle \pi^m \rangle$, so this map is well-defined and is easily seen to be a morphism of R-modules. We then have

$$\varphi \circ \psi(F_*^e (xy^{p^e})) = \varphi(F_*^e x \otimes_R y) = F_*^e (xy^{p^e})$$

and

$$\psi \circ \varphi(F^e_* x \otimes_R y) = \psi(F^e_*(xy^{p^e})) = F^e_* x \otimes_R y,$$

so we conclude that φ is an isomorphism. $\square$

We now proceed to the following proposition, which provides a means of manipulating expressions involving tensor products, reflexifications, and scalar-restrictions of divisorial ideals.

Proposition 4.11. *Suppose $(R, \mathfrak{m}, k)$ is a Noetherian normal domain of prime characteristic $p > 0$ and let D_1 and D_2 be Weil divisors. Note that for an R-module M, we denote by M^{**} the reflexification of M $\mathrm{Hom}_R(\mathrm{Hom}_R(M,R),R)$. The following are true:*

(a) $\mathrm{Hom}_R(R(D_1), R(D_2)) \cong R(D_2 - D_1)$

(b) $(R(D_1) \otimes R(D_2))^{**} \cong R(D_1 + D_2)$

(c) $(F^e_* R(D_1) \otimes_R R(D_2))^{**} \cong F^e_* R(D_1 + p^e D_2)$.

Proof. We first prove (a). Suppose $f \in R(D_2 - D_1)$, and define a map $\varphi_f : R(D_1) \to K^\times$ by $g \longmapsto f \cdot g$. Since $f \in R(D_2 - D_1)$, $\mathrm{div}(f) + D_2 \geq D_1$, and so for any $x \in R(D_1)$,

$$\mathrm{div}(xf) + D_2 = \mathrm{div}(x) + \mathrm{div}(f) + D_2 \geq \mathrm{div}(x) + D_1 \geq 0,$$

so $xf \in \mathrm{div}(D_2)$. Each $f \in R(D_2 - D_1)$ defines a map $\varphi_f : R(D_1) \to R(D_2)$, so $R(D_2 - D_1) \subseteq \mathrm{Hom}_R(R(D_1), R(D_2))$.

Now fix a map $\varphi \in \mathrm{Hom}_R(R(D_1), R(D_2))$. Each divisorial ideal $R(D)$ is rank 1, so tensoring $\varphi : R(D_1) \to R(D_2)$ gives us a commutative diagram

$$\begin{array}{ccc} R(D_1) & \xrightarrow{\quad\varphi\quad} & R(D_2) \\ \downarrow & & \downarrow \\ K \cong R(D_1) \otimes_R K & \xrightarrow{\varphi'} & R(D_2) \otimes_R K \cong K. \end{array}$$

The map φ' is linear as a map of k-vector spaces, so there is some element $f \in k$ such that $\varphi'(x) = xf$ for every $x \in k$. Tracing through the diagram and using the fact that each divisorial ideal is a submodule of k, we realize $\varphi(x) = xf$ as well. This means $R(D_1 - \operatorname{div}(f)) = f \cdot R(D_1) \subseteq R(D_2)$, so $D_1 - \operatorname{div}(f) \leq D_2 \implies D_2 - D_1 + \operatorname{div}(f) \geq 0$, giving us the second inclusion.

Given (a), the proof of (b) follows from the fact that $\operatorname{Hom}(M, -)$ and $- \otimes M$ form an adjoint pair, i.e. that $\operatorname{Hom}(A \otimes B, C) = \operatorname{Hom}(A, \operatorname{Hom}(B, C))$. Indeed,

$$\begin{aligned}
\operatorname{Hom}_R\left(\operatorname{Hom}_R(R(D_1) \otimes R(D_2), R), R\right) &\cong \operatorname{Hom}_R\left(\operatorname{Hom}_R(R(D_1), \operatorname{Hom}(R(D_2), R)), R\right) \\
&\cong \operatorname{Hom}_R\left(\operatorname{Hom}_R(R(D_1), R(-D_2)), R\right) \\
&\cong \operatorname{Hom}_R\left(R(-(D_2 + D_1)), R\right) \\
&\cong R(D_1 + D_2).
\end{aligned}$$

To prove (c), for two divisors D_1 and D_2 we first notice that the map

$$\varphi : F^e_* R(D_1) \otimes_R R(D_2) \to F^e_* R(D_1 + p^e D_2), \quad F^e_* x \otimes y \mapsto F^e_*(x \cdot y^{p^e})$$

is a homomorphism. Indeed, if $x \in R(D_1)$ and $y \in R(D_2)$, then

$$\operatorname{div}(x \cdot y^{p^e}) + D_1 + p^e D_2 = \operatorname{div}(x) + D_1 + p^e(\operatorname{div}(y) + D_2)) \geq 0,$$

so $F^e_* x \otimes y$ lands in $F^e_* R(D_1 + p^e D_2)$. It's R-multiplicative: taking $r \in R$, we see

$$\varphi\big(r \cdot (F^e_* x \otimes y)\big) = \varphi\big(F^e_* x \otimes r \cdot y\big) = F^e_*(x \cdot r^{p^e} y^{p^e}) = r \cdot F^e_*(x \cdot y^{p^e}) = r \cdot \varphi\big(F^e_* x \otimes y\big),$$

and by extending additive to arbitrary tensors we have that φ is R-linear. By localizing at some height one prime $\mathfrak{p} \in \operatorname{Spec}(R)$, we get a map

$$\varphi_{\mathfrak{p}} : F^e_* R(D_1)_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} R(D_2)_{\mathfrak{p}} \to F^e_* R(D_1 + p^e D_2)_{\mathfrak{p}}$$

where we have taken advantage of the fact $(F^e_* R(D_1) \otimes_R R(D_2))_{\mathfrak{p}} \cong F^e_* R(D_1)_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} R(D_2)_{\mathfrak{p}}$. We claim $\varphi_{\mathfrak{p}}$ is an isomorphism.

Let $n\mathfrak{p}$ and $m\mathfrak{p}$ be the components of $\mathfrak{p}$ in D_1 and D_2 respectively, where n and m are integers. Because $\mathfrak{p}$ is height one, we see $R(D_1)_{\mathfrak{p}} \cong \mathfrak{p}^{-n}R_{\mathfrak{p}} \cong \langle \pi^{-n} \rangle$ and $R(D_2)_{\mathfrak{p}} \cong \mathfrak{p}^{-m}R_{\mathfrak{p}} \cong \langle \pi^{-m} \rangle$, where $\langle \pi \rangle$ is the maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$ in $R_{\mathfrak{p}}$. After localization and composition with the above isomorphisms, the map $\varphi_{\mathfrak{p}}$ is defined

$$\varphi_{\mathfrak{p}} : F_*^e \langle \pi^{-n} \rangle \otimes_R \langle \pi^{-m} \rangle \to F_*^e \langle \pi^{-n-mp^e} \rangle, \; F_*^e x \otimes y \mapsto F_*^e (xy^{p^e}),$$

and applying Lemma 4.10 tells us it is an isomorphism.

Since $\mathfrak{p}$ was chosen arbitrarily, φ is an isomorphism after localizing at *any* height 1 prime. Thus, since φ is an isomorphism at the level of height one primes, by reflexifying, we see that

$$(F_*^e R(D_1) \otimes_R R(D_2))^{**} \xrightarrow{\varphi^{**}} \left(F_*^e R(D_1 + p^e D_2)\right)^{**}$$

is an isomorphism by [Har94, Theorem 1.12]. Since reflexification commutes with $F_*^e(-)$ and every divisorial ideal is reflexive, $F_*^e R(D_1 + p^e D_2)$ is reflexive as well. This gives us

$$(F_*^e R(D_1) \otimes_R R(D_2))^{**} \cong \left(F_*^e R(D_1 + p^e D_2)\right)^{**} \cong F_*^e R(D_1 + p^e D_2)$$

as desired. $\square$

5. Preliminary Results and Notation

For R-modules M and N, denote by $a^M(N)$ the maximal number of M summands appearing in a direct sum decomposition of N. In the case that $N = F_*^e R$, we say that $a_e^M(R) := a^M(F_*^e R)$. We use $\mathrm{T}(\mathrm{Cl}(R))$ to denote the torsion subgroup of $\mathrm{Cl}(R)$, the divisor class group of R. We now present a refinement of [Pol20, Corollary 2.2], stated as Lemma 5.2, which features the same techniques employed by Polstra. We state [Pol20, Lemma 2.1] for convenience.

Lemma 5.1 ([Pol20, Lemma 2.1])**.** *Let $(R,\mathfrak{m},k)$ be a local normal domain. Let C be a finitely generated (S_2)-module, M a rank 1 module, and suppose that $C \cong M^{\oplus a_1} \oplus N_1 \cong M^{\oplus a_2} \oplus N_2$ are choices of direct sum decompositions of C so that M cannot be realized as a direct summand of either N_1 or N_2. Then $a_1 = a_2$.*

Lemma 5.2. *Let $(R,\mathfrak{m},k)$ be a local normal domain and C a finitely generated (S_2)-module. If $D_1,...,D_t$ are divisors representing distinct elements of the divisor class group and $R(D_i)$ is a direct summand of C for each $1 \le i \le t$, then*

$$R(D_1)^{a^{R(D_1)}(C)} \oplus \ldots \oplus R(D_t)^{a^{R(D_t)}(C)}$$

is a direct summand of C.

Proof. Suppose $C \cong R(D_1)^{n_1} \oplus ... \oplus R(D_i)^{n_i} \oplus N$ for some $1 \le i \le t$ and $n_j \le a^{R(D_j)}(C)$ for $1 \le j \le i$. We induct in two ways: first we show that $R(D_{i+1})$ is necessarily a summand of N when $i < t$, and second we show that if $n_j < a^{R(D_j)}(C)$, then $R(D_j)$ is a summand of N. In this way, we may refine N until $C \cong R(D_1)^{a^{R(D_1)}(C)} \oplus \ldots \oplus R(D_t)^{a^{R(D_t)}(C)} \oplus N$.

We claim that $R(D_{i+1})$ is a summand of N, and by Lemma 5.1, it suffices to show that $R(D_{i+1})$ is not a summand of $R(D_1)^{n_1} \oplus ... \oplus R(D_i)^{n_i}$. Suppose $R(D_{i+1})$ is indeed a summand of $R(D_1)^{n_1} \oplus ... \oplus R(D_i)^{n_i}$, for the sake of contradiction. By applying $\mathbf{Hom}(-,R(D_{i+1}))$ and using Proposition 4.11 we get that $R(D_{i+1}-D_1)^{n_1} \oplus ... \oplus R(D_{i+1}-D_i)^{n_i}$ has an R summand, hence there must exist a surjective R-linear map $R(D_{i+1}-D_1)^{n_1} \oplus ... \oplus R(D_{i+1}-D_i)^{n_i} \to R$. By the locality of R, there must exist some j such that the image of $R(D_{i+1}-D_j)$ under this map contains a unit. This means $R(D_{i+1}-D_j)$ has free rank 1, and since every divisorial ideal is also rank 1, $R(D_{i+1}-D_j) \cong R$ as R-modules. Thus, D_{i+1} and D_j are linearly equivalent divisors, a contradiction to our assumption that they represent different equivalence classes in the divisor class group.

Now suppose for some $1 \leq j \leq i$ that $n_j < a^{R(D_j)}(C)$, and remember that we have the following direct sum decomposition of C:

$$C \cong R(D_1)^{n_1} \oplus \ldots \oplus R(D_i)^{n_i} \oplus N.$$

We claim that $R(D_j)$ must be a summand of N. We proceed by contradiction, as before, and assume instead that $R(D_1)^{n_1} \oplus \ldots \oplus R(D_i)^{n_i}$ has $n_j + 1$ $R(D_j)$ summands (the n_j summands already present in addition to one extra). Applying $\mathrm{Hom}(-, R(D_j)$ to C and using Proposition 4.11 show that

$$R(D_j - D_1)^{n_1} \oplus \ldots \oplus R^{\oplus n_j} \oplus \ldots \oplus R(D_j - D_i)^{n_i}$$

must have $n_j + 1$ R-summands. There must then exist an R-linear map

$$R(D_j - D_1)^{n_1} \oplus \ldots \oplus R^{n_j} \oplus \ldots \oplus R(D_j - D_i)^{n_i} \to R^{n_j+1}.$$

Taking the homomorphic image modulo $R^{\oplus n_j}$ induces a map

$$\bigoplus_{1 \leq k \leq i, k \neq j} R(D_j - D_k) \to R,$$

and by locality there must be some k such that the image of $R(D_j - D_k)$ contains a unit. As before, by rank considerations, this contradicts the assumption that D_j and D_k are represent distinct classes in the divisor class group, which concludes the proof. $\square$

It is known that the divisorial ideals of torsion divisors in strongly F-regular rings are maximal Cohen-Macaulay modules due to [PS14] and [DS16], but we present a novel proof here. If M is a finitely generated module over a local ring $(R, \mathfrak{m}, k)$ of prime characteristic p and $e \in \mathbb{N}$, then we let

$$I_e(M) = \left\{ \eta \in M \;\middle|\; \varphi(F^e_* \eta) \in \mathfrak{m}, \forall \varphi \in \operatorname{Hom}_R(F^e_* M, R) \right\}.$$

Lemma 5.3. *Let $(R, \mathfrak{m}, k)$ be an F-finite strongly F-regular ring and M_i a finitely generated torsion free R-module for $1 \leq i \leq n$. Then there exists an $e_0 \in \mathbb{N}$ such that $F^e_* M_i$ has a free summand for all $1 \leq i \leq n$ and $e > e_0$.*

Proof. Observe that $F^e_* M$ has a free summand exactly when there is some $\varphi \in \operatorname{Hom}_R(F^e_* M, R)$ such that $\varphi(m) = 1$. To see this, suppose we have such a $\varphi(m) = 1$. If this is the case, then the map $\alpha : R \to M$ defined $\alpha(1) = m$ is a morphism such that $\varphi \circ \alpha = \mathrm{id}_R$, so the exact sequence

$$0 \to \ker \varphi \to M \to R \to 0$$

splits and $M \cong \ker \varphi \oplus R$.

Assume that M is a torsion free R-module. Lemma 2.3 (4) in [Pol20] gives us that

$$\bigcap_{e \in \mathbb{N}} I_e(M) = 0. \tag{7}$$

Thus, for every $0 \neq \eta \in M$, there is some $e(\eta) \in \mathbb{N}$ such that $\eta \notin I_{e(\eta)}(M)$ and therefore some $\varphi \in \operatorname{Hom}_R(F^{e(\eta)}_* M, R)$ such that $\varphi(\eta) \notin \mathfrak{m}$. Without loss of generality we take $\varphi(\eta) = 1$.

Now suppose $M_1, ..., M_n$ are torsion free R-modules. For each M_i, choose $0 \neq \eta_i \in M_i$ and let $e(\eta_i)$ be a natural number depending on η_i such that $\eta_i \notin I_{e(\eta_i)}(M_i)$. Set

$$e_0 = \max\{e(\eta_1), ..., e(\eta_n)\}.$$

By part (3) of Lemma 2.3 in [Pol20], $I_{e_0}(M_i) \subseteq I_{e(\eta_i)}(M_i)$ since $e_0 \geq e(\eta_i)$. Thus, for each $1 \leq i \leq n$ we may find a $\varphi_i \in \operatorname{Hom}_R(F^{e_0}_* M_i, R)$ such that $\varphi_i(\eta_i) = 1$, and conclude that $F^{e_0}_* M_i$ has a free summand for each $1 \leq i \leq n$. $\square$

Proposition 5.4. *Let $(R,\mathfrak{m},k)$ be an F-finite strongly F-regular ring. If D is a torsion divisor, then $R(D)$ is a maximal Cohen-Macaulay module.*

Proof. Since $R(D) \subseteq K$, $R(D)$ is torsion free. Furthermore, since D is a torsion divisor, up to linear equivalence $nD = 0$ for some $0 \neq n \in \mathbb{Z}$ and the list $\{nD\}_{n\in\mathbb{Z}}$ is finite. By Lemma (5.3) there is some $e \in \mathbb{N}$ such that for all $n \in \mathbb{Z}$, $F^e_* R(nD)$ has a free summand. This means we may write $F^e_* R(-p^e D) = R \oplus M$ for some module M. Taking the tensor product with $R(D)$ and applying the reflexification functor yields

$$F^e_* R \cong R(D) \oplus \operatorname{Hom}_R(\operatorname{Hom}_R(M \otimes_R R(D), R), R)$$

by Proposition 4.11. Thus, $R(D)$ is a summand of the maximal Cohen-Macaulay R-module $(F^e_* R)$ so we conclude that $R(D)$ is a maximal Cohen-Macaulay R-module. □

For the final result in this section, we discuss the F-signature of a finitely generated R-module (defined in Proposition 5.7). This invariant was first introduced by Yongwei Yao in [Yao06], and was proven to exist for arbitrary finitely generated modules over F-finite local domains by Tucker in [Tuc12]. We refrain from referencing Hilbert-Kunz multiplicity in our proof, which is the key distinction between our proof and Tucker's. We first state a result from [PT18], which we use to prove Lemma 5.6, before concluding the section with the proof of Proposition 5.7.

Lemma 5.5. *Suppose M, N, and P are finitely generated modules over a local ring R, and that*

$$M \longrightarrow N \longrightarrow P \longrightarrow 0$$

is exact. Then $\operatorname{frk}_R(N) \leq \operatorname{frk}_R(M) + \mu(P)$, where $\mu(P)$ is the minimal cardinality of a generating set for P.

Proof. [PT18, Lemma 2.1]. □

Lemma 5.6. *Suppose M is a finitely generated R-module where R is a domain, and let $N \subseteq M$ be a maximal rank free submodule of M (i.e.) $N \cong R^{\oplus \operatorname{rank}_R(M)}$). Then there exist exact sequences*

$$0 \longrightarrow N \overset{\Phi}{\hookrightarrow} M \longrightarrow C_1 \longrightarrow 0 \tag{8}$$

$$M \xrightarrow{\Phi} N \longrightarrow C_2 \longrightarrow 0 \tag{9}$$

where C_1 and C_2 are torsion modules.

Proof. Fix a linearly independent generating set $\{b_1,...,b_\ell\}$ for N in M. Letting C_1 be the cokernel of the inclusion $N \hookrightarrow M$ makes it clear that (8) is exact. Suppose there existed a non-torsion element $x \in C_1 = M/N$, then for all $r \in R$, $rx \notin N$. However, lifting x to M, we see that $rx \neq r_1 b_1 + ... + r_\ell b_\ell$ for any $r_1,...,r_\ell \in R$ and $\{x, b_1,...,b_\ell\}$ is a linearly independent set, which contradicts the maximal rank of N. Therefore C_1 is a torsion module.

Localizing at the zero ideal gives us the exact sequence

$$0 \longrightarrow N_{(0)} \overset{\Phi}{\hookrightarrow} M_{(0)} \longrightarrow (C_1)_{(0)} \longrightarrow 0 \ ,$$

since C_1 is a torsion module, $(C_1)_{(0)}$ is zero and $N_{(0)} \cong M_{(0)}$. Now consider the diagram

$$\begin{tikzcd} M \arrow[r, dotted, "\Phi"] \arrow[d] & N \arrow[d] \\ M_{(0)} \arrow[r, "\phi"] & N_{(0)} \end{tikzcd}$$

where the vertical maps are given by $x \mapsto \frac{x}{1}$ and ϕ is an isomorphism. Fix a generating set $\{m_1,...,m_\ell\}$ for M, and let $x_1,...,x_\ell \in N$ be elements such that $\phi\left(\frac{m_i}{1}\right) = \frac{x_i}{1}$. Defining Φ by $m_i \mapsto x_i$ and extending by linearity makes it clear this diagram commutes. Since

ϕ is precisely Φ localized at the zero ideal, $C_2 = \operatorname{coker}(\Phi)$ must be a torsion module, giving us (9). $\square$

Proposition 5.7. *Let $(R, \mathfrak{m}, k)$ be a local F-finite domain with $\dim(R) = d$ and M a finitely generated R-module. Then the limit*

$$s(M) = \lim_{e \to \infty} \frac{\operatorname{frk}_R(F_*^e M)}{\operatorname{rank}_R(F_*^e R)} \tag{10}$$

exists, and $s(M) = s(R) \cdot \operatorname{rank}_R(M)$, where $s(R)$ is the F-signature of R. We call $s(M)$ the F-signature of M.

Proof. Set $\alpha(R) = [k : k^p]$, so that $\operatorname{rank}_R(F_*^e R) = p^{e(d+\alpha(R))}$ [Kun76, Proposition 2.3]. Let $N \subseteq M$ be a free-submodule of M of maximal rank, i.e. $N \cong R^{\oplus \operatorname{rank}_R(M)}$. We have the exact sequence

$$0 \longrightarrow N \xhookrightarrow{\Phi} M \longrightarrow C \longrightarrow 0$$

as in Lemma 5.6. $F_*^e(-)$ is exact, so

$$0 \longrightarrow F_*^e N \xhookrightarrow{\Phi} F_*^e M \longrightarrow F_*^e C \longrightarrow 0$$

is exact as well. Applying Lemma 5.5 we see that

$$\operatorname{frk}_R(F_*^e M) \leq \operatorname{frk}_R(F_*^e N) + \mu(F_*^e C).$$

Since C is a torsion module, $\dim(C) < d$, and there exists a real number $c \in \mathbb{R}$ such that $\mu(F_*^e C) \leq cp^{e \dim(C)} \leq cp^{e(d-1)}$ [PT18, Lemma 3.1 and (2)]. Recalling that $N \cong R^{\oplus \operatorname{rank}_R(M)}$, notice

$$\operatorname{frk}_R(F_*^e N) = \operatorname{frk}_R(F_*^e R^{\oplus \dim(M)}) = \operatorname{frk}_R(F_*^e R) \cdot \operatorname{rank}_R(M).$$

Dividing both sides of the above inequality by $p^{e(d+\alpha(R))}$ and taking the limit as $e \to \infty$, we have

(11)
$$s(M) = \lim_{e\to\infty} \frac{\mathrm{frk}_R(F_*^e M)}{p^{e(d+\alpha(R))}} \le \lim_{e\to\infty} \left(\frac{\mathrm{frk}_R(F_*^e R)\cdot \mathrm{rank}_R(M)}{p^{e(d+\alpha(R))}} + \frac{cp^{e(d-1)}}{p^{e(d+\alpha(R))}} \right) = s(R)\cdot \mathrm{rank}_R(M).$$

Using the second exact sequence in Lemma 5.6 and following the same proof, we obtain

$$s(M) \ge s(R)\cdot \mathrm{rank}_R(M), \tag{12}$$

and conclude $s(M) = s(R)\cdot \mathrm{rank}_R(M)$. □

6. Main Result

Throughout this section, $(R,\mathfrak{m},k)$ is a local F-finite strongly F-regular ring. If M is a finitely generated R-module and $e \in \mathbb{N}$ then we define

$$I_e(M) = \left\{ \eta \in M \;\middle|\; \varphi(F_*^e \eta) \in \mathfrak{m}, \forall \varphi \in \mathrm{Hom}_R(F_*^e M, R) \right\}.$$

If $e \ge e'$, then $I_e(M) \subseteq I_{e'}(M)$ [Pol20, Lemma 2.3], a fact we use in the following proof:

Lemma 6.1. *Let D be any torsion divisor. There exists an e_0 such that if $e \ge e_0$, then $a_e^{R(D)}(R) \ge 1$.*

Proof. Since every torsion divisor is maximal Cohen Macaulay by Lemma 5.4, we may apply Theorem 3.1 from [Pol20] to find an e_0 such that there is a morphism $\varphi \in \mathrm{Hom}_R(F_*^e R(-p^e D), R)$ and an element $\eta \in M$ such that $\varphi(F_*^{e_0}\eta) = 1$. Suppose $e \ge e_0$. Because $I_e(R(-p^e D)) \subseteq I_{e_0}(-p^e D)$, $\eta \notin I_{e_0}(R(-p^e D))$ implies $\eta \notin I_e(R(-p^e D))$. There must then exist some map $\psi \in \mathrm{Hom}_R(F_*^e R(-p^e D), R)$ such that

$$\psi(F_*^e \eta) = 1.$$

The assignment $1 \mapsto F^e_* \eta$ defines a map $\sigma : R \to F^e_* R(-p^e D)$ such that $\psi \circ \sigma = \mathbf{1}_R$. Therefore the exact sequence

$$0 \to M \to F^e_* R(-p^e D) \xrightarrow{\psi} R \to 0$$

splits giving us $F^e_* R(-p^e D) \cong R \oplus M$. Taking the tensor product with $R(D)$ and applying the reflexification functor yields

$$F^e_* R \cong R(D) \oplus \mathrm{Hom}_R(\mathrm{Hom}_R(M \otimes_R R(D), R), R).$$

D and e were chosen arbitrarily, and we conclude that $a_e^{R(D)}(R) \geq 1$ for every torsion divisor D and $e \geq e_0$. $\square$

Note that Lemma 6.1 can be seen to follow immediately from the proof of Proposition 5.4.

Lemma 6.2. *Let D be a torsion divisor. Then*

$$\lim_{e \to \infty} \frac{a_e^{R(D)}(R)}{\mathrm{rank}\, F^e_* R} = s(R),$$

where $s(R)$ is the F-signature of R.

Proof. This proof consists of two parts. We first show that $\mathrm{frk}_R F^e_* R(-p^e D) = a_e^{R(D)}(R)$, and then we calculate the limit.

First $e \in \mathbb{N}$ and let $n = a_e^{R(D)}(R)$. We have $F^e_* R \cong R(D)^n \oplus M$, where M is a finitely generated R-module without an $R(D)$ summand. By Proposition 4.11, applying $- \otimes_R R(-D)$ and then $\mathrm{Hom}_R(\mathrm{Hom}_R(-, R), R)$ to this isomorphism we obtain

$$F^e_* R(-p^e D) \cong R^n \oplus N \tag{13}$$

where $N = \mathrm{Hom}_R(\mathrm{Hom}_R(M \otimes_R R(-D), R), R)$. We claim $n = \mathrm{frk}\, F^e_* R(-p^e D)$. Suppose for the sake of contradiction that N had a free summand, i.e. that $N \cong R \oplus P$ for some

R-module P. We take the tensor product of equation 13 with $R(D)$ and apply to the reflexification functor to obtain

$$F_*^e R \cong R(D)^n \oplus R(D) \oplus \operatorname{Hom}_R(\operatorname{Hom}_R(P \otimes_R R(D), R), R).$$

This means $R(D)^{n+1}$ appears as a summand in a direct sum decomposition of $F_*^e R$, which contradicts the maximality of n. Thus, $\operatorname{frk}_R F_*^e R(-p^e D) = a_e^{R(D)}(R)$.

For the second part of the proof, we first establish notation. Polstra proved that the torsion subgroup $\mathrm{T}(\mathrm{Cl}(R))$ of the divisor class group of a strongly F-regular ring is finite [Pol20], so we may enumerate them: $\mathrm{T}(\mathrm{Cl}(R)) = \{D_1, \ldots, D_k\}$. We denote the "$e$th" term in the sequence defining the F-signature of $R(D_i)$ as follows:

$$s_e(R(D_i)) = \frac{\operatorname{frk} F_*^e R(D_i)}{\operatorname{rank} F_*^e R}.$$

Since each divisorial ideal is a finitely generated rank 1 module, Tucker tells us

$$\lim_{e \to \infty} s_e(R(D_i)) = s(R(D_i)) = s(R) \cdot \operatorname{rank} R(D_i) = s(R)$$

for each $1 \leq i \leq k$ [Tuc12, Theorem 4.11]. In particular, $s_e(R(D_i))$ and $s_e(R(D_j))$ are equivalent Cauchy sequences for each $1 \leq i, j \leq k$. Now set

$$b_e = \frac{a_e^{R(D)}(R)}{\operatorname{rank} F_*^e R}$$

for sake of clarity. We show that the sequence $\{b_e\}$ is equivalent to $\{s_e(R(D_1))\}$ as a Cauchy sequence and conclude that $\lim b_e = s(R)$.

Fix $\varepsilon > 0$. By the equivalence of Cauchy sequences, for each $1 \leq i \leq k$, we may find $N_i \in \mathbb{N}$ such that for all $e \geq N_i$, $|s_e(R(D_i)) - s_e(R(D_j))| < \varepsilon$. Notice that since $a_e^{R(D)}(R) = \operatorname{frk} F_*^e R(-p^e D)$ and $-p^e D$ is a torsion divisor, b_e is equal to $s_e(R(D_i))$ for some $1 \leq i \leq k$. If we let $N = \max\{N_1, ..., N_k\}$, then for all $e \geq N$, we have

$$|s_e(R(D_1)) - b_e| \leq \max\left\{|s_e(R(D_1)) - s_e(R(D_i))| \ : \ 1 \leq i \leq k\right\} < \varepsilon.$$

Thus, $\{s_e(R(D_1)) - b_e\}$ is equivalent to the 0 sequence, so $\{b_e\}$ is equivalent to $\{s_e(R(D_1))\}$ as a Cauchy sequence. We conclude that $\lim_{e\to\infty} b_e = s(R)$. $\square$

Theorem 6.3. *Let $(R, \mathfrak{m}, k)$ be a local F-finite and strongly F-regular ring of prime characteristic $p > 0$. Then*

$$|\mathrm{T}(\mathrm{Cl}(R))| \leq 1/s(R),$$

where $\mathrm{T}(\mathrm{Cl}(R))$ is the torsion subgroup of the divisor class group of R.

Proof. Set

$$n_e = \sum_{D \in \mathrm{T}(\mathrm{Cl}(R))} a_e^{R(D)} x(R).$$

Fix e_0 as in Lemma 6.1, and let $e \geq e_0$. For each torsion divisor D, $R(D)$ is a summand of $F_*^e R$, so by Lemma 5.2 and the fact that $R(D)$ is rank 1 for any torsion divisor, we have that

$$n_e = \sum_{D \in \mathrm{T}(\mathrm{Cl}(R))} a_e^{R(D)}(R) \cdot \operatorname{rank} R(D) \leq \operatorname{rank} F_*^e R.$$

By Lemma 6.2,

$$\begin{aligned}
\lim_{e\to\infty} \frac{n_e}{\operatorname{rank} F_*^e R} &= \lim_{e\to\infty} \sum_{D \in \mathrm{T}(\mathrm{Cl}(R))} \frac{a_e^{R(D)}(R)}{\operatorname{rank} F_*^e R} \\
&= \sum_{D \in \mathrm{T}(\mathrm{Cl}(R))} \lim_{e\to\infty} \frac{a_e^{R(D)}(R)}{\operatorname{rank} F_*^e R} \\
&= \sum_{D \in \mathrm{T}(\mathrm{Cl}(R))} s(R) \\
&= |\mathrm{T}(\mathrm{Cl}(R))| \cdot s(R).
\end{aligned}$$

The limit commutes with the sum since $|\operatorname{T}(\operatorname{Cl}(R))| < \infty$ by Corollary 3.3 in [Pol20]. Because $n_e \leq \operatorname{rank} F_*^e R$,

$$|\operatorname{T}(\operatorname{Cl}(R))| \cdot s(R) = \lim_{e \to \infty} \frac{n_e}{\operatorname{rank} F_*^e R} \leq 1,$$

and we conclude

$$|\operatorname{T}(\operatorname{Cl}(R))| \leq \frac{1}{s(R)}.$$

□

We immediately obtain the following corollary to Theorem 6.3. This is not surprising, since results regarding local rings transfer easily to graded rings via localization at the homogeneous maximal ideal.

Corollary 6.4. *Let R be a $\mathbb{N}$-graded F-finite and strongly F-regular ring of prime characteristic $p > 0$ such that R_0 is a field. Then*

$$|\operatorname{T}(\operatorname{Cl}(R))| \leq \frac{1}{s(R)}.$$

Proof. Let $\mathfrak{m}$ denote the unique homogeneous maximal ideal of R. Strong F-regularity is a local property, so the localization $R_\mathfrak{m}$ is strongly F-regular and therefore $|\operatorname{T}(\operatorname{Cl}(R_\mathfrak{m}))| \leq \frac{1}{s(R_\mathfrak{m})}$ by Theorem 6.3.

We know $\operatorname{Cl}(R) \to \operatorname{Cl}(R_\mathfrak{m})$ is a bijection by [Fos73, Corollary 10.3] and that $s(R) = s(R_\mathfrak{m})$ by [SPY18, Corollary 6.19], so we have the desired result. □

6.1. **Examples.** Here we provide two examples of local strongly F-regular rings R of prime characteristic $p > 0$ to illustrate cases for which the inequality in 6.3 can be strengthened to either an equality or a strict inequality.

Example 6.1. Suppose $p > 0$ is prime and $R = \frac{\mathbb{F}_p[w,x,y,z]}{(wx - yz)}$. This is a determinantal ring with $r = s = 2$, in the notation of Singh [Sin05, Example 3.1], and therefore has dimension $d = r + s - 1 = 3$. By Singh's example, we have that

$$s(R) = \frac{1}{d!}\sum_{i=0}^{s}(-1)^i \binom{d+1}{i}(s-i)^d = \frac{1}{3!}\sum_{i=0}^{2}(-1)^i \binom{4}{i}(2-i)^3 = \frac{2}{3}.$$

Since R is a determinant ring satisfying the hypotheses of [BH93, p. 7.3.5], we have that $\mathrm{Cl}(R) = \mathbb{Z}$ and in particular $|\mathrm{T}(\mathrm{Cl}(R))| = 1$. The ring R therefore satisfies the conclusion of Theorem 6.3 but is clearly not local, so to match the hypotheses of the Theorem we let $\mathfrak{m} = (w,x,y,z)$ and consider $R_{\mathfrak{m}}$. By [Fos73, Corollary 10.3] we immediately see $\mathrm{Cl}(R) = \mathrm{Cl}(R_{\mathfrak{m}})$ and by [SPY18, Corollary 6.19] $s(R_{\mathfrak{m}}) = s(R)$, so

$$|\mathrm{T}(\mathrm{Cl}(R_{\mathfrak{m}}))| < \frac{1}{s(R_{\mathfrak{m}})}.$$

Example 6.2. Suppose $p > 0$ is prime and $n \geq 2$ and set $R = \frac{\mathbb{F}_p[x,y,z]}{xy-z^n}$. The class group of R is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ by [SS07, Corollary 3.4], so it remains to find $s(R)$. Notice that we have the isomorphism $\frac{\mathbb{F}_p[x,y,z]}{xy-z^n} \cong \mathbb{F}_p[x^n, xy, y^n]$. The latter ring lends itself well to the calculation of F-signature as it is an affine semigroup ring, so we redefine $R = \mathbb{F}_p[x^n, xy, y^n]$, set $A = \mathbb{F}_p[x,y]$ and note that $R \subseteq A$.

Fix $e \in \mathbb{N}$ and set $q = p^e$ and let $\mathfrak{m} \subsetneq A$ denote the homogeneous maximal ideal. By [Sin05, Lemma 4],

$$a_e(R) = \ell\left(\frac{R}{\mathfrak{m}^{[q]} \cap R}\right).$$

Let S_e denote the ring $\frac{R}{\mathfrak{m}^{[p^e]} \cap R}$. We can form a maximal chain of submodules of S_e entirely from ideals generated by monomials. To see this, let T denote the collection of distinct monomials in S_e and let

$$(0) = I_0 \subsetneq I_1 \subsetneq \ldots \subsetneq I_n = S_e$$

be a maximal chain of ideals in S_e whose generators are in T. Suppose $0 \leq i \leq n$, and choose elements $f_1, \ldots, f_m \in T$ so that $(f_1, \ldots, f_m) = I_i$. If I_{i+1} contained two monomials not in I_i, then the above chain would not be maximal, so we can find a monomial $f_m \in T$ so that $(f_1, ..., f_m, f_{m+1}) = I_{i+1}$.

Now suppose we have a nonzero coset $\overline{g} \in I_{i+1}/I_i$. The representative g must be a nonzero element in $I_{i+1} \setminus I_i$, and therefore $g = a_1 f_1 + ... + a_{m+1} f_{m+1}$ with $a_{m+1} \neq 0$. This means the set $\{f_1, ..., f_m, g\}$ generates I_{i+1} as an ideal, and so $\langle \overline{g} \rangle = I_{i+1}/I_i$. Since any nonzero element of I_{i+1}/I_i generates the entire group, I_{i+1}/I_i is simple. This means the above maximal sequence is a composition series, and it therefore suffices to count the number of distinct monomials in S_e to determine $\ell(S_e)$.

The nonzero monomials $x^a y^b$ in S_e are precisely those monomials in R which are not killed by $\mathfrak{m}^{[q]}$. A monomial $x^a y^b \in R$ must satisfy $x^a y^b = x^{ni}(xy)^j y^{nk} = x^{ni+j} y^{nk+j}$ for some positive integers i, j and k, which implies that $a \equiv b \mod (n)$. If $x^a y^b$ is nonzero in S_e then it is not contained in $\mathfrak{m}^{[q]} = F^e((x,y))A = (x^q, y^q)$ and hence $a < q$ and $b < q$. Likewise, it can be easily seen that any monomial $x^a y^b$ in A for which $a < q$, $b < q$, and $a \equiv b \mod (n)$ is a monomial in S_e, hence there is a bijection between the set of distinct monomials in S_e and pairs of nonnegative integers (a, b) satisfying these conditions.

Suppose for a moment that $q = mn$ for some $m \in \mathbb{N}$, and fix a so that $0 \leq a \leq q-1$. The integers congruent to a modulo n are of the form $ni + a$ for some $i \in \mathbb{N}$, and there are exactly m such distinct integers b such that $0 \leq b \leq q-1$. As there are mn choices for a and m choices for b given a, there are exactly $m^2 n$ pairs of integers (a, b) such that $a < q$, $b < q$, and $a \equiv b \mod (n)$. Therefore $a_e(R) = m^2 n$.

Now suppose q is once again arbitrary and pick m_q to be the maximal integer such that $m_q n \leq q$. By the special case addressed above we know $m_q^2 n \leq a_e(R) \leq (m_q + 1)^2 n$. The ring R has Krull dimension 2, therefore $\operatorname{rank} F_*^e R = p^{ed} = q^2$. We have the equality

$$q^2 - (m_q n)^2 = 2q(q - m_q n) - (q - m_q n)^2$$

from which we obtain

$$q^2 - (m_q n)^2 \leq 2qn - (q - m_q n)^2 \leq 2qn.$$

Using this inequality we see

$$\frac{1}{n} - \frac{m_q^2 n}{q^2} \leq \frac{2}{q}$$

and

$$\frac{(m_q+1)^2 n}{q^2} - \frac{1}{n} \leq \frac{2qn + n^2 + 2q}{q^2}$$

both of which approach 0 as $q \to \infty$. We therefore have that

$$\frac{1}{n} = \lim_{q\to\infty} \frac{(m_q+1)^2 n}{q^2} \leq \lim_{q\to\infty} \frac{a_e(R)}{q^2} \leq \lim_{q\to\infty} \frac{(m_q+1)^2 n}{q^2} = \frac{1}{n},$$

and hence $\lim_{e\to\infty} \frac{a_e(R)}{p^{2e}} = \frac{1}{n}$.

We conclude that $s(R) = 1/n$ and $|\mathrm{T}(\mathrm{Cl}(R))| = n$, and in particular, that

$$|\mathrm{T}(\mathrm{Cl}(R))| = 1/s(R).$$

7. Globalization and Future Work

The reader may have noticed that the definition of F-signature

$$s(R) = \lim_{e\to\infty} \frac{a_e(R)}{\mathrm{rank}(F_*^e R)}$$

did not require R to be a local ring; in fact, we needed only that R was F-finite in order to avoid $a_e(R)$ and $\mathrm{rank}(F_*^e R)$ from evaluating to infinity. This observation begs the question: can the results involving F-signature be globalized? We used the local hypothesis in several key places, most notably in our proof of a case of Kunz's theorem. Somewhat surprisingly, the answer is yes. We direct the reader to the following theorem of De Stefani, Polstra, and Yao, which in addition to providing new statements regarding F-signature of non-local rings globalizes both the existence of F-signature and the characterization theorems involving F-signature.

Theorem 7.1. *Let R and T be a F-finite domain, not necessarily local. Then the following are true:*

(1) The limit

$$s(R) = \lim_{e\to\infty} \frac{a_e(R)}{\operatorname{rank}(F_*^e R)}$$

exists (globalization of [Tuc12, Theorem 4.9]).

(2) We have $s(R) = \min\{s(R_{\mathfrak{p}}) \mid \mathfrak{p} \in \operatorname{Spec}(R)\}$.

(3) The ring R is regular if and only if $s(R) = 1$ (globalization of [HL02, Corollary 16]).

(4) The ring R is strongly F-regular if and only if $s(R) > 0$ (globalization of [AL03, Theorem 1.7]).

(5) If $R \to T$ is faithfully flat then $s(R) \geq s(T)$.

Theorem 6.3 holds true if the local hypothesis is replaced with the graded hypothesis, as seen in corollary 6.4, but it remains unclear whether the result holds true for arbitrary F-finite rings.

Strong F-regularity is a local property, and hence theorem 6.3 holds for $R_{\mathfrak{p}}$ where R is an arbitrary F-finite strongly F-regular ring and $\mathfrak{p} \in \operatorname{Spec}(R)$. By theorem 7.1 part (2), we have $s(R) < s(R_{\mathfrak{p}})$ and hence

$$|T(\operatorname{Cl}(R_{\mathfrak{p}}))| \leq \frac{1}{s(R_{\mathfrak{p}})} \leq \frac{1}{s(R)}.$$

However, it is *not* the case that $|T(\operatorname{Cl}(R))| \leq |T(\operatorname{Cl}(R_{\mathfrak{p}}))|$ in general. In fact, every Weil divisor of $R_{\mathfrak{p}}$ is a Weil divisor of R because $\operatorname{Spec}(R_{\mathfrak{p}}) \hookrightarrow \operatorname{Spec}(R)$, so the best we can hope for is $|T(\operatorname{Cl}(R))| = |T(\operatorname{Cl}(R_{\mathfrak{p}}))|$. The naive approach to globalizing 6.3 will therefore not work, and hence more work is required.

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