ALGEBRAIC DE RHAM COHOMOLOGY AND THE HODGE SPECTRAL SEQUENCE

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ABSTRACT. This is an expository write up and modernization of a classical result in Hodge theory, developed over the duration of the 2020 REU program at the University of Chicago. We provide the necessary background to understand the degeneration of the Hodge spectral sequence for a smooth and proper scheme X over a field k. The case in which k is of prime characteristic $p > 0$ is the primary focus of these notes, but we briefly illustrate how these results may be extended to the char $= 0$ case.

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1. INTRODUCTION

Taking inspiration from differential geometry, we would like to develop a cohomology theory for algebraic varieties with the intent to capture topological data. We seek to adapt the classic de Rham isomorphism

$$H^i_{sing}(M, \mathbb{R}) \cong H^i_{dR}(M)$$

which allows us to access the topological (left side) via the analytic (right side) when M is a real smooth manifold. Serre's GAGA principle says that the analytic and the algebraic

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are closely related, so it is natural to adapt the above isomorphism to fit our needs:

$$\text{Topological Data} \xleftrightarrow{\textit{de Rham}} \text{Analytic Data} \xleftrightarrow{\textit{GAGA}} \text{Algebraic Data}$$

However, for a complex manifold X where Ω^i_{hol} denotes the ith degree holomorphic forms, the isomorphism

$$H^i_{sing}(X,\mathbb{C}) \cong h^i(\Omega^\bullet_{hol})$$

cannot hope to hold. One reason for this is simply that the complex $\Omega^\bullet_{hol}$ is too short. For $i \geq \dim_{\mathbb{C}} X$, we see that $H^i(X,\Omega^i_{X/\mathbb{C}}) = 0$, but $H^i_{sing}(X,\mathbb{C}) = 0$ only for $i > \dim_{\mathbb{R}} X = 2 \cdot \dim_{\mathbb{C}} X$. We therefore cannot hope to adapt de Rham's theorem for real smooth manifolds to varieties – at least not directly.

These notes walk the reader through the necessary facts in "fixing" this error. This paper does not comprise original research – rather, it serves as an expository overview of the theory surrounding Hodge and algebraic de Rham cohomology, and focuses on proving the following result:

Theorem (Illusie, Hodge Degeneration Theorem)**.** *Let k be a field, and X a smooth and proper k-scheme. Then*

$$H^\ell_{Hodge}(X/k) \cong H^\ell_{dR}(X/k).$$

2. Background, Motivation, and Definitions

This section is intended as a "catch all" for material that does not neatly fit into other sections. The first subsection is intended to provide accessibility for less experienced readers, and includes the rudimentary definitions encountered in a first course in scheme theory. The other two subsections motivate the definition of algebraic de Rham cohomology and and list fundamental results respectively.

2.1. **Scheme theory fundamentals.** Before we may discuss schemes, we need to introduce sheaves, and before we may discuss sheaves, we need to discuss *presheaves*. Here we list two equivalent definition for presheaves.

Definition 2.1. Let X be a topological space. A presheaf $\mathcal{F}$ of Abelian groups on X consists of the data

(a) for every open subset $U \subseteq X$, an Abelian group $\mathcal{F}(U)$
(b) for every inclusion $V \hookrightarrow U$ of open subsets of X, a morphism of Abelian groups $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ subject to the conditions
  (0) $\mathcal{F}(\emptyset) = 0$
  (1) ρ_{UU} is the identity map $\mathcal{F}(U) \to \mathcal{F}(U)$
  (2) if $W \subseteq V \subseteq U$ are there open subsets, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$

Definition 2.2. Let X be a topological space and define a category $\mathcal{T}\text{op}(X)$ whose objects are open sets in X and whose morphisms are inclusions. So $\text{Hom}(V,U)$ is empty if $V \nsubseteq U$ and $\text{Hom}(V,U)$ has one element if $V \subseteq U$. A presheaf $\mathcal{F}$ ob objects in a category $\mathcal{C}$ is a contravariant functor from the category $\mathcal{T}\text{op}(X)$ to $\mathcal{C}$.

As a matter of terminology, We refer to $\mathcal{F}(U)$ as the *sections* of the presheaf $\mathcal{F}$ over the open set U.

Note that for $V \subseteq U$, if $\varphi : V \hookrightarrow U$ is the inclusion map, then $\mathcal{F}(\varphi) = \rho_{UV}$ is called the **restriction map** and we write $s|_V$ to denote $\rho_{UV}(s)$ when $s \in \mathcal{F}(U)$.

Definition 2.3. A **sheaf** is a presheaf $\mathcal{F}$ on a topological space X which additionally satisfies

(3) if U is an open set, $\{V_i\}$ is an open covering of U, and $s \in \mathcal{F}(U)$ is an element satisfying $s|_{V_i} = 0$ for all i, then $s = 0$.
(4) if U is an open set, $\{V_i\}$ is an open covering of U, and if for each i we have elements $s_i \in \mathcal{F}(V_i)$ with the property that for each i, j $s_i|V_i \cap V_j = s_j|_{V_i \cap V_j}$, then there is an element $s \in \mathcal{F}(U)$ (necessarily unique) such that $s|_{V_i} = s_i$ for each i.

To see why the s in condition (4) is unique, take two elements s and t that satisfy (4). Then $(s-t)|_{V_i} = 0$ so condition (3) implies $s - t = 0$. In this way, we think of sheaves as presheaves whose sections are defined by local data.

We ought to discuss morphisms on sheaves as well.

Definition 2.4. If $\mathcal{F}$ and $\mathcal{G}$ are presheaves on X of objects in a category $\mathcal{C}$, a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ is a natural transformation $\mathcal{F} \implies \mathcal{G}$. That is, φ consists of a morphism $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ in $\mathcal{C}$ for each open set $U \subseteq X$, such that for any inclusion $V \subseteq U$, the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \Big\downarrow{\scriptstyle \rho_{UV}} & & \Big\downarrow{\scriptstyle \rho'_{UV}} \\ \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \end{array}$$

commutes. If $\mathcal{F}$ and $\mathcal{G}$ are fully-fledged sheaves, then we use the same definition for a morphism of sheaves (we don't require anything extra for a morphism of sheaves). We say that φ is an isomorphism of sheaves if it is a natural isomorphism.

It's worth noting that if $\mathcal{F}$ is a sheaf on X and $f : X \to Y$ is a continuous map of topological spaces, then f induces a sheaf on Y.

Definition 2.5. Let $f : X \to Y$ be a map of topological spaces and $\mathcal{F}$ be a sheaf on X. We define the **direct image** sheaf $f_*\mathcal{F}$ on Y by $(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$ for any open set $V \subseteq Y$.

Definition 2.6. Let A be a ring, $\mathfrak{a} \subseteq A$ an ideal. $V(\mathfrak{a}) \subseteq \operatorname{Spec}(A)$ is the set of all primes in A which contain $\mathfrak{a}$.

Lemma 2.7.
(a) If $\mathfrak{a}, \mathfrak{b}$ are two ideals in A, then $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ (Finite union still closed)
(b) If $\{a_i\}$ is any set of ideals in A, then $V(\sum \mathfrak{a}_i) = \bigcap V(\mathfrak{a}_i)$ (Arbitrary intersection still closed)
(c) If $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals, $V(\mathfrak{a}) \subseteq V(\mathfrak{b}) \iff \sqrt{\mathfrak{a}} \supseteq \sqrt{\mathfrak{b}}$.

Proof. See [Har77, Lemma 2.1]; however, the proof is elementary and the reader is encouraged to write it themselves. □

This means the collection $\{\operatorname{Spec} A \setminus V(\mathfrak{a})\}_{\mathfrak{a} \text{ an ideal in } A}$ is a topology on $\operatorname{Spec} A$. Note that $V(A) = \emptyset$ and $V(\{0\}) = \operatorname{Spec} A$.

Given this, we may define the sheaf of rings $\mathcal{O}$ on $\operatorname{Spec} A$. For an open set $U \subseteq \operatorname{Spec} A$, we define $\mathcal{O}(U)$ to be the set of functions

$$s : U \to \bigsqcup_{\mathfrak{p} \in U} A_{\mathfrak{p}}$$

such that $s(\mathfrak{p}) \in A_{\mathfrak{p}}$ for each $\mathfrak{p} \in U \subseteq \operatorname{Spec} A$, that is, s is locally a quotient of elements of A. More precisely, this means

$\forall \mathfrak{p} \in U$, there is a neighborhood $V \subseteq U$ of $\mathfrak{p}$ and elements $a, f \in A$ such that for each $q \in V$, we have $f \notin q$ and $s(\mathfrak{q}) = \frac{a}{f} \in A_{\mathfrak{q}}$.

Pointwise sums and products of these functions yield back functions of this same form. The functions s_0 and s_1 in $\mathcal{O}(U)$ which send everything to 0 and 1 respectively are additive and multiplicative identities respectively. This means $\mathcal{O}(U)$ is indeed a ring with identity, and is commutative exactly when A is commutative. If $V \subseteq U$ is an open set in $\operatorname{Spec} A$, then the natural restriction map $\mathcal{O}(U) \to \mathcal{O}(V)$ (which sends a function $s \in \mathcal{O}(U)$ to $s|_V$) is a ring homomorphism. It is clear $\mathcal{O}$ is a presheaf we send $V \hookrightarrow U$ to $\mathcal{O}(U) \to \mathcal{O}(V)$. Finally, because of the local nature of the sections $s \in \mathcal{O}(U)$, $\mathcal{O}$ is a full-fledged sheaf. We quickly check that:

(a) Suppose $U \subseteq \operatorname{Spec} A$ is open, $\{V_i\}$ is an open cover of U, and $s \in \mathcal{O}(U)$ satisfies $s|_V = 0$ for all i. Choose an arbitrary element $\mathfrak{p} \in U$. This is contained in V_j for some j, and therefore $s(\mathfrak{p}) = s|_{V_j}(\mathfrak{p}) = 0$. The section s sends everything in U to 0, so it must be the 0 function on U. This proves property (3) of a sheaf.

(b) Suppose $U \subseteq \operatorname{Spec} A$ is open, $\{V_i\}$ is an open cover of U, and for each i we have elements $s_i \in \mathcal{O}(V_i)$ with the property that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$. For each $\mathfrak{p} \in U$, there is some V_i such that $\mathfrak{p} \in V_i$, so we may define a function

$$s : U \to \bigsqcup_{\mathfrak{p} \in U} A_{\mathfrak{p}}$$

by $s(\mathfrak{p}) = s_i(\mathfrak{p})$. If $\mathfrak{p} \in V_i \cap V_j$, then $s_i(\mathfrak{p}) = s_i|_{V_i \cap V_j}(\mathfrak{p}) = s_j|_{V_i \cap V_j}(\mathfrak{p}) = s_j(\mathfrak{p})$, so s is well-defined. Finally, $s|_{V_i} = s_i$ by definition, so we have property (4) of a sheaf.

This leads us to a definition:

Definition 2.8. Let A be a ring. The **spectrum** of A is the pair $(\operatorname{Spec} A, \mathcal{O})$ consisting of the set $\operatorname{Spec} A$ endowed with the Zariski topology and the sheaf of rings $\mathcal{O}$ defined above.

More generally we may talk about a ringed space:

Definition 2.9. A **ringed space** is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X. A **morphism** of ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\sharp)$ of a continuous map $f : X \to Y$ and a map $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings on Y.

Example 2.10. Any arbitrary topological space X can be made a ringed space by defining a sheaf $\mathcal{O}_X$ which takes an open subset $U \subseteq X$ to the set of real-valued continuous functions $f : U \to \mathbb{R}$, or complex-valued continuous functions $f : U \to \mathbb{C}$. That is, if we define

$$\mathcal{O}_X(U) = \{\ f : U \to \mathbb{C} \ \mid\ f \text{ is continuous}\ \},$$

then $(X, \mathcal{O}_X)$ is a ringed space. To see this, remember first that the set $\mathcal{O}_X(U)$ is indeed a commutative ring with $(f+g)(x) = f(x) + g(x)$ and $(f \cdot g)(x) = f(x) \cdot g(x)$ where the constant maps $f_0 : x \mapsto 0$ and $f_1 : x \mapsto 1$ are the additive identity and the multiplicative identity respectively. Next, given an inclusion $V \hookrightarrow U$ we get a morphism of rings $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$ which does the obvious thing: $f \mapsto f|_V$. Thus, $\mathcal{O}_X$ is a presheaf. We apply the same arguments above which show the sheaf of rings $\mathcal{O}$ on $\operatorname{Spec} A$ satisfy the "local data" requirements of a sheaf to show that $\mathcal{O}_X$ is a sheaf.

Definition 2.11. A ringed space $(X, \mathcal{O}_X)$ is a **locally ringed space** if for each point $P \in X$, the stalk $\mathcal{O}_{X,P}$ is a local ring.

Example 2.12.

- Given an arbitrary topological space, the ringed space $(X, \mathcal{O}_X)$ defined in example 2.10 is also a locally ringed space. We claim that for any point $P \in X$, the stalk $\mathcal{O}_{X,P}$ at P is a local ring whose unique maximal ideal $\mathfrak{m}$ is the set of functions which are zero at P.

  Proof. Clearly, $\mathfrak{m}$ is an ideal of $\mathcal{O}_{X,P}$: for $f, g \in \mathfrak{m}$ and $h \in \mathcal{O}_{X,P}$, $f+g$ and $h \cdot f$ are still zero at P. We show that if $f \in \mathcal{O}_{X,P} \setminus \mathfrak{m}$ then f is a unit, and conclude that $\mathfrak{m}$ is the unique maximal ideal of $\mathcal{O}_{X,P}$.

  To see this, first take $U \subseteq X$ with U open and notice that a function $f \in \mathcal{O}_X(U)$ is a unit if and only if f is nonzero on U. Its inverse is the function f^{-1} defined $f^{-1}(x) = 1/f(x)$.

  Next, notice that if f is not zero at P, i.e. if $f \in \mathcal{O}_{X,P} \setminus \mathfrak{m}$, then there is some neighborhood U of P such that f is nonzero on U. The germ of f is therefore invertible, so f is a unit in $\mathcal{O}_{X,P}$. We conclude that $(X, \mathcal{O}_X)$ is a locally ringed space. □
- If X is additionally a manifold (with the correct additional structure), then we may take $\mathcal{O}_X$ to be the sheaf of differentiable or complex-analytic functions. In either case, $(X, \mathcal{O}_X)$ is a locally ringed space.
- If A is a commutative ring, then the spectrum $(\operatorname{Spec} A, \mathcal{O})$ of A is a locally ringed space. For any $\mathfrak{p} \in \operatorname{Spec} A$, the stalk $\mathcal{O}_{\mathfrak{p}}$ is isomorphic to $A_{\mathfrak{p}}$ [Har77, Prop 2.3].

Definition 2.13. An **affine scheme** is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic to the spectrum of a commutative ring. A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ in which every point has an open neighborhood U such that U (as a topological space under the subset topology) together with the restricted sheaf $\mathcal{O}_X \mid_U$ is an affine scheme. That is, a scheme is a locally ringed space which is *locally* realized as the spectra of commutative rings. We call X the *underlying topological space* of the scheme $(X, \mathcal{O}_X)$ and $\mathcal{O}_X$ its *structure sheaf*. We often use X to refer to the entire scheme, and write $\operatorname{sp}(X)$ (read "space" X) to refer to the topological space X devoid of its scheme structure. A **morphism** of schemes is a morphism of locally ringed spaces.

Finally, in order to discuss algebraic de Rham cohomology, we require Kähler differentials. We summarize Matsumura's treatment of the topic found in [MRB86, Chapter 9] and [Mat70, Chapter 10]. Kähler differentials generalize the idea of a derivative of a polynomial, and provide a purely algebraic realization of differentiation.

Definition 2.14. Suppose A is a commutative ring, B is an A-algebra via the ring homomorphism $\varphi : A \to B$, and M is a B-module. An A-derivation on B is an A-module homomorphism $d : B \to M$ which satisfies the Leibniz rule

$$d(fg) = f\, dg + g\, df$$

for all $f, g \in B$. We denote the set of all such functions $\operatorname{Der}_A(B, M)$. In the case that $A = \mathbb{Z}$, we sometimes write $\operatorname{Der}(B, M)$ for $\operatorname{Der}_{\mathbb{Z}}(B, M)$.

The following is sometimes included in the definition of a derivation, but it is equivalent to the assumption that d is an A-module homomorphism:

Proposition 2.15. *Let $d : B \to M$ be an A-derivation, where $\varphi : A \to B$ is the structure map of B as an A-algebra. Then $\varphi(A) \subseteq \ker(d)$.*

Proof. By the Leibniz rule,

$$d(1) = d(1 \cdot 1) = d(1) + d(1),$$

so $d(1) = 0$. Since d is a morphism of A-modules, for $a \in A$, $d(\varphi(a)) = \varphi(a) \cdot d(1) = 0$. □

Lemma 2.16. *Suppose R is a ring of characteristic p. If $d \in \mathrm{Der}(A,M)$, then $d(a^p) = 0$.*

Proof. For any $a \in A$, we have $d(a^n) = na^{n-1}d(a)$. Thus, $d(a^p) = pa^{p-1}d(a) = 0$. □

Definition 2.17. Let A, B, φ, and M be as in definition 2.14. The module of **Kähler differentials** or the **module of relative differential forms** of B over A is the B-module $\Omega_{A/B}$ for which there is a *universal* derivation $d : B \to \Omega_{A/B}$ which satisfies the following universal property:

> for any B-module M and any derivation $d' : B \to M$, there exists a unique B-module homomorphism $\alpha : \Omega_{B/A} \to M$ such that $d' = \alpha \circ d$.

In other words, the composition with d yields an isomorphism

$$\mathrm{Hom}_B(\Omega_{B/A}, M) \to \mathrm{Der}_A(B, M)$$

for every B-module M.

Alternatively, we may define $\Omega_{B/A}$ as follows.

Definition 2.18. Let A, B, φ, and M be as in definition 2.14. Define I to be the kernel of the map

$$\begin{cases} B \otimes_A B \to B \\ \sum s_i \otimes t_i \mapsto \sum s_i \cdot t_i \end{cases}$$

Then the module of Kähler differentials of B can be equivalently defined by

$$\Omega_{B/A} = I/I^2$$

and the universal derivation is the homomorphism d given by

$$ds = 1 \otimes s - s \otimes 1.$$

The module of differentials can be carried over to schemes. Let $f : X \to Y$ be a morphism of schemes, and let $\Delta : X \to X \times_Y X$. The morphism Δ is an isomorphism of X onto $\Delta(X)$, which is a *locally closed* subscheme of $X \times_Y X$, i.e., a subscheme of an open subset W of $X \times_Y X$. [Har77, Corollary 2.4.2].

Definition 2.19. Let $\mathcal{J}$ be the sheaf of ideals of $\Delta(X)$ in W. We define the sheaf of relative differentials of X over Y to be the sheaf $\Omega_{X/Y} = \Delta^*(\mathcal{J}/\mathcal{J}^2)$ on X.

2.2. **Algebraic de Rham Cohomology and Hodge Cohomology.** In this subsection, we define algebraic de Rham Cohomology and Hodge Cohomology.

Definition 2.20. *Differential forms of higher degree* are defined as the exterior powers over $\mathcal{O}_X$:

$$\Omega^n_{X/Y} := \bigwedge^n \Omega_{X/Y}.$$

The universal derivation $d : \mathcal{O}_X \to \Omega_{X/Y}$ extends in a natural way to a sequence of maps

$$0 \to \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/Y} \xrightarrow{d} \Omega^2_{X/Y} \xrightarrow{d} \dots$$

which satisfies $d \circ d = 0$. This forms a cochain complex known as the *algebraic de Rham complex*.

It is tempting to define the algebraic de Rham cohomology to be the cohomology of the algebraic de Rham complex, but as mentioned in the introduction, taking the cohomology of the complex $(\Omega^\bullet_{X/Y}, d)$ does not provide anything particularly useful. Instead, the correct definition involves *hypercohomology*: $\mathbb{H}^i(X, \Omega^\bullet_{X/k})$. For a complex $\mathcal{F}^\bullet$ of coherent sheaves on a variety X,

- **(Loosely)** hypercohomology is to a complex of sheaves as "normal" cohomology is to a single sheaf, or rather,
- **(Rigorously)** hypercohomology of $\mathcal{F}^\bullet$ is $h^i(Rf_*\mathcal{F}^\bullet)$, where $f : X \to \operatorname{Spec} k$ is the structure morphism and $Rf_* : D(X) \to D(\operatorname{Spec} k)$ is the induced map on category of abelian sheaves.

A deep understanding of hypercohomology is not necessary to read these notes. For a rigorous treatment of the subject, we recommend the book [Huy06]. Instead, we primarily care that hypercohomology satisfies the following two properties:

(1) Any quasi-isomorphism of complexes $\alpha : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ induces an isomorphism $\alpha_* : \mathbb{H}^i(X,\mathcal{F}^\bullet) \to \mathbb{H}^i(X,\mathcal{G}^\bullet)$.
(2) Letting $\mathcal{F}$ denote a complex "concentrated in degree 0" (a complex in which every term is 0 except for at $i = 0$, which is $\mathcal{F}$), then $\mathbb{H}^i(X,\mathcal{F}) = H^i(X,\mathcal{F})$.

Definition 2.21. Suppose X is a finite dimensional scheme over a field k. We define the *algebraic de Rham cohomology* of X to be the hypercohomology of the algebraic de Rham complex:

$$H^i_{dR}(X/k) := \mathbb{H}^i(X,\Omega^\bullet_{X/k}).$$

To see that this is indeed the "correct" definition, consider that, at the very least, we hope to recover the isomorphism

$$H^i_{sing}(X,\mathbb{C}) \cong H^i_{dR}(X/\mathbb{C})$$

for a variety over $\mathbb{C}$. Given some sheaf $\mathcal{F}_0$ and a resolution

$$0 \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{F}^0 \longrightarrow \mathcal{F}^1 \longrightarrow ...,$$

we can easily verify that we have a quasi-isomorphism between the complexes $\mathcal{F}_0$ and $\mathcal{F}^\bullet$:

$$(0 \to \mathcal{F}_0 \to 0 \to ...) \simeq (0 \to \mathcal{F}^0 \to \mathcal{F}^1 \to ...),$$

which by the properties listed above gives us

$$H^i(X,\mathcal{F}_0) \overset{(2)}{\cong} \mathbb{H}^i(X,\mathcal{F}_0) \overset{(1)}{\cong} \mathbb{H}^i(X,\mathcal{F}^\bullet).$$

For a variety $X/\mathbb{C}$, we then have

$$H^i_{sing}(X,\mathbb{C}) \cong H^i(X,\underline{\mathbb{C}}) \cong \mathbb{H}^i(X,\mathcal{O}^\bullet_{X/k}) = H^i_{dR}(X,\mathbb{C}),$$

as desired, where $\underline{\mathbb{C}}$ denotes the constant sheaf on X whose stalks are all $\mathbb{C}$. Note that the first isomorphism is due to [AG66].

We now define a slightly more accessible, albeit more topological, cohomology for schemes.

Definition 2.22. Let X be a scheme, k a field, and $f : X \to \operatorname{Spec} k$ a morphism of schemes. The Hodge cohomology of a variety X/k is

$$H^\ell_{\text{Hodge}}(X/k) = \bigoplus_{i+j=\ell} H^i(X,\Omega^j_{X/k}),$$

where $H^i(X,\Omega^j_{X/k})$ is the ith sheaf cohomology of $\Omega^j_{X/k}$ [Har77, pg 207]

2.3. **Miscellaneous Results.** This section proceeds in no particular order; rather, it is a list of results included for convenience. References are provided where proofs do not appear.

Lemma 2.23. *Assume $f : X \to Y$ is a smooth map of schemes. Then f factors as*

$$X \xrightarrow{g} \mathbb{A}^n_Y \xrightarrow{h} Y,$$

where h is the canonical projection and g is étale.

Proof. Fix a point $x \in X$ and denote by $k(x)$ the residue field of the local ring $\mathcal{O}_{X,x}$. Let $s_1,...,s_n$ be sections of $\mathcal{O}_X$ in a neighborhood of x for which the differentials form a basis of $\Omega^1_{X/Y}$ at x, i.e. chosen such that the images $(ds_i)_x$ of ds_i in $\Omega^1_{X/Y}$ form a basis of this module over $\mathcal{O}_{X,x}$. Note that this is equivalent to choosing $s_1,...,s_n$ such that the images $(ds_i)_x$ in $\Omega^1_{X/Y,x} \otimes k(x)$ form a $k(x)$ basis when considering $\Omega^1_{X/Y,x}$ to be a $k(x)$ vector space. Since f is smooth, $\Omega^1_{X/Y}$ is locally free of finite type as a $\mathcal{O}_X$ module. This means that there exists an open neighborhood U of x for which the s_i are defined over U and that the ds_i form a basis of $\Omega^1_{X/Y|U}$. The s_j then define a Y-morphism of U in the affine space of dimension n over Y:

$$s = (s_1,...,s_n) : U \to \mathbb{A}^n_Y = Y[t_1,...,t_n].$$

If we do this over each $x \in X$, this yields a well defined étale map $g : X \to \mathbb{A}^n_Y$ (see [Ill96, Section 2.7] for details). Composition with the projection map $h : \mathbb{A}^n_Y \to Y$ gives f. □

Proposition 2.24.

(a) *If $X \to Y$ is a smooth morphism of schemes, the $\mathcal{O}_X$-module $\Omega^1_{X/Y}$ is locally free of finite type.*

(b) *Let $X \xrightarrow{f} Y \xrightarrow{g} S$ be morphisms of schemes. If f is smooth, then we have a sequence*

$$0 \to f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0$$

which is exact and locally split. In particular, if f is étale, the canonical homomorphism $f^*\Omega^1_{Y/X} \to \Omega^1_{X/S}$ is an isomorphism.

Proof. The proof is provided in [AG66, IV, 17.2.3]. □

Assertion (b) in Proposition 2.24 has a converse, which is deduced in the same manner.

Proposition 2.25. *In the situation of 2.24 (b), assume gf to be a smooth map. If the canonical homomorphism $f^*\Omega^1_{Y/S} \to \Omega^1_{X/S}$ is an isomorphism, then f is étale.*

Proposition 2.26. *Let $f : X \to Y$ be a morphism of schemes, locally of finite presentation. The following conditions are equivalent:*

(i) *f is smooth*

(ii) *f is flat and the geometric fibers of f are regular schemes.*

Proof. This is a verification of the discussion found in [Ill96, Sections 2.7 and 2.8]. □

3. The Hodge Spectral Sequence

In this section, we introduce a spectral sequence, the *Hodge spectral sequence*, which provides a connection between Hodge and algebraic de Rham cohomology. We will see that these two cohomology theories are equivalent exactly when the Hodge specral sequence degenerates on the first page. This section is broken up into two parts. First, the necessary spectral sequence is constructed for arbitrary Abelian categories. We then check that this spectral sequence does indeed give us the desired connection between Hodge and algebraic de Rham cohomology.

Throughout this section and indeed the entire document we treat the theory of spectral sequences as a black box. As Ravi Vakil says, "It has been suggested that the name ‘spectral’ was given because, like spectres, spectral sequences are terrifing, evil, and dangerous," [Vak08]. For a gentle introduction to spectral sequences, we suggest [Vak08], and for a more thorough treatment, [McC00].

3.1. **General Setup.** We will care only about spectral sequences which arise from filtrations of complexes, which we construct here. We first require a few definitions.

A *decreasing filtration* $\mathrm{Fil}^\bullet$ of an object M in an Abelian category is a family of objects $(\mathrm{Fil}^n)_{n\in\mathbb{Z}}$ such that

$$A \supseteq \ldots \supseteq F^n \supseteq F^{n+1} \supseteq \ldots 0.$$

Whenever we have a decreasing filtered object $\mathrm{Fil}^\bullet M$, we may associate to it a graded object

$$\mathrm{gr}_\bullet \mathrm{Fil}^\bullet M := \bigoplus_{i\in\mathbb{Z}} \mathrm{gr}_i \mathrm{Fil}^\bullet M$$

where

$$\mathrm{gr}_i \mathrm{Fil}^\bullet M := \mathrm{Fil}^i M / \mathrm{Fil}^{i+1} M.$$

Creatively, we call $\mathrm{gr}_\bullet \mathrm{Fil}^\bullet M$ the *associated graded object* of $\mathrm{Fil}^\bullet M$, and $\mathrm{gr}_i \mathrm{Fil}^\bullet M$ the ith graded piece of $\mathrm{Fil}^\bullet M$.

Now, given a bounded below complex $\mathcal{F}^\bullet$ of objects in an Abelian category $\mathcal{A}$, assume we have a decreasing filtration

$$\mathrm{Fil}^\bullet \mathcal{F}^\bullet, \tag{3.1}$$

and to be clear, since this is a decreasing filtration, for all $i, j \in \mathbb{Z}$ with $i < j$ we have the inclusions

$$\mathcal{F}^\bullet \supseteq \mathrm{Fil}^i \mathbb{F}^\bullet \supseteq \mathrm{Fil}^j \mathbb{F}^\bullet.$$

We will further assume that this complex is *biregular*, which only means that for all $i \ll 0$, $\mathrm{Fil}^i \mathcal{F}^\bullet = \mathcal{F}^\bullet$ and for all $i \gg 0$, $\mathrm{Fil}^i \mathcal{F}^\bullet = 0$ – that is, except for a "middle bit" the filtration is constant. Together, the assumption that $\mathcal{F}^\bullet$ is bounded below and that $\mathrm{Fil}^\bullet \mathcal{F}^\bullet$ is biregular mean we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \xrightarrow{d} & 0 & \xrightarrow{d} & 0 & \xrightarrow{d} & 0 & \xrightarrow{d} & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \xrightarrow{d} & \mathrm{Fil}^N \mathcal{F}^0 & \xrightarrow{d} & \mathrm{Fil}^N \mathcal{F}^1 & \xrightarrow{d} & \mathrm{Fil}^N \mathcal{F}^2 & \xrightarrow{d} & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \xrightarrow{d} & \mathrm{Fil}^1 \mathcal{F}^0 & \xrightarrow{d} & \mathrm{Fil}^1 \mathcal{F}^1 & \xrightarrow{d} & \mathrm{Fil}^1 \mathcal{F}^2 & \xrightarrow{d} & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \xrightarrow{d} & \mathrm{Fil}^0 \mathcal{F}^0 & \xrightarrow{d} & \mathrm{Fil}^0 \mathcal{F}^1 & \xrightarrow{d} & \mathrm{Fil}^0 \mathcal{F}^2 & \xrightarrow{d} & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \xrightarrow{d} & \mathcal{F}^0 & \xrightarrow{d} & \mathcal{F}^1 & \xrightarrow{d} & \mathcal{F}^2 & \xrightarrow{d} & \dots
\end{array} \tag{3.2}$$

(here we have shifted all indices to be nonnegative, but apriori they need not be).

Since we're headed toward spectral sequences, for a functor $F : \mathcal{A} \to \mathcal{B}$ on Abelian categories, we ought to consider how the right derived functors of $F(\mathcal{F}^\bullet)$ behave under this filtration. Indeed, the filtration $\mathrm{Fil}^\bullet \mathcal{F}^\bullet$ gives a (decreasing) filtration on

$$\mathbb{R}^\ell F(\mathcal{F}^\bullet) := h^\ell(\mathbb{R}F(\mathcal{F}^\bullet))$$

which is defined

$$\mathrm{Fil}^i \mathbb{R}^\ell F(\mathcal{F}^\bullet) := \mathrm{im}\left(\mathbb{R}^\ell F(\mathrm{Fil}^i \mathcal{F}^\bullet) \to \mathbb{R}^\ell F(\mathcal{F}^\bullet)\right),$$

and since $\mathrm{Fil}^\bullet \mathcal{F}^\bullet$ is biregular, so is $\mathrm{Fil}^\bullet \mathbb{R}^\ell F(\mathcal{F}^\bullet)$.

From our initial discussion, we now have two natural graded objects, $\mathrm{gr}_\bullet \mathrm{Fil}^\bullet \mathcal{F}^\bullet$ and $\mathrm{gr}_\bullet \mathrm{Fil}^\bullet \mathbb{R}^\ell F(\mathcal{F}^\bullet)$. The first is a bounded below complex of $\mathcal{A}$ and the second is an object of $\mathcal{B}$. Passing $\mathrm{gr}_\bullet \mathrm{Fil}^\bullet \mathcal{F}^\bullet$ through the ℓth right hyperderived functor $\mathbb{R}^\ell F$ yields another object in $\mathcal{B}$, which leads to the question: do we have an isomorphism of graded objects

$$\mathbb{R}^\ell F(\mathrm{gr}_\bullet \mathrm{Fil}^\bullet \mathcal{F}^\bullet) \overset{?}{\cong} \mathrm{gr}_\bullet \mathrm{Fil}^\bullet \mathbb{R}^\ell F(\mathcal{F}^\bullet) \tag{3.3}$$

(Note that an isomorphism of graded objects means that the isomorphism restricts to the individual gr_i's.) In other words, do $\mathbb{R}^\ell$ and $\mathrm{gr}_\bullet \mathrm{Fil}^\bullet$ commute?

This question is indeed important. In essentially every situation, computing the object $\mathbb{R}^\ell F(\mathrm{gr}_\bullet \mathrm{Fil}^\bullet \mathcal{F}^\bullet)$ is far easier than computing the hyperderived functor $\mathbb{R}^\ell F(\mathcal{F}^\bullet)$ (which one needs to compute before computing $\mathrm{gr}_\bullet \mathrm{Fil}^\bullet \mathbb{R}^\ell F(\mathcal{F}^\bullet)$). This is because computing the graded object first and before moving to $\mathcal{B}$ under F allows us to compute an honest cohomology group, rather than a hyper-cohomology group.

As suggested by diagram (3.2), we may form a spectral sequence known as the *spectral sequence of a filtered complex*

$$E_1^{ab} := \mathbb{R}^{a+b} F(\mathrm{gr}_a \mathrm{Fil}^\bullet \mathcal{F}^\bullet) \implies \mathbb{R}^{a+b} F(\mathcal{F}^\bullet). \tag{F.C.S.S}$$

From the theory of spectral sequences, we know that

$$E_\infty^{ab} = \mathrm{gr}_a \mathrm{Fil}^\bullet \mathbb{R}^{a+b} F(\mathcal{F}^\bullet).$$

Thus, the terms of the first page constitute the left side (the "easy" side) of equation (3.3), while the terms of the ∞-page consitute the right side (the "hard" side) of equation (3.3). Note that since E_∞^{ab} is a subquotient of E_1^{ab}, if the functor F takes values in the category of finite dimensional k-vector spaces then we always have the inequality:

Lemma 3.4. *If the functor F takes values in the category of finite dimensional k-vector spaces, then*

$$\dim_k \mathbb{R}^\ell F(\mathcal{F}^\bullet) = \dim_k \operatorname{gr}_\bullet \operatorname{Fil}^\bullet \mathbb{R}^\ell F(\mathcal{F}^\bullet) = \sum_{a+b=\ell} \dim_k E_\infty^{ab} \leq \sum_{a+b=\ell} \dim_k E_\ell^{ab}$$

with equality if and only if the (F.C.S.S) degenerates on the first page.

This simple observation is invaluable, as it reduces the task of proving the degeneration of the (F.C.S.S) on the first page to a dimension computation.

3.2. **The Hodge filtration.** With this general construction out of the way, we now consider a particular filtration which works on any Abelian category. We call this the *Hodge filtration*, or more aptly, the *stupid filtration.*

Suppose $\mathcal{F}^\bullet$ is a complex in an Abelian category $\mathcal{A}$, and *further* suppose that $\mathcal{F}^\bullet$ is bounded both above *and* below. The Hodge filtration is defined

$$\operatorname{Fil}_h^i \mathcal{F}^j := \begin{cases} 0 & \text{if} \quad j < i \\ \mathcal{F}^j & \text{if} \quad j \geq i \end{cases}. \tag{3.5}$$

Since $\mathcal{F}^\bullet$ is bounded, it is clear that $\operatorname{Fil}_h^\bullet \mathcal{F}^\bullet$ is decreasing and biregular. If the first nonzero term of $\mathcal{F}^\bullet$ occurs at index 0 and the last at index N, then $\operatorname{Fil}_h^\bullet \mathcal{F}^\bullet$ looks like

$$\begin{array}{ccccccccccccc}
0 & \xrightarrow{d} & 0 & \xrightarrow{d} & 0 & \xrightarrow{d} & 0 & \xrightarrow{d} & \dots & \xrightarrow{d} & 0 & \xrightarrow{d} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
0 & \xrightarrow{d} & 0 & \xrightarrow{d} & 0 & \xrightarrow{d} & 0 & \xrightarrow{d} & \dots & \xrightarrow{d} & \mathcal{F}^N & \xrightarrow{d} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots & & & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
0 & \xrightarrow{d} & 0 & \xrightarrow{d} & 0 & \xrightarrow{d} & \mathcal{F}^2 & \xrightarrow{d} & \dots & \xrightarrow{d} & \mathcal{F}^N & \xrightarrow{d} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
0 & \xrightarrow{d} & 0 & \xrightarrow{d} & \mathcal{F}^1 & \xrightarrow{d} & \mathcal{F}^2 & \xrightarrow{d} & \dots & \xrightarrow{d} & \mathcal{F}^N & \xrightarrow{d} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
0 & \xrightarrow{d} & \mathcal{F}^0 & \xrightarrow{d} & \mathcal{F}^1 & \xrightarrow{d} & \mathcal{F}^2 & \xrightarrow{d} & \dots & \xrightarrow{d} & \mathcal{F}^N & \xrightarrow{d} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
0 & \xrightarrow{d} & \mathcal{F}^0 & \xrightarrow{d} & \mathcal{F}^1 & \xrightarrow{d} & \mathcal{F}^2 & \xrightarrow{d} & \dots & \xrightarrow{d} & \mathcal{F}^N & \xrightarrow{d} & 0
\end{array} \tag{3.6}$$

Note that

$$\operatorname{gr}_a \operatorname{Fil}_h^\bullet \mathcal{F}^\bullet = \operatorname{Fil}^a \mathcal{F}^\bullet / \operatorname{Fil}^{a+1} \mathcal{F}^\bullet = \mathcal{F}^a[-a],$$

where $[-a]$ denotes the complex concentrated at the zero index obtained by shifting $\mathcal{F}^\bullet$ a indices to the right and replacing every term except for $\mathcal{F}^a$ with a zero. We may easily compute

$$\mathbb{R}^{a+b}F(\mathcal{F}^a[-a]) = \mathbb{R}^b F(\mathcal{F}^a) = R^b F(\mathcal{F}^a),$$

and so the F.C.S.S becomes

$$E_1^{ab} = R^b F(\mathcal{F}^a) \Rightarrow \mathbb{R}^{a+b}F(\mathcal{F}^\bullet), \tag{H.S.S.}$$

which we call the *Hodge Spectral Sequence.*

We are interested in the case where k is some field and X is a k-scheme (that is, we have a structure morphism $f : X \to \operatorname{Spec} k$). If we take F to be the functor $\Gamma(X, \bullet)$ and the complex $\mathcal{F}^\bullet$ to be the algebraic de Rham complex $\Omega^\bullet_{X/k}$ then

$$E_1^{ab} = R^b\Gamma(X, \Omega^a_{X/k}) = H^b(X, \Omega^a_{X/k})$$

and

$$\mathbb{R}^{a+b}\Gamma(X, \Omega^\bullet_{X/k}) = H^{a+b}_{dR}(X/k).$$

This is a spectral sequence of k-vectors spaces, so Lemma 3.4 applies, and since

$$H^\ell_{\text{Hodge}}(X/k) \cong H^\ell_{dR}(X/k) \iff \text{the H.S.S. degenerates on the first page,}$$

it tells us the following:

Lemma 3.7. *Let X be a proper scheme over a field k.*

$$H^\ell_{Hodge}(X/k) \cong H^\ell_{dR}(X/k) \iff \sum_{i+j=\ell} \dim_k H^j(X, \Omega^i_{X/k}) = \dim_k H^\ell_{dR}(X/k).$$

Proof. Since X is a proper scheme, $H^\ell_{\text{Hodge}}(X/k)$ and $H^\ell_{dR}(X/k)$ are finite-dimensional (see [Har77, Chapter III Theorem 5.2] or [Ser55] for the projective case, [AG66, III chapter 3] for the general case). This means

$$H^\ell_{\text{Hodge}}(X/k) \cong H^\ell_{dR}(X/k)$$

if and only if

$$\dim_k H^\ell_{dR}(X/k) = \dim_k H^\ell_{Hodge}(X/k) = \sum_{i+j=\ell} \dim_k H^j(X, \Omega^i_{X/k}).$$

□

4. Equivalence of Hodge and algebraic de Rham Cohomology for Prime Characteristic Schemes

Once the Hodge spectral sequence has been defined, one typically proves the equivalence of Hodge and algebraic de Rham cohomology for schemes by first proving it for complex manifolds and extending to schemes over fields of characteristic 0 via Serre's GAGA principle. However, this technique fails for schemes of characteristic p. We instead make use of the relative Frobenius map to build the Cartier isomorphism, which allows us to cleverly circumvent the use of GAGA. In this section, we briefly introduce prime characteristic schemes, outline the construction of the Cartier isomorphism, and use it to prove the equivalence of Hodge and algebraic de Rham cohomology for prime characteristic schemes under certain assumptions.

4.1. **Frobenius action and Cartier Isomorphism.** Throughout this section, p denotes a fixed prime integer. We say that a scheme X is of *characteristic* p if $p\mathcal{O}_X = 0$, i.e. if the morphism of schemes $X \to \operatorname{Spec}\mathbb{Z}$ factors (uniquely) through $\operatorname{Spec}\mathbb{F}_p$:

$$\begin{array}{ccc} X & \longrightarrow & \operatorname{Spec}\mathbb{Z} \\ & \searrow & \nearrow \\ & \operatorname{Spec}\mathbb{F}_p & \end{array}$$

Here, $p\mathcal{O}_X = 0$ precisely means that $p\mathcal{O}_X(U)$ is the zero ring for any open set $U \subseteq X$. Notice that this means $\mathcal{O}_X(U)$ is a ring of prime characteristic for each open $U \subseteq X$.

If X is a scheme of prime characteristic, then the *absolute Frobenius morphism* of X (or simply the Frobenius endomorphism if there is no risk of ambiguity) is the endomorphism $F_X : X \to X$ which is the identity on $\operatorname{sp}(X)$ and is the restriction of scalars along Frobenius on $\mathcal{O}_X$. To be painfully explicit, the absolute Frobenius morphism on a scheme $(X, \mathcal{O}_X)$ of prime characteristic is a map $F_X = (id_X, F^\sharp)$ where id_X is the identity on the topological space X and $F^\sharp : \mathcal{O}_X \to \mathcal{O}_X$ is a map of sheaves whose components $F : \mathcal{O}_X(U) \to \mathcal{O}_X(U)$ are all the Frobenius endomophism $r \mapsto r^p$

$$\begin{array}{ccc} \mathcal{O}_X(U) & \xrightarrow{r \mapsto r^p} & \mathcal{O}_X(U) \\ \Big\downarrow \rho_{UV} & & \Big\downarrow \rho_{UV} \\ \mathcal{O}_X(V) & \xrightarrow{s \mapsto s^p} & \mathcal{O}_X(V). \end{array}$$

Let $X \to Y$ be a morphism of prime characteristic schemes. There is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{F_X} & X \\ \Big\downarrow f & & \Big\downarrow f \\ Y & \xrightarrow{F_Y} & Y, \end{array} \tag{4.1}$$

which allows us to construct the scheme $(Y, F_Y) \times_Y X$ (see [Har77, page 87] for a definition of the *fibered product*) induced from X by the change of base F_Y. The scheme $(Y, F_Y) \times_Y X$ is denoted $X^{(p)}$ and is known as the *Frobenius twist* of X. The morphism F_X defines a unique Y-morphism $F = F_{X/Y} : X \to X^{(p)}$, giving rise to yet another commutative diagram:

$$\begin{array}{ccccc} X & \xrightarrow{F} & X^{(p)} & \longrightarrow & X \\ & \overset{f}{\searrow} & \Big\downarrow & & \Big\downarrow f \\ & & Y & \xrightarrow{F_Y} & Y. \end{array} \tag{4.2}$$

The map $F : X^{(p)} \to$ is known as the *relative Frobenius map* of X. As it turns out, the Frobenius twist has no affect on cohomology.

Lemma 4.3. *Let $S = \operatorname{Spec} k$ be a perfect field, and let X/S be smooth projective. Then,*

$$H^\ell_{Hodge}(X/k) \cong H^\ell_{Hodge}(X^{(p)}/k).$$

Proof. Take $i \geq 0$. In the language of 2.24, we have morphisms $X^{(p)} \xrightarrow{f} X \xrightarrow{g} \operatorname{Spec} k$ where f is an isomorphism and therefore smooth. This gives us an exact sequence

$$0 \to f^*\Omega^1_{X/k} \to \Omega^1_{X/k} \to \Omega^1_{X^{(p)}/X} \to 0$$

in which $\Omega^1_{X^{(p)}/X} \cong 0$, since f is an isomorphism. This means

$$f^*\Omega^1_{X/k} \cong \Omega^1_{X/k},$$

and by taking wedge products we obtain isomorphisms

$$f^*\Omega^i_{X/k} \to \Omega^i_{X/k}$$

for each $i \geq 0$. This gives us an isomorphism on sheaf cohomology

$$H^j(X^{(p)}, \Omega^i_{X^{(p)}/k}) \cong H^j(X^{(p)}, f^*\Omega^i_{X/k}) \cong H^j(X, \Omega^i_{X/k})$$

where the second isomorphism follows, once again, from the fact that f is an isomorphism. These are precisely the summands which appear in Hodge cohomology, so we conclude

$$H^\ell_{\text{Hodge}}(X/k) = \bigoplus_{i+j=\ell} H^j(X, \Omega^i_{X/k}) \cong \bigoplus_{i+j=\ell} H^j(X^{(p)}, \Omega^i_{X^{(p)}/k}) = H^\ell_{\text{Hodge}}(X^{(p)}/k).$$

□

4.2. **Cartier Isomorphism.** The key to proving the degeneration of the H.S.S. in the prime characteristic case lies in the use of the Cartier isomorphism. Before we construct it, we will need the following proposition:

Proposition 4.4. *Let Y be a scheme of characteristic p, and $f : X \to Y$ a smooth morphism of pure relative dimension n (see Proposition 2.26). Then the relative Frobenius $F : X \to X^{(p)}$ is a finite and flat morphism, and the $\mathcal{O}_{X^{(p)}}$-algebra $F_*\mathcal{O}_X$ is locally free of rank p^n. In particular, if f is étale, F is an isomorphism, i.e. the square (4.2) is Cartesian.*

Proof. Suppose first that $n = 0$. Since F is a smooth map of relative dimension 0, it is étale [AG66, IV, Corollary 17.10.2], which certainly means it is flat and finite [AG66, IV, Corollary 17.6.2].

Once we know the $n = 0$ case holds, the case in which $X = \mathbb{A}^n_Y = \operatorname{Spec} Y[t_1, \ldots, t_n]$ is immediate. The monomials $\prod t_i^{m_i}$ with $0 \leq m_i < p - 1$ form a basis for $F_*\mathcal{O}_X$ over $\mathcal{O}_X$, which gives us the desired result.

Finally, the general case is deduced by factoring f as $f = hg$ according to Lemma 2.23:

$$X \xrightarrow{g} \mathbb{A}^n_Y \xrightarrow{h} Y.$$

Here the map h is the canonical projection map, which is smooth and of pure relative dimension n. The map g is étale by construction, and is therefore finite and flat. □

Theorem 4.5 (Cartier)**.** *Let Y be a scheme of characteristic p and $f : X \to Y$ a morphism.*

(a) There exists a unique homomorphism of graded $\mathcal{O}_X$-algebras

$$\gamma : \bigoplus \Omega^i_{X^{(p)}/Y} \to \bigoplus \mathcal{H}^i F_*\Omega^*_{X/Y}$$

satisfying the following two conditions:

(i) for $i = 0$, γ is given by the homomorphism $F^* : \mathcal{O}_{X^{(p)}} \to F_*\mathcal{O}_X$;

(ii) for $i = 1$, γ sends $1 \otimes ds$ to the class of $s^{p-1}ds$ in $\mathcal{H}^1 F_*\Omega^*_{X/Y}$ (where $1 \otimes ds$ denotes the image of the section ds of $\Omega^1_{X/Y}$ in $\Gamma^1_{X^{(p)}/Y}$.

(b) If f is smooth, γ is an isomorphism.

Proof. As is done in [Ill96], we only briefly outline the proof of (a). It amounts to constructing a homomorphism of graded $\mathcal{O}_X$-algebras

$$\gamma_{\text{abs}} : \bigoplus \Omega^i_{X/Y} \to \bigoplus H^i F_{X*}\Omega^\bullet_{X/Y}.$$

The only difference between γ and γ_{abs} is the appearance of the absolute Frobenius map $F_X : X \to X$ in the place of the relative Frobenius F in the codomain, and we construct γ_{abs} to satisfy conditions analogous to (i) and (ii):

(i') for $i = 0$, γ_{abs} is given by the homomorphism F_X^*;
(ii') for $i = 1$, γ_{abs} is given by sending ds in $\Omega^1_{X/Y}$ to $s^{p-1}ds$.

For the complete argument, see [Kat70, Theorem 7.2].

To prove (b), we first factor the map $f : X \to Y$ through the affine n space $\mathbb{A}^n_Y$ based at Y, argue that we may reduce to the case $Y = \mathbb{A}^n_k$, and then further reduce to $n = 1$ and $k = \mathbb{F}_p$. By 2.23, one may assume that f factors as

$$X \xrightarrow{g} \mathbb{A}^n_Y \xrightarrow{h} Y.$$

Here, g is étale, $\mathbb{A}^n_Y$ is the *affine n space* over the base scheme Y, defined

$$\mathbb{A}^n_Y = \operatorname{Spec}_Y(\mathcal{O}_Y[T_1, ..., T_n]),$$

and the map h is simply the canonical projection. The commutative square (4.1) gives us a new commutative diagram,

$$\begin{array}{ccc} X & \xrightarrow{F_X} & X^{(p)} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g^{(p)}} \\ \mathbb{A}^n_Y & \xrightarrow{F_Y} & (A^n_Y)^{(p)} \end{array}$$

and from Proposition 2.24 (b) we know the homomorphism $g^*\Omega^i_{Z/Y} \to \Omega^i_{X/Y}$ to be an isomorphism. By Proposition 4.4, we know the relative Frobenius map F to be finite and the above square to be Cartesian, so we see that the map of complexes induced by $g^{(p)}$ is an isomorphism:

$$g^{(p)*}F_*\Omega^\bullet_{Z/Y} \xrightarrow{\sim} F_*\Omega^\bullet_{X/Y}.$$

This means it is also an isomorphism on the level of cohomology:

$$g^{(p)*}H^i\big(F_*\Omega^\bullet_{Z/Y}\big) \xrightarrow{\sim} H^i\big(F_*\Omega^\bullet_{X/Y}\big).$$

Using the functoriality of γ reduces to $\mathbb{A}^n_Y$, and using extension of scalars and Künneth's formula reduces it to $n = 1$ and $Z = \mathbb{F}_p$ (see [Ill96, (3.6.3)] for more details). It therefore suffices to prove (b) for the case where $Y = \operatorname{Spec} \mathbb{F}_p[t]$. We show that $H^i\big(F_*\Omega^\bullet_{Y/\mathbb{F}_p}\big)$ is a free module over $\mathcal{O}_Y$ with rank 1, and conclude that γ is an isomorphism.

In this case, $Y^{(p)} = (Y, F_Y) \times_Y = Y$ and $\mathcal{O}_Y = \mathbb{F}_p[t]$. We claim that $1, t, ..., t^{p-1}$ forms a basis for $F_*\mathcal{O}_Y$. Indeed, recall that the module $F_*\mathcal{O}_Y$ is obtained by restricting the $\mathcal{O}_Y$ action through the Frobenius endomorphism, and note that we denote by F_*r the element in $F_*\mathcal{O}_Y$ corresponding to $r \in \mathcal{O}_Y$. For any monomial $F_*at^\ell \in F_*\mathcal{O}_Y$, choose $m \in \{0, ..., p-1\}$ such that for some q, $p \cdot q + m = \ell$, giving us

$$at^q \cdot F_*t^m = F_*a^p t^{q \cdot p} \cdot t^m = F_*at^{q \cdot p + m} = F_*at^\ell.$$

Since we may obtain any monomial in $F_*\mathcal{O}_Y$ from the basis $\{1, ..., t^{p-1}\}$, by taking linear combinations over $\mathcal{O}_Y$ we see this does indeed span $F_*\mathcal{O}_Y$. Now, we have that

$$H^0\big(F_*\Omega^\bullet_{Y/\mathbb{F}_p}\big) = \ker d,$$

and since the differential $d : F_*\mathcal{O}_Y \to F_*\mathcal{O}_Y^1 = (F_*\mathcal{O}_Y)dt$ sends t^i to $i \cdot t^{i-1}\ dt$, the element 1 generates $\ker d$. This means $H^0\big(F_*\Omega^\bullet_{Y/\mathbb{F}_p}\big)$ is rank 1, and therefore isomorphic to $\mathcal{O}_Y$ as a module. Similarly, $H^1\big(F_*\Omega^\bullet_{Y/\mathbb{F}_p}\big)$ is rank 1. Finally, as in part (a), the cases $i = 0$ and $i = 1$ fully determine $H^i\big(F_*\Omega^\bullet_{Y/\mathbb{F}_p}\big)$, and we conclude that γ is an isomorphism. □

4.3. **Degeneration of the Hodge Spectral Sequence.** The Cartier isomorphism is the key to showing the equivalence of algebraic de Rham and Hodge cohomology in the case that X is of characteristic p. The key idea is to construct a quasi-isomorphism

$$(\Omega^\bullet_{X^{(p)}/k}, 0) \to (F_*\Omega^\bullet_{X/k}, d)$$

which induces the Cartier isomorphisms on cohomology. In the interest of brevity, if such a quasi-isomorphism exists for a scheme X, we say that X satisfies (QI).

(QI): there exists a quasi-isomorphism $(\Omega^\bullet_{X^{(p)}/k}, 0) \to (F_*\Omega^\bullet_{X/k}, d)$ inducing the Cartier isomorphisms on cohomology.

It is easy to see why (QI) implies degeneration on the first page of the H.S.S.. This is illustrated in the following proposition.

Proposition 4.6. *Let X be a scheme for which (QI) holds. Then the H.S.S. degenerates on the first page.*

Proof. If (QI) holds, then

$$\begin{aligned} H^\ell_{\text{Hodge}}(X/k) &\overset{(1)}{\cong} H^\ell_{\text{Hodge}}(X^{(p)}/k \\ &\cong \mathbb{H}^\ell\big((\Omega^\bullet_{X^{(p)}/k}, d)\big) \\ &\overset{(2)}{\cong} \mathbb{H}^\ell\big((F_*\Omega^\bullet_{X/k}, d)\big) \\ &\overset{(3)}{\cong} \mathbb{H}^\ell\big((\Omega^\bullet_{X/k}, d)\big) \\ &\cong H^\ell_{dR}(X/k). \end{aligned}$$

Isomorphism (1) is simply Lemma 4.3, (2) is given by (QI), and (3) follows from the fact that F is a finite map. □

It is now our task to determine exactly when (QI) holds. There two known cases:

(1) X/k has a smooth lift to $W_2(k)$ and $\dim(X) < \operatorname{char}(k)$ (when $k = \mathbb{F}_p$, $W_2(k) = \mathbb{Z}/p^2\mathbb{Z}$)
(2) X/k has a smooth lift to $W_2(k)$ and the relative Frobenius map $F : X \to X^{(p)}$ lifts to a map $F : \widetilde{X} \to \widetilde{X^{(p)}}$, where $\widetilde{X^{(p)}} = \widetilde{X} \times_{W_2(k)} W_2(k)$.

Here $W_2(k)$ is the ring k^2 with the addition

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2 + p^{-1}(a_1^{p-1} + b_1^{p-1} - (a_1 + b_1)^p)$$

and the multiplication

$$(a_1, a_2) \cdot (b_1, b_2) = (a_1 b_1, b_1^p a_2 + b_2 a_1^p).$$

It is known as the ring of two Witt vectors of k.

Of the two cases where (QI) holds, we handle only the former. The latter is significantly more technical. Our goal, remember, is to build a quasi-isormophism of complexes

$$\varphi^\bullet : \Omega^\bullet_{X^{(p)}/k}[-1] \to F_*\Omega^\bullet_{X/k} \tag{4.7}$$

which induces γ_1 in cohomology. We use the lift of X to $W_2(k)$ in this first case to build the map.

Theorem 4.8. *Let k be a perfect field of prime characteristic $p > 0$, and let X be a smooth and proper k-scheme such that $\dim X < p$. If X is lifted over $W_2(k)$, the H.S.S. degenerates on the first page.*

Proof. By Proposition 4.6, it suffices to show that there exists a quasi-isomorphism $\varphi^\bullet$: $\Omega^\bullet_{X^{(p)}/k}[-1] \to F_*\Omega^\bullet_{X/k}$ which induces the Cartier isomorphism on cohomology. First, suppose we had this map in degree 1:

$$\varphi^1 : \Omega^1_{X^{(p)}/k}[-1] \to F_*\Omega^1_{X/k}.$$

From this, it is possible to build the maps

$$\varphi^i : \Omega^i_{X^{(p)}/k}[-1] \to F_*\Omega^1_{X/k}.$$

Indeed, if $i \geq 1$, then we may define

$$\varphi^i : \Omega^i_{X^{(p)}/k} \xrightarrow{a} \left(\Omega^1_{X^{(p)}/k}\right)^{\otimes i} \xrightarrow{(\varphi^1)^{\otimes i}} \left(F_*\Omega^\bullet_{X/k}\right)^{\otimes i} \to F_*\Omega^\bullet_{X/k}.$$

Here, a is the antisymmetrization map:

$$a(\omega_1 \wedge \ldots \wedge \omega_i) = \frac{1}{i!}\sum_{\sigma \in S_i} (-1)^{\mathrm{sgn}(\sigma)}\omega_{\sigma(1)} \otimes \ldots \otimes \omega_{\sigma(i)}.$$

Since k is of characteristic p, this map is undefined for $i \geq p$. It is crucial, then, that $\Omega^i_{X^{(p)}}/k$ and $\Omega^i_{X/k}$ are trivial for $i \geq p$, which happens exactly when $\dim X < p$.

It now remains to build φ^0 and φ^1. The former map is easy, we simply define $\varphi^0 = F^*$: $\mathcal{O}_X \to F_*\mathcal{O}_X$, $r \mapsto F_*r$. The latter map requires several additional steps.

Let X_0 denote the lift of X over $W_2(k)$. We first notice that the map $p : F_*\Omega^1_{X/k} \to p \cdot F_*\Omega^1_{X_0/k_0}$ is an isomorphism. Furthermore, by [Ill96, 3.4 (a)], the image of $F^* : \Omega^1_{X/k} \to F^*\Omega^1_{X/k}$ is contained in $pF_{0,*}\Omega^1_{X/k}$ as long as the relative Frobenius map $F : X \to X^{(p)}$ lifts to X_0. This lift always exists Zariski-locally [Ill96], and by gluing the local lifts of the relative Frobenius using the Cech complex, considering only the local case suffices. See [Ill96, Lemma 5.4] for the full argument. Since $p : F_*\Omega^1_{X/k} \to p \cdot F_*\Omega^1_{X_0/k_0}$ is an isomorphism and $F^*(\Omega^1_{X/k})$ is contained in $pF_{0,*}\Omega^1_{X/k}$, we have the following diagram:

$$\begin{array}{ccc} \Omega^1_{X^{(p)}/k} & \xrightarrow{F^*} & pF_{0,*}\Omega^1_{X/k} \\ \downarrow & & \uparrow{\cdot p} \\ \Omega^1_{X_0^{(p)}/k_0} & \xrightarrow{\psi} & F_{0,*}\Omega^1_{X_0/k_0}. \end{array}$$

The map ψ is the homomorphism induced from F^* by division by p, that is, it is the unique map that makes the above diagram commute. If x is a local section of $\mathcal{O}_{X_0}$ whose reduction modulo p is x_0, and x' lifts in $\mathcal{O}_{X^{(p)}}$ the image x'_0 of x_0 in $\mathcal{O}_{X'_0}$, we see that

$$F^*x' = x^p + pb$$

for some $b \in \mathcal{O}_X$. If we let $d : \mathcal{O}_X \to \Omega^1_{X/k}$ be the differential in the algebraic de Rham complex, then

$$F^*(d(x)) = pa^{p-1}da + pdb,$$

so

$$\psi(da_0') = a^{p-1}da_0 + db_0.$$

Define $\varphi^1 = p^{-1} \circ F^* : \Omega^1_{X^{(p)}/k} \to F_*\Omega^1_{X/k}$, or equivalently, the map obtained by tracing down and right in the above diagram. By the above, $H^1\varphi^1 = \gamma^1$, the Cartier map. Since this extends to φ^i, we are done. □

5. From characteristic p to characteristic 0

As previously stated, GAGA allows one to use the equivalence of Hodge and de Rham cohomology for complex manifolds to prove it for finite dimensional schemes over characteristic 0 fields. However, using the techniques of Illusie, it is possible to extend the result in the characteristic p case to the characteristic 0 case, and therefore entirely avoid the use of GAGA. We briefly outline those results here.

Suppose k is a field of characteristic zero, and $X \to \operatorname{Spec} k$ a scheme over k. Illusie's idea was to realize k as an inductive limit of finite-type $\mathbb{Z}$-algebras, in order to quotient by a prime p such that $\dim X < p$. To do this, one must first know an arbitrary field k is indeed the inductive limit of $\mathbb{Z}$ algebras.

Lemma 5.1. *Suppose k is a field. Then there exists a system $(A_i)_{i\in I}$ of $\mathbb{Z}$-algebras of finite type such that for each $i \in I$, $A_i \subseteq k$, and*

$$\varinjlim A_i = k.$$

Proof. Let Γ be the collection of all $\mathbb{Z}$-algebras of finite type contained in k. This is a directed poset under inclusion. If we have two finite type $\mathbb{Z}$-algebras $A = \mathbb{Z}[a_1,\dots,a_n]$ and $B = \mathbb{Z}[b_1,\dots,b_m]$ then $A, B \subseteq \mathbb{Z}[a_1,\dots,a_n,b_1,\dots.,b_m]$, so any pair of elements in Γ have an upper bound. This means Γ is a directed set. Furthermore, if $A \subseteq B$, we may define $f_{A,B} : A \to B$ to be the inclusion map. The map $f_{A,A} : A \to A$ is the identity on A, and for $A \subseteq B \subseteq C$, $f_{B,C} \circ f_{A,B} = f_{A,C}$, so Γ is a direct system. Certainly $\varinjlim\{A\}_{A\in\Gamma} \subseteq k$, since $A \subseteq k$ for all $A \in \Gamma$. Furthermore, for each $x \in k$, $\mathbb{Z}[x]$ is a finite $\mathbb{Z}$-algebra, meaning it must be contained in Γ. This gives us $x \in \varinjlim\{A\}_{A\in\Gamma}$, and we conclude $\varinjlim\{A\}_{A\in\Gamma} = k$. □

Having established this lemma, we now sketch Illusie's proof.

Theorem 5.2 (Illusie, Hodge Degeneration Theorem)**.** *Let k be a field of characteristic zero, and X a smooth and proper k-scheme. Then the Hodge spectral sequence of X over k,*

$$E_1^{ab} = H^b(X, \Omega^a_{X/k}) \Rightarrow H^\bullet_{dR}(X/k)$$

degenerates at E_1.

Proof. Set $\dim_k H^j(X, \Omega^b_{X/k}) = h^{ab}$, $\dim H^n_{dR}(X/k) = h^n$. It suffices to prove that for all n, $h^n = \sum_{a+b=n} h^{ab}$ by Lemma 3.4. Applying Lemma 5.1, we may write k as the limit of a system $(A_i)_{i\in I}$ where each A_i is a $\mathbb{Z}$-algebra of finite type which lives in k. According to [Ill96, 6.3], there is some $\alpha \in I$ as well as a smooth and proper S_α-scheme X_α (where $S_\alpha = \operatorname{Spec} A_\alpha$) for which X is induced by base change $\operatorname{Spec} k \to S_\alpha$. Write $A = A_\alpha$ and $S = S_\alpha$. If $Z = \operatorname{Spec} A[1/N]$ for suitably large N, one may choose a point s of S for which the residue field $k = k(s)$ is a finite field of characteristic $p > \dim X$. One may show that the dimension of cohomology is preserved, so that

$$\sum_{i+j=\ell} \dim_k H^j(X, \Gamma^i_{X/k}) = \dim_k H^\ell_{dR}(X/k),$$

and conclude by Lemma 3.7. □

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