Higher order extensions of the Boltzmann equation

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DISSERTATION
Presented to the Faculty of the Graduate School of
The University of Texas at Austin
in Partial Fulfillment
of the Requirements
for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT AUSTIN
August 2020
Dedicated to my beloved grandma Elpida who peacefully left this world on 2/11/2020 to start her eternal journey
Acknowledgments

There have been many people that without their help I would not be able to complete this academic journey. First and most important of all, I am grateful to my advisor Nataša Pavlović for the inspiration and support she provided throughout my studies at UT Austin. Besides her tremendous mathematical knowledge and charisma, her unbelievable integrity and character set her as an example for my development. Her mentorship and all I learned from her will be a stalemate in my professional and personal life. I would also like to thank my collaborators I. M. Gamba and Maja Tasković, with whom Part II of the dissertation is joint. Besides being productive, my collaboration and interaction with them was very inspiring, providing me a deep insight in kinetic theory and PDE. I would also like to thank Thomas Chen, Luis Caffarelli, Philip Morrison, Alexis Vasseur, Bob Strain, Ryan Denlinger for all the interesting conversations we had during class or private communication. Their interest in my progress and research is something I am really grateful for. Many thanks to the UT administrative staff Elisa Armendariz and Jenny Kondo for their outstanding support throughout all these years. Finally, I greatly appreciate financial support from the NSF grants DMS-1516228, DMS-1840314 and DMS-2009549.

In a more personal level, thank you Haley, Jason and Georgia for being there during this journey. Last but not least, I want to thank from the deeps of my heart my family, my parents Stavros and Dimitra and my grandparents Panagiotis and
Elpida, for making this whole distance feel so much shorter. Thank you for growing me up to be the person I am.
Higher order extensions of the Boltzmann equation

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The University of Texas at Austin, 2020

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This dissertation investigates extensions of the Boltzmann equation to higher order interactions and consists of two parts, which are submitted separately for publication, see [6, 4].

In the first part of the dissertation, we present a rigorous derivation of a novel kinetic equation, which we call ternary Boltzmann equation, describing the limiting behavior of a classical system of particles with three particle instantaneous interactions. Derivation of such an equation required development of new conceptual and geometrical ideas to treat interactions among three particles and their evolution in time. We also show that a symmetrized version of the ternary Boltzmann equation has the same conservation laws and entropy production properties as the classical binary operator. The superposition of this ternary equation with the classical Boltzmann equation, which we call the binary-ternary Boltzmann equation, could be understood as a step towards modeling a dense gas in non-equilibrium, since both binary and ternary interactions between particles are taken into account.
In the second part of the dissertation, we show global well-posedness near vacuum for the binary-ternary Boltzmann equation for monoatomic gases with a wide range of hard and soft potentials. Well-posedness of the ternary equation for these potentials follows as a special case. This is the first global well-posedness result for the binary-ternary Boltzmann equation and for the ternary Boltzmann equation. To prove global well-posedness, we implement a Kaniel-Shinbrot iteration and related works to the ternary correction of the Boltzmann equation to approximate the solution of the nonlinear equation by monotone sequences of supersolutions and subsolutions which converge, for small initial data, to the global in time solution of the binary-ternary equation. This analysis required establishing new convolution type estimates to control the contribution of the ternary collisional operator to the model. We show that the ternary operator allows consideration of softer potentials than the binary operator, consequently our solution to the ternary correction of the Boltzmann equation preserves all the properties of the binary interactions solution.
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Chapter 1

Beyond the Boltzmann equation

The Boltzmann equation is the central equation of collisional kinetic theory. It is a nonlinear integro-differential equation giving the statistical description of a dilute monoatomic gas in non-equilibrium in $\mathbb{R}^d$, for $d \geq 2$. It is given by

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = Q_2(f, f), & (t, x, v) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d, \\ f(0, x, v) = f_0(x, v), & (x, v) \in \mathbb{R}^d \times \mathbb{R}^d, \end{cases} \quad (1.1)$$

where the unknown function $f : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ represents the probability density of finding a particle of the gas in position $x \in \mathbb{R}^d$, with velocity $v \in \mathbb{R}^d$, at time $t \geq 0$, and $f_0 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is the initial probability density of the gas. Without loss of generality, we assume that the particles are of equal mass $m = 1$. The expression $Q_2(f, f)$ on the right hand side of (1.1) is the collisional operator which is an appropriate quadratic integral operator acting on $f$, taking into account binary interactions of a pair of gas particles. It is given by

$$Q_2(f, f) = \int_{S_1^{d-1} \times \mathbb{R}^d} B_2(u, \omega) \left( f(t, x, v') f(t, x, v'_1) - f(t, x, v) f(t, x, v_1) \right) \, d\omega \, dv_1, \quad (1.2)$$

where

$$u := v_1 - v, \quad (1.3)$$

is the relative velocity of a pair of interacting particles with velocities $v, v_1 \in \mathbb{R}^d$ before the binary interaction with respect to an impact direction $\omega \in S_1^{d-1}$ representing the
rescaled relative position of the particles and

\[ v' := v + (\omega \cdot u)\omega, \]
\[ v'_1 := v_1 - (\omega \cdot u)\omega, \]  \hspace{1cm} (1.4)

are the outgoing velocities after the binary interaction.

One can easily verify that the binary energy-momentum conservation system is satisfied in this case

\[ v' + v'_1 = v + v_1, \] \hspace{1cm} (1.5)
\[ |v'|^2 + |v'_1|^2 = |v|^2 + |v_1|^2. \] \hspace{1cm} (1.6)

Either (1.4) or (1.5)-(1.6) imply

\[ |u'| = |u|, \quad \text{where} \quad u' := v'_1 - v'. \] (1.7)

In addition, equation (1.4) yields the specular reflection with respect to the impact direction \( \omega \)

\[ \omega \cdot u' = -\omega \cdot u. \] (1.8)

In fact it is not hard to show that, given \( v, v_1 \in \mathbb{R}^d \), expression (1.4) provides the general solution of the system (1.5)-(1.6) parametrized by \( \omega \in \mathbb{S}^{d-1} \). The factor \( B_2 \) in the integrand of (1.2) is referred as the binary interaction differential cross-section which depends on relative velocity \( u \) and the impact direction \( \omega \). It expresses the statistical repartition of binary interactions, and its exact form depends on the type of binary interactions between particles. For more details on the binary collisional operator and its properties, see [16, 17].
Since the gas is assumed to be very dilute, interactions among three particles or higher order interactions are neglected due to much lower probability of occurring compared to binary. However, when the gas is dense enough, higher order interactions are much more likely to happen, therefore they produce a significant effect to the evolution of the gas and one needs to take them into consideration. An example of such a situation is a colloid, which is a homogeneous non-crystalline substance consisting of either large molecules or ultramicroscopic particles of one substance dispersed through a second substance. As pointed out in [56], interactions among three particles significantly contribute to the grand potential of a colloidal gas, therefore they play a crucial role in its evolution. A surprising but very important result of [56] is that interactions among three particles actually depend on the sum of the distances between particles, as opposed to depending on different geometric configurations among interacting particles. This observation is apparently of invaluable computational importance since it significantly simplifies numerical calculations on three particle interactions. The results of [56] have been further verified experimentally e.g. [26] and numerically e.g. [44].

1.1 The program introduced and the goal of this dissertation

Motivated by the observations of [56] and the fact that the Boltzmann equation is valid only for very dilute gases, we aim to introduce, and rigorously derive from systems of finitely many particles, a kinetic model which goes beyond binary interactions. In particular, our long term goal is to incorporate a sum of higher order interaction terms in (1.1), so that the new equation gives a more accurate description
of denser gases in non-equilibrium. Such an equation would be of the form

\[
\begin{aligned}
\frac{\partial f}{\partial t} + v \cdot \nabla_x f &= \sum_{k=2}^{m} Q_k(f, f, \cdots, f), \quad (t, x, v) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d, \\
f(0, x, v) &= f_0(x, v), \quad (x, v) \in \mathbb{R}^d \times \mathbb{R}^d,
\end{aligned}
\] (1.9)

where, for \( k = 1, \ldots, m \), the expression \( Q_k(f, f, \cdots, f) \) is the \( k \)-th order collisional operator and \( m \in \mathbb{N} \) is the accuracy of the approximation depending on the density of the gas. Notice that for \( m = 2 \), equation (1.9) reduces to the classical Boltzmann equation (1.1).

We note that equations similar to (1.9) were studied for Maxwell molecules in the works of Bobylev, Gamba and Cercignani [9, 8] using Fourier transform methods.

Also, we note that attempts for generalization of the Boltzmann equation to denser gases using formal density expansions were made by physicists in the past, see e.g. [23, 21, 38, 42, 58], but in a different context than ours. These attempts have not been further developed since they lead to divergence of the fourth order term and higher.

The task of rigorously deriving an equation of the form (1.9) from a classical many particle system, even for the case \( m = 2 \), is a challenging problem that has been settled only in certain situations; for hard sphere interactions, the analysis was pioneered by Lanford [49] and recently completed by Gallagher, Saint-Raymond, Texier [33], while for short-range potentials, it has been done in [48, 33, 55]. Up to our knowledge, the case \( m = 3 \) i.e. the equation

\[
\begin{aligned}
\frac{\partial f}{\partial t} + v \cdot \nabla_x f &= Q_2(f, f) + Q_3(f, f, f), \quad (t, x, v) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d, \\
f(0, x, v) &= f_0(x, v), \quad (x, v) \in \mathbb{R}^d \times \mathbb{R}^d,
\end{aligned}
\] (1.10)
has not been studied at all. We refer to it as the binary-ternary Boltzmann equation. In addition to understanding binary interactions and interactions among three particles, derivation of (1.10) requires careful analysis of their mutual interactions. This challenging task is a work in progress [5] and requires a deep understanding of interactions between three particles and their connection to binary interactions. For this reason, in the first part of this dissertation, we focus on understanding interactions among three particles and rigorously deriving a purely ternary equation, which itself brings a lot of challenges due to combinatorial and configurational intricacies of evolving in time interactions among three particles. We derive an equation of the form

\[
\begin{cases}
\partial_t f + v \cdot \nabla_x f = Q_3(f, f, f), & (t, x, v) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d, \\
f(0, x, v) = f_0(x, v), & (x, v) \in \mathbb{R}^d \times \mathbb{R}^d,
\end{cases}
\]

(1.11)

where \(Q_3(f, f, f)\) is the ternary collisional operator which is an integral operator of cubic order in \(f\). We refer to (1.11) as the ternary Boltzmann equation. The rigorous derivation of the ternary equation (1.11), which required development of new conceptual and geometrical ideas, is presented in Part I of the dissertation. We also note that Part I of the dissertation is a basis for a paper submitted for publication [6] with Pavlović.

Motivated by the work in progress [5] towards the rigorous derivation of the binary-ternary Boltzmann equation (1.10), in the second part of this dissertation, we investigate well-posedness of the equation (1.10). This part of the dissertation is based on a joint work with Gamba, Pavlović and Tasković [4]. We show global
well-posedness near vacuum in space-velocity Maxwellian-weighted $L^\infty$ spaces of non-negative functions. To achieve that, we adapt the method of solving the Boltzmann equation developed by Kaniel-Shinbrot [47] and Illner-Shinbrot [46] to the superposition of the binary and ternary collisional operators. This analysis requires establishment of appropriate new convolution type estimates for the ternary operator.

### 1.2 Organization of the dissertation

Let us briefly describe the main structure of the dissertation.

**Part I**

- Chapter 2 serves as an introduction to the first part of the dissertation.

- In Chapter 3 we define the collisional transformation of three interacting particles. This will be the law under which interacting particles instantaneously transform velocities after an interaction.

- In Chapter 4 we define the appropriate phase space which models ternary interactions. We then establish a measure-preserving interaction flow on this phase space and a Liouville-type equation. This is our first main result. Existence of the interaction flow is crucial because it guarantees that almost all configurations evolve “well” in time.

- In Chapter 5 we derive the BBGKY hierarchy after integrating by parts the Liouville equation. We then formally take the limit under the scaling (2.8) to heuristically derive the Boltzmann hierarchy. In the case of factorized initial
data, we reduce the Boltzmann hierarchy to the ternary Boltzmann equation, and study some of its main properties.

- In Chapter 6, we define mild solutions to the BBGKY hierarchy and Boltzman hierarchy and the ternary Boltzmann equation in appropriate exponentially weighted $L^\infty$-spaces. We then derive some a-priori bounds to show local well posedness for both hierarchies and the ternary equation. In the special case of factorized initial data, we show that the Boltzmann hierarchy reduces to the ternary Boltzmann equation.

- In Chapter 7, we give the definition of convergence in observables, and introduce the necessary vocabulary to address the convergence question.

- In Chapter 8, we use the a-priori bounds developed in Chapter 6 to reduce the convergence proof to term by term convergence of the corresponding BBGKY hierarchy and Boltzmann hierarchy series expansions terms.

Chapters 9-12 are devoted to the elimination of recollisions and the convergence proof and are the heart of our contribution regarding this part.

- In Chapter 9, we develop the necessary geometric tools to eliminate recollisions. Many of these estimates are new, and are inspired by the geometric nature of ternary interactions.

- In Chapter 10, we subtract a pathological set of initial data such that the
backwards interaction flow and the backwards free flow coincide on the complement. We then employ the geometric results developed in Chapter 9 to show that the pathological set subtracted is of small measure up to some truncation parameters.

- In Chapters 11-12, we conclude the convergence proof. In particular, in Chapter 11 we eliminate recollisions, using results from Chapter 10, and in Chapter 12 we combine results from Chapters 8, 11 to prove Theorem 7.4.1.

**Part II**

- Chapter 13 serves as an introduction to the second part of the dissertation.

- In Chapter 14, we decompose the binary and ternary collisional operators into gain and loss form. We then introduce the functional spaces used and state our main well-posedness result (Theorem 14.3.1).

- In Chapter 15, we prove the convolution estimate and derive essential a-priori bounds for the gain and loss operators.

- In Chapter 16, we inductively construct monotone sequences of supersolutions and subsolutions which are shown to converge to a common limit which solves the binary-ternary Boltzmann equation (13.1), as long as a beginning condition is satisfied.

- In Chapter 17 we provide the proof of our main well-posedness result (Theorem 14.3.1).
1.3 Notation

For convenience, we introduce some basic notation which will be frequently used throughout the dissertation:

- $d \in \mathbb{N}$ will be a fixed dimension with $d \geq 2$.

- Given $x, y \in \mathbb{R}$, we write
  \[
  x \lesssim y \iff \exists C_d > 0 : x \leq C_d y,
  \]
  \[
  x \simeq y \iff \exists C_d > 0 : x = C_d y,
  \]
  \[
  x \approx y \iff \exists C_{1,d}, C_{2,d} > 0 : C_{1,d} y \leq x \leq C_{2,d} y.
  \]

  In particular, we write
  \[
  x \approx_{C_{1,d}, C_{2,d}} y \iff C_{1,d} y \leq x \leq C_{2,d} y.
  \]

- Given $n \in \mathbb{N}$ and $A \subseteq \mathbb{R}^n$, Lebesgue measurable, we write $|A|^n$ for the $n$-dimensional Lebesgue measure of $A$. Given a hypersurface $S \subseteq \mathbb{R}^n$ and a measurable $A \subseteq S$, we write $|A|_S$ for the surface measure of $S$ induced by the Lebesgue measure in $\mathbb{R}^n$.

- Given $n \in \mathbb{N}$, $\rho > 0$ and $w \in \mathbb{R}^n$, we write $B^n_\rho(w)$ for the $n$-closed ball of radius $\rho > 0$, centered at $w \in \mathbb{R}^n$

  \[
  B^n_\rho(w) = \{x \in \mathbb{R}^n : |x - w| \leq \rho\}.
  \]

  In particular, we write
  \[
  B^n_\rho := B^n_\rho(0),
  \]

  for the $\rho$-ball centered at the origin.
Given $n \in \mathbb{N}$ and $\rho > 0$, we write $S^{n-1}_\rho$ for the $(n-1)$-sphere of radius $\rho > 0$:

$$S^{n-1}_\rho = \{ x \in \mathbb{R}^n : |x| = \rho \}.$$ 

When we write $x \ll y$, we mean that there is a small enough constant $c > 0$, independent of $x, y$, such that $x < cy$. This constant $c$ is appropriately chosen for the calculations to make sense.

We use the following notation for the pull-back of a function. Given sets $X, Y \neq \emptyset$, a function $\Psi : X \to Y$ and $B \subseteq Y$ we write

$$[\Psi \in B] := \Psi^{-1}(B) = \{ x \in A : \Psi(x) \in B \},$$

for the pullback of $B$ under $\Psi$. 

10
Part I

Rigorous derivation of a ternary Boltzmann equation for a classical system of particles
Chapter 2

Introduction to the derivation of the ternary Boltzmann equation

As mentioned in Chapter 1, the goal of this first part of the dissertation is the rigorous derivation of a purely ternary equation of the form (1.11) which takes into account interactions among three particles. The results obtained in this direction are submitted for publication [6] in a joint work with Pavlović.

We start by clarifying what we mean by a ternary interaction.

2.1 Ternary interactions

In a typical, dilute hard-sphere gas, the probability of a simultaneous contact of three hard-spheres is very small compared to e.g. the situation when one of the three particles is in simultaneous contact with the other two particles. Motivated by this observation and the fact that, according to [56], interactions among three particles are determined by the sum of the distances of the interacting particles, we introduce the notion of an interaction of three particles based on a non-symmetric version of a ternary distance. More precisely, we introduce the ternary distance:

\[ d(x_1; x_2, x_3) := \sqrt{|x_1 - x_2|^2 + |x_1 - x_3|^2}, \quad x_1, x_2, x_3 \in \mathbb{R}^d. \]  

(2.1)
Having defined the ternary distance, we introduce the notion of a ternary interaction. Let $\epsilon > 0$ and consider three particles $i, j, k$ with positions and velocities $(x_i, v_i), (x_j, v_j), (x_k, v_k) \in \mathbb{R}^{2d}$. We say that the particles $i, j, k$ are in $(i; j, k)$ ternary $\epsilon$-interaction if the following geometric condition holds:

$$d^2(x_i; x_j, x_k) = |x_i - x_j|^2 + |x_i - x_k|^2 = 2\epsilon^2. \quad (2.2)$$

The parameter $\epsilon$ above is called interaction zone. The $i$-th particle is called the central collisional particle, while the particles $j, k$ are called adjacent collisional particles.

Heuristically speaking, an $(i; j, k)$ interaction expresses the interaction of the central particle $i$ with the pair of the uncorrelated adjacent particles $(j, k)$ with respect to the interaction zone $\epsilon$. By uncorrelated, we mean that particles $j, k$ are not directly affected by each other. For example, Figure 2.1 shows particles that are not in ternary interaction, while Figure 2.2 offers two examples of particles which

![Figure 2.1](image)

are in ternary interaction.

\footnote{when not ambiguous, we will refer to $(i; j, k)$ ternary $\epsilon$-interaction as $(i; j, k)$ interaction.}
Figure 2.2

Let us now describe how velocities instantaneously transform when a ternary interaction happens. Consider an \((i; j, k)\) ternary \(\epsilon\)-interaction. Let \(v_i^*\), \(v_j^*\), \(v_k^*\) denote the velocities of the interacting particles after the interaction. Assuming the particles are of equal mass \(m = 1\), we consider the interaction to be elastic i.e. the three particle momentum-energy conservation system is satisfied:

\[
v_i^* + v_j^* + v_k^* = v_i + v_j + v_k, \tag{2.3}
\]

\[
|v_i^*|^2 + |v_j^*|^2 + |v_k^*|^2 = |v_i|^2 + |v_j|^2 + |v_k|^2. \tag{2.4}
\]

Now we introduce the relative positions re-scaled vectors \((\tilde{\omega}_1, \tilde{\omega}_2) := \left(\frac{x_j - x_i}{\sqrt{2} \epsilon}, \frac{x_k - x_i}{\sqrt{2} \epsilon}\right)\).

Notice that (2.2) implies \((\tilde{\omega}_1, \tilde{\omega}_2) \in S_{1}^{2d-1}\) i.e. \(|\tilde{\omega}_1|^2 + |\tilde{\omega}_2|^2 = 1\). We shall call the vectors \(\tilde{\omega}_1, \tilde{\omega}_2\) impact directions of the interaction. Since the \(i\) particle interacts with the pair of uncorrelated particles \((j, k)\), we assume the velocities \(v_j, v_k\) transform with respect to the impact directions unit vector i.e.

\[
\begin{pmatrix}
  v_j^* \\
  v_k^*
\end{pmatrix}
= \begin{pmatrix}
  v_j \\
  v_k
\end{pmatrix} - c \begin{pmatrix}
  \tilde{\omega}_1 \\
  \tilde{\omega}_2
\end{pmatrix}, \tag{2.5}
\]

for some \(c \in \mathbb{R}\). We note that once we added condition (2.5) to the system (2.3)-(2.4), the new system has a unique solution that algebraically characterizes the conservation of momentum and energy for the type of ternary interaction defined in

\footnote{we note that (2.5) is the ternary analogue of the condition that appears when one considers binary interactions, see e.g. [33].}
It is straightforward to verify that (2.3)-(2.5) yield that $v_i^*, v_j^*, v_k^*$ are given by the collisional formulas presented in (3.4).

### 2.2 Phase space, existence of a flow and the Liouville equation

Now we are ready to describe the evolution of a system of $N$-particles of $\epsilon$-interaction zone. Recall that in this paper we pursue only ternary interactions analysis, thus the phase space will take into account only those.

**Definition 2.1.** Let $d \in \mathbb{N}$, with $d \geq 2$, $N \in \mathbb{N}$ and $\epsilon > 0$. The phase space of the $N$-particle system of $\epsilon$-interaction zone is defined as:

$$\mathcal{D}_{N,\epsilon} = \left\{ Z_N = (X_N, V_N) \in \mathbb{R}^{2dN} : d^2(x_i; x_j, x_k) \geq 2\epsilon^2 \quad \forall (i, j, k) \in I_N \right\}, \quad (2.6)$$

where

$$d(x_i; x_j, x_k) = \sqrt{|x_i - x_j|^2 + |x_i - x_k|^2},$$

$X_N = (x_1, ..., x_N) \in \mathbb{R}^{dN}$, $V_N = (v_1, ..., v_N) \in \mathbb{R}^{dN}$, represent the positions and velocities of the $N$-particles, and $I_N$ is the index set

$$I_N = \left\{(i, j, k) \in \{1, ..., N\}^3 : i < j < k \right\}.$$

**Remark 2.1.** The phase space (2.6) will produce the kinetic equation (2.9), in which the tracked particle is always the central particle of the interactions occurring. Alternatively, by working in the phase space

$$\tilde{\mathcal{D}}_{N,\epsilon} = \left\{ Z_N = (X_N, V_N) \in \mathbb{R}^{2dN} : d^2_l(x_i, x_j, x_k) \geq 2\epsilon^2, \quad \forall (i, j, k, l) \in \tilde{I}_N \right\}, \quad (2.7)$$
where

$$\tilde{J}_N = \{(i,j,k,l) : (i,j,k) \in J_N \text{ and } l : \{i,j,k\} \rightarrow \{i,j,k\} \text{ is a permutation}\},$$

and using similar arguments as in Part I of this dissertation, one can derive a symmetrized version of (2.9) (see (5.55)), in which the tracked particle can be either central or adjacent. For simplicity, we opt to work with the phase space (2.6). However, we would like to mention that all intermediate results needed for the derivation of the symmetrized ternary equation can be obtained after some minor changes (see Remark 4.5 for more details).

Let us now describe the evolution in time of a system of particles in the phase space (2.6). Consider an initial configuration $Z_N \in \mathcal{D}_{N,\epsilon}$. The motion is described as follows:

(I) Particles are assumed to perform rectilinear motion as long as there is no interaction i.e.

$$\dot{x}_i = v_i, \quad \dot{v}_i = 0, \quad \forall i \in \{1, ..., N\}$$

(II) Assume now that an initial configuration $Z_N = (X_N, V_N)$ has evolved until time $t > 0$, reaching $Z_N(t) = (X_N(t), V_N(t))$, and there is an $(i;j,k)$ interaction at time $t$. Then the velocities $(v_i(t), v_j(t), v_k(t))$ instantaneously transform to $(v^*_i(t), v^*_j(t), v^*_k(t))$.

We remark that it is not at all obvious that (I)-(II) produce a well defined dynamics, since the evolution is not smooth in time, and the system can possibly
run into pathological configurations. In the case of binary interactions, the analogous result has been established in the work of Alexander [1], but that work does not apply to the notion of ternary interactions used here.

We informally state the first main result of this paper, for a rigorous statement see Theorem 4.9.1.

Existence of a global flow: Let \( m \in \mathbb{N} \) and \( 0 < \sigma << 1 \). There is a global in time measure-preserving flow \((\Psi^t_m)_{t \in \mathbb{R}} : D_{m,\sigma} \rightarrow D_{m,\sigma}\) which preserves kinetic energy. This flow is called the \( \sigma \)-interaction zone flow of \( m \)-particles or simply the interaction flow.

The main difficulty in proving Theorem 4.9.1 is the elimination of configurations following pathological trajectories in time. In particular, in order to go from local to global in time flow we establish the following crucial fact - when an \((i; j, k)\) interaction happens, then the subsequent interaction cannot involve the same triplet of particles. This observation enables us to develop ellipsoidal coverings and new geometric estimates to control the measure of these pathological sets.

The global measure-preserving interaction flow established yields a Liouville equation (see (4.60)) for the evolution \( f_N \) of an initial \( N \)-particle of \( \epsilon \)-interaction zone probability density \( f_{N,0} \).

### 2.3 The ternary equation derived

Although Liouville’s equation is a linear transport equation, efficiently solving it is almost impossible in case where the particle number \( N \) is very large. This is
why an accurate statistical description is welcome, and to obtain it one wants to understand the limiting behavior of it as $N \to \infty$ and $\epsilon \to 0^+$, with the hope that qualitative properties will be revealed for a large but finite $N$. Letting the number of particles $N \to \infty$ and the interaction zone $\epsilon \to 0^+$ in the new scaling:

$$N \epsilon^{d-\frac{1}{2}} \simeq 1,$$

we derive the ternary Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f = Q_3(f, f, f), \quad (t, x, v) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d,$$

The expression $Q_3(f, f, f)$ is the ternary cubic order collisional operator, given by:

$$Q_3(f, f, f) = \int_{\mathbb{R}^{2d-1}} b_+(f^* f_1^* f_2^* - f f_1 f_2) \, d\omega_1 \, d\omega_2 \, dv_1 \, dv_2,$$

where

$$b = b(\omega_1, \omega_2, v_1 - v, v_2 - v) := \langle \omega_1, v_1 - v, v_2 - v \rangle, \quad b_+ = \max\{b, 0\},$$

$$f^* = f(t, x, v^*), \quad f = f(x, t, v), \quad f_1^* = f_1^*(t, x, v_1^*), \quad f_i = f(t, x, v_i), \quad \text{for} \ i \in \{1, 2\}.$$

It is important to point out that the symmetrized version (5.55) of the ternary equation (2.9), which can be derived from the phase space (2.7), enjoys similar statistical and entropy production properties and conservation laws as the classical Boltzmann equation, see Section 5.4 for more details.

### 2.4 Strategy of the derivation and statement of the main result

Now the natural question is: how do we pass from the $N$-particle dynamics to the kinetic equation (2.9)? We implement the program pioneered by Lanford [49]
and recently refined by Gallagher, Saint-Raymond, Texier [33] for deriving, for short times, the classical Boltzmann equation (1.1) for hard-spheres in the Boltzmann-Grad scaling $N\epsilon^{d-1} \simeq 1$. This program has been implemented in the case of short range potentials too e.g. [48, 33, 55]. However, to the best of our knowledge, the program has not been explored outside of the context of binary interactions. By generalizing the program to allow consideration of ternary particle interactions, we illustrate that the program is universal enough. However to make it applicable to ternary interactions we follow evolution in time of ternary particle interactions, that inform new mathematical arguments described below.

We first derive a finite two-step coupled hierarchy of equations for the marginals densities of the solution to the Liouville equation, which we call the BBGKY hierarchy. We then formally let $N \to \infty$ and $\epsilon \to 0^+$ in the scaling (2.8) to obtain an infinite two-step coupled hierarchy of equations, which we call the Boltzmann hierarchy. It can be observed that for factorized initial data, the Boltzmann hierarchy reduces to the ternary Boltzmann equation (2.9). This observation connects the Boltzmann hierarchy with the ternary Boltzmann equation.

To make this argument rigorous, we first need to show that the BBGKY and Boltzmann hierarchy are well-posed, at least for short times, and then that if the BBGKY initial data converge to the Boltzmann hierarchy initial data, then this convergence propagates in time in the scaling (2.8). Local well-posedness is shown

\footnote{the two-step refers to the coupling between the $k$-th element of the hierarchy and the $(k+2)$-th element of the hierarchy.}

\footnote{Bogoliubov, Born, Green, Kirkwood, Yvon}
in Chapter 6, see Theorem 6.4.1, Theorem 6.8.1. Showing convergence is a very challenging task and is the heart of our contribution. We informally state our main result here. For a rigorous statement of the result see Theorem 7.4.1.

**Statement of the main result:** Let \( F_0 \) be initial data for the Boltzmann hierarchy, and \( F_{N,0} \) be some BBGKY hierarchy initial data which “approximate” \( F_0 \) as \( N \to \infty, \epsilon \to 0^+ \) under the scaling (2.8). Let \( F_N \) be the solution to the BBGKY hierarchy with initial data \( F_{N,0} \), and \( F \) the solution to the Boltzmann hierarchy, with initial data \( F_0 \), up to short time \( T > 0 \). Then \( F_N \) converges in observables to \( F \) in \([0,T]\) as \( N \to \infty, \epsilon \to 0^+ \), under the scaling (2.8).

The proof of this result is achieved by repeatedly using Duhamel’s formula for the finite and infinite hierarchy respectively and comparing the corresponding series expansions. However this a delicate point because of the divergences of the finite particle flow and the free flow, due to the ternary interactions of particles in the finite particle case. The problem of divergence is present in the derivation of the classical Boltzmann equation as well, see [49, 33], but our case is significantly harder due the complexity of ternary interactions. To overcome this problem, we develop new geometric and combinatorial estimates, that help us extract small measure sets of initial data which lead to these diverging trajectories. In particular the main difficulty is to control post-collisional configurations and it requires completely new treatment. To achieve that, we need to explicitly calculate the Jacobian of ternary interactions with respect to impact directions, and estimate the surface measure of sets of the form \((K_\rho^d \times \mathbb{R}^d) \cap S\), where \( K_\rho^d \) is a \( d \)-dimensional solid cylinder of radius
\( \rho \) and \( S \) is an appropriate ellipsoid in \( \mathbb{R}^{2d} \). These results are thoroughly presented in Chapter 9.
Chapter 3

Collisional transformation of three particles

In this first chapter, we define the collisional transformation of three particles, induced by a given pair of impact directions, and investigate its properties. The collisional transformation will be the law under which the velocities \((v_1, v_2, v_3)\) of three interacting particles, with impact directions \((\omega_1, \omega_2) \in S^{2d-1}_1\), instantaneously transform. The impact directions will in general represent the re-scaled relative positions of the particles. We also prove that the collisional transformation provides the general solution of the three particle momentum-energy conservation system, parametrized by the impact directions.

For convenience, given \((\omega_1, \omega_2, v_1, v_2, v_3) \in S^{2d-1}_1 \times \mathbb{R}^3\), let us write

\[
c_{\omega_1, \omega_2, v_1, v_2, v_3} = \frac{\langle \omega_1, v_2 - v_1 \rangle + \langle \omega_2, v_3 - v_1 \rangle}{1 + \langle \omega_1, \omega_2 \rangle}.
\]

Notice that \(c_{\omega_1, \omega_2, v_1, v_2, v_3}\) is well-defined for all \((\omega_1, \omega_2, v_1, v_2, v_3) \in S^{2d-1}_1 \times \mathbb{R}^3\), since

\[
1 + \langle \omega_1, \omega_2 \rangle \geq 1 - |\omega_1||\omega_2| \geq 1 - \frac{1}{2}(|\omega_1|^2 + |\omega_2|^2) = \frac{1}{2}.
\]

**Definition 3.1.** Consider impact directions \((\omega_1, \omega_2) \in S^{2d-1}_1\). We define the collisional transformation induced by \((\omega_1, \omega_2) \in S^{2d-1}_1\) as

\[
T_{\omega_1, \omega_2} : (v_1, v_2, v_3) \in \mathbb{R}^3 \longrightarrow (v_1^*, v_2^*, v_3^*) \in \mathbb{R}^3,
\]
where
\[
\begin{align*}
v_1^* &= v_1 + c_{\omega_1,\omega_2,v_1,v_2,v_3}(\omega_1 + \omega_2), \\
v_2^* &= v_2 - c_{\omega_1,\omega_2,v_1,v_2,v_3}\omega_1, \\
v_3^* &= v_3 - c_{\omega_1,\omega_2,v_1,v_2,v_3}\omega_2,
\end{align*}
\]
and \(c_{\omega_1,\omega_2,v_1,v_2,v_3}\) is given by (3.1).

**Remark 3.1.** Notice that the collisional transformation \(T_{\omega_1,\omega_2} : \mathbb{R}^{3d} \rightarrow \mathbb{R}^{3d}\) crucially depends on the choice of the impact directions \((\omega_1, \omega_2) \in S^{2d-1}_1\). In general different impact directions induce different collisional transformations.

In the following definition, we introduce the notion of the cross-section which will have a prominent role in the rest of this part of the dissertation.

**Definition 3.2.** We define the cross-section \(b : S^{2d-1}_1 \times \mathbb{R}^{2d} \rightarrow \mathbb{R}\) as:
\[
b(\omega_1, \omega_2, \nu_1, \nu_2) = \langle \omega_1, \nu_1 \rangle + \langle \omega_2, \nu_2 \rangle, \quad (\omega_1, \omega_2, \nu_1, \nu_2) \in S^{2d-1}_1 \times \mathbb{R}^{2d}.
\]

**Remark 3.2.** Notice that, given \((\omega_1, \omega_2, v_1, v_2, v_3) \in S^{2d-1}_1 \times \mathbb{R}^{3d}\), we have
\[
b(\omega_1, \omega_2, v_2 - v_1, v_3 - v_1) = (1 + \langle \omega_1, \omega_2 \rangle) c_{\omega_1,\omega_2,v_1,v_2,v_3}.
\]

For convenience, given \((\omega_1, \omega_2, v_1, v_2, v_3) \in S^{2d-1}_1 \times \mathbb{R}^{3d}\), we will write
\[
c = c_{\omega_1,\omega_2,v_1,v_2,v_3}, \quad b = b(\omega_1, \omega_2, v_2 - v_1, v_3 - v_1),
\]
\[
c^* = c_{\omega_1,\omega_2,v_1^*,v_2^*,v_3^*}, \quad b^* = b(\omega_1, \omega_2, v_2^* - v_1^*, v_3^* - v_1^*).
\]

**Remark 3.3.** Notice that Cauchy-Schwartz inequality yields
\[
1 + \langle \omega_1, \omega_2 \rangle \approx 1, \quad \forall (\omega_1, \omega_2) \in S^{2d-1}_1.
\]

Indeed, we have
\[
|\langle \omega_1, \omega_2 \rangle| \leq |\omega_1||\omega_2| \leq \frac{1}{2}(|\omega_1|^2 + |\omega_2|^2) = \frac{1}{2}, \quad \forall (\omega_1, \omega_2) \in S^{2d-1}_1.
\]
thus
\[
\frac{1}{2} \leq 1 + \langle \omega_1, \omega_2 \rangle \leq \frac{3}{2}, \quad \forall (\omega_1, \omega_2) \in S_{1}^{2d-1}.
\]
Hence (3.6) yields
\[
b(\omega_1, \omega_2, v_2 - v_1, v_3 - v_1) \approx c_{\omega_1, \omega_2,v_1,v_2,v_3}, \quad \forall (\omega_1, \omega_2, v_1, v_2, v_3) \in S_{1}^{2d-1} \times \mathbb{R}^{3d}. \quad (3.9)
\]

Direct algebraic calculations illustrate the main properties of the collisional transformation.

**Proposition 3.1.** Consider a pair of impact directions $(\omega_1, \omega_2) \in S_{1}^{2d-1}$. The induced collisional transformation $T_{\omega_1, \omega_2}$ has the following properties:

(i) **Conservation of momentum**

\[
v_1^* + v_2^* + v_3^* = v_1 + v_2 + v_3. \quad (3.10)
\]

(ii) **Conservation of energy:**

\[
|v_1^*|^2 + |v_2^*|^2 + |v_3^*|^2 = |v_1|^2 + |v_2|^2 + |v_3|^2. \quad (3.11)
\]

(iii) **Conservation of relative velocities magnitude:**

\[
|v_1^* - v_2^*|^2 + |v_1^* - v_3^*|^2 + |v_2^* - v_3^*|^2 = |v_1 - v_2|^2 + |v_1 - v_3|^2 + |v_2 - v_3|^2. \quad (3.12)
\]

This is the ternary analog of (14.7). We also have
\[
\frac{\sqrt{3}}{3} |u| \leq |u^*| \leq \sqrt{3} |u|, \quad \forall v_1, v_2, v_3 \in \mathbb{R}^d, \quad (3.13)
\]

where
\[
u = \begin{pmatrix} v_2 - v_1 \\ v_3 - v_1 \end{pmatrix} \quad \text{and} \quad u^* = \begin{pmatrix} v_2^* - v_1^* \\ v_3^* - v_1^* \end{pmatrix}.
\]
(iv) Micro-reversibility of the cross-section:

\[ b(\omega_1, \omega_2, v_2^* - v_1^*, v_3^* - v_1^*) = -b(\omega_1, \omega_2, v_2 - v_1, v_3 - v_1). \] (3.14)

(v) \( T_{\omega_1, \omega_2} \) is a linear involution i.e. \( T_{\omega_1, \omega_2} \) is linear, and

\[ T_{\omega_1, \omega_2}^{-1} = T_{\omega_1, \omega_2}. \] (3.15)

In particular,

\[ |\det T_{\omega_1, \omega_2}| = 1, \] (3.16)

so \( T_{\omega_1, \omega_2} \) is measure-preserving.

Proof. (i) and (ii) are guaranteed by construction. Relation (3.12) of (iii) comes immediately after combining (i) and (ii). To prove (3.13), notice that

\[
|\mathbf{u}|^2 = |v_1 - v_2|^2 + |v_1 - v_3|^2
\leq |v_1 - v_2|^2 + |v_1 - v_3|^2 + |v_2 - v_3|^2
= |v_1^* - v_2^*|^2 + |v_1^* - v_3^*|^2 + |v_2^* - v_3^*|^2
\leq |v_1^* - v_2^*|^2 + |v_1^* - v_3^*|^2 + (|v_1^* - v_2^*| + |v_1^* - v_3^*|)^2
\leq 3|v_1^* - v_2^*|^2 + 3|v_1^* - v_3^*|^2
= 3|\mathbf{u}^*|^2. \] (3.17)

Similarly, a symmetric argument shows \(|\mathbf{u}^*|^2 \leq 3|\mathbf{u}|^2\) and (3.13) follows. To prove (iv), we use (3.4) to obtain

\[ v_2^* - v_1^* = v_2 - v_1 - 2c\omega_1 - c\omega_2, \]
\[ v_3^* - v_1^* = v_3 - v_1 - 2c\omega_2 - c\omega_1. \]
Pairing with $\omega_1$ and $\omega_2$ respectively, adding, using notation from (3.7) and the fact that $(\omega_1, \omega_2) \in S^{2d-1}$, we get

$$b^* = \langle \omega_1, v_2^* - v_1^* \rangle + \langle \omega_2, v_3^* - v_1^* \rangle = \langle \omega_1, v_2 - v_1 \rangle + \langle \omega_2, v_3 - v_1 \rangle - 2c (1 + \langle \omega_1, \omega_2 \rangle)$$

$$= b - 2b$$

$$= -b,$$

by (3.6). To prove $(v)$, first notice that $T_{\omega_1, \omega_2}$ is linear in velocities. Recalling notation from (3.7), $(iv)$ together with (3.7) imply that

$$c^* + c = 0.$$

Hence,

$$\begin{cases} v_1^{**} = v_1^* + c^* (\omega_1 + \omega_2) = v_1 + (c^* + c)(\omega_1 + \omega_2) = v_1, \\ v_2^{**} = v_2^* - c^* \omega_1 = v_2 - (c^* + c)\omega_1 = v_2, \\ v_3^{**} = v_3^* - c^* \omega_2 = v_3 - (c^* + c)\omega_2 = v_3, \end{cases}$$

which implies

$$T^{-1}_{\omega_1, \omega_2} = T_{\omega_1, \omega_2}.$$

Moreover,

$$|\det T_{\omega_1, \omega_2}| = 1,$$

so $T_{\omega_1, \omega_2}$ is measure-preserving.

In the rest of this chapter, we will show that the collisional transformation produces the general solution of the momentum-energy conservation system of $(v_1, v_2, v_3) \in \mathbb{R}^{3d}$ which is given by:

$$v'_1 + v'_2 + v'_3 = v_1 + v_2 + v_3,$$

$$|v'_1|^2 + |v'_2|^2 + |v'_3|^2 = |v_1|^2 + |v_2|^2 + |v_3|^2,$$
parametrized by the impact directions \((\omega_1, \omega_2) \in S^{2d-1}_1\). In (3.18)-(3.19), \((v_1', v_2', v_3') \in \mathbb{R}^{3d}\) represents a possible solution of the system. For this purpose, we define the set of solutions to the system (3.18)-(3.19)

\[
S_{v_1, v_2, v_3} := \{ (v_1', v_2', v_3') \in \mathbb{R}^{3d} \mid (v_1', v_2', v_3') \text{ satisfy the system given by (3.18)-(3.19)} \}.
\]

**Proposition 3.2.** Let \((v_1, v_2, v_3) \in \mathbb{R}^{3d}\). Then the solutions to the momentum-energy system (3.18)-(3.19) are given by

\[
S_{v_1, v_2, v_3} = \left\{ \begin{pmatrix} v_1 + c_{\omega_1, \omega_2, v_1, v_2, v_3} (\omega_1 + \omega_2) \\ v_2 - c_{\omega_1, \omega_2, v_1, v_2, v_3} \omega_1 \\ v_3 - c_{\omega_1, \omega_2, v_1, v_2, v_3} \omega_2 \end{pmatrix} : (\omega_1, \omega_2) \in S^{2d-1}_1 \right\}.
\]

**Proof.** For convenience, let us write

\[
A_{v_1, v_2, v_3} := \left\{ \begin{pmatrix} v_1 + c_{\omega_1, \omega_2, v_1, v_2, v_3} (\omega_1 + \omega_2) \\ v_2 - c_{\omega_1, \omega_2, v_1, v_2, v_3} \omega_1 \\ v_3 - c_{\omega_1, \omega_2, v_1, v_2, v_3} \omega_2 \end{pmatrix} : (\omega_1, \omega_2) \in S^{2d-1}_1 \right\}.
\]

A direct calculation shows that the triplet

\[
\begin{pmatrix} v_1 + c_{\omega_1, \omega_2, v_1, v_2, v_3} (\omega_1 + \omega_2) \\ v_2 - c_{\omega_1, \omega_2, v_1, v_2, v_3} \omega_1 \\ v_3 - c_{\omega_1, \omega_2, v_1, v_2, v_3} \omega_2 \end{pmatrix},
\]

is a solution to (3.18)-(3.19) i.e.

\[
S_{v_1, v_2, v_3} \supseteq A_{v_1, v_2, v_3}.
\]

Let us now show the opposite inclusion. Consider \((v_1', v_2', v_3') \in S_{v_1, v_2, v_3}\).

(i) Assume first that \((v_1', v_2', v_3') \neq (v_1, v_2, v_3)\). Conservation of momentum (3.18) implies that \((v_2', v_3') \neq (v_2, v_3)\). Therefore there exists \(c \neq 0\) and \((\omega_1, \omega_2) \in S^{2d-1}_1\) such that

\[
\begin{pmatrix} v_2' \\ v_3' \end{pmatrix} = \begin{pmatrix} v_2 \\ v_3 \end{pmatrix} - c \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.
\]  

(3.20)
Conservation of momentum then yields

\[
\begin{align*}
    v'_1 &= v_1 + c(\omega_1 + \omega_2), \\
    v'_2 &= v_2 - c\omega_1, \\
    v'_3 &= v_3 - c\omega_2,
\end{align*}
\]  

(3.21)

where \( c \neq 0 \). Substituting (3.21) into (3.19), and using the fact that \( c \neq 0 \), we obtain

\[
c = c_{\omega_1,\omega_2,v_1,v_2,v_3},
\]

given by (3.1). Therefore \((v'_1,v'_2,v'_3) \in \mathcal{A}_{v_1,v_2,v_3}\).

(ii) Assuming now that \((v'_1,v'_2,v'_3) = (v_1,v_2,v_3)\), we choose \((\omega_1,\omega_2) \in \mathbb{S}_1^{2d-1}\) such that

\[
\begin{bmatrix}
\omega_1 \\
\omega_2
\end{bmatrix} \perp \begin{bmatrix}
v_2-v_1 \\
v_3-v_1
\end{bmatrix} \iff c_{\omega_1,\omega_2,v_1,v_2,v_3} = 0.
\]

Notice that there are infinitely many \((\omega_1,\omega_2) \in \mathbb{S}_1^{2d-1}\) with this property, since the intersection of \(\mathbb{S}_1^{2d-1}\) with the hyperplane \( V = \left\{ \begin{bmatrix} v_2-v_1 \\ v_3-v_1 \end{bmatrix} \right\} \perp \) is an ellipse in \(\mathbb{R}^{2d}\).

That immediately implies \((v'_1,v'_2,v'_3) \in \mathcal{A}_{v_1,v_2,v_3}\).

Combining (i)-(ii) we obtain

\[
\mathbb{S}_{v_1,v_2,v_3} \subseteq \mathcal{A}_{v_1,v_2,v_3}.
\]

The result is proved. \(\square\)
Chapter 4

Dynamics of $m$-particles

In this chapter we rigorously define the dynamics of $m$-particles of small interaction zone $0 < \sigma << 1$. Heuristically speaking particles perform free motion as long as they are not interacting, and instantaneously transform velocities according to the collisional transformation, defined in Chapter 3, when they interact. Intuitively, the dynamics is well-defined as long as we have well-separated in time interactions, such that each of those interactions involves only one triplet. However it is far from obvious that such a dynamics can be globally defined.

To overcome this problem, we first define the flow up to the first collision time for a.e. configurations, and this flow is measure-preserving. Removing a small measure set, up to truncation parameters, of pathological initial configurations, we are showing that the motion can be continued up to the second collision time, the flow being measure-preserving. Inductively repeating this process, we define a global in time measure-preserving flow for all initial configurations except a small measure pathological set, up to the truncation. In the limit, it is shown that the measure of this pathological set is negligible, therefore a measure-preserving flow is established for almost any choice of initial configurations, see Theorem 4.9.1. The strategy of our proof is inspired by Alexander [1], where a measure-preserving flow is
a.e. established for the binary hard spheres interaction flow. In our case though, the proof is significantly more complicated due to the higher combinatorial and geometric complexity induced by ternary collisions.

Throughout this chapter we consider $m \in \mathbb{N}$ and $0 < \sigma << 1$.

### 4.1 Phase space definitions

Let $0 < \sigma << 1$. For $m \geq 3$, we define the $m$-index set of ordered triplets as:

$$I_m = \{(i, j, k) \in \{1, ..., m\}^3 : i < j < k\}.$$  \hspace{1cm} (4.1)

For $m \geq 3$, we define the phase space of $m$-particles of $\sigma$-interaction zone as

$$\mathcal{D}_{m,\sigma} = \{Z_m = (X_m, V_m) \in \mathbb{R}^{2dm} : d^2(x_i; x_j, x_k) \geq 2\sigma^2, \quad \forall (i, j, k) \in I_m\},$$  \hspace{1cm} (4.2)

where

$$X_m = (x_1, ..., x_m) \in \mathbb{R}^{dm}, \quad V_m = (v_1, ..., v_m) \in \mathbb{R}^{dm},$$

represent the positions and velocities of the $m$-particles respectively, and

$$d(x_i; x_j, x_k) = \sqrt{|x_i - x_j|^2 + |x_i - x_k|^2},$$  \hspace{1cm} (4.3)

is the distance in positions of the particles $i, j, k$.

For convenience we also define

$$\mathcal{D}_{1,\sigma} \equiv \mathbb{R}^{2d} \quad \text{and} \quad \mathcal{D}_{2,\sigma} \equiv \mathbb{R}^{4d}.$$  \hspace{1cm} (4.4)

Elements of $\mathcal{D}_{m,\sigma}$ are called configurations.
For \( m \geq 3 \), the phase space \( \mathcal{D}_{m,\sigma} \) decomposes to the interior

\[
\hat{\mathcal{D}}_{m,\sigma} = \left\{ Z_m = (X_m, V_m) \in \mathbb{R}^{2dm} : d^2(x_i; x_j, x_k) > 2\sigma^2, \quad \forall (i, j, k) \in \mathcal{I}_m \right\},
\]

and the boundary

\[
\partial \mathcal{D}_{m,\sigma} = \left\{ Z_m = (X_m, V_m) \in \mathcal{D}_{m,\sigma} : \exists (i, j, k) \in \mathcal{I}_m, \quad \text{with} \quad d^2(x_i; x_j, x_k) = 2\sigma^2 \right\}
= \bigcup_{(i,j,k)\in\mathcal{I}_m} \Sigma_{ijk},
\]

(4.6)

where \( \Sigma_{ijk} \) are the collisional surfaces given by:

\[
\Sigma_{ijk} = \left\{ Z_m = (X_m, V_m) \in \mathcal{D}_{m,\sigma} : d^2(x_i; x_j, x_k) = 2\sigma^2 \right\}.
\]

(4.7)

Elements of \( \hat{\mathcal{D}}_{m,\sigma} \) are called non-collisional configurations or just non-collisional, and elements of \( \partial \mathcal{D}_{m,\sigma} \) are called collisional configurations or just collisions.

We further decompose the boundary to simple collisions:

\[
\partial_{\mathrm{sc}} \mathcal{D}_{m,\sigma} = \left\{ Z_m = (X_m, V_m) \in \partial \mathcal{D}_{m,\sigma} : \text{there is unique} \ (i, j, k) \in \mathcal{I}_m : Z_m \in \Sigma_{ijk} \right\},
\]

and multiple collisions:

\[
\partial_{\mathrm{mc}} \mathcal{D}_{m,\sigma} = \left\{ Z_m = (X_m, V_m) \in \partial \mathcal{D}_{m,\sigma} : \text{there are} \ (i, j, k) \neq (i', j', k') \in \mathcal{I}_m : Z_m \in \Sigma_{ijk} \cap \Sigma_{i'j'k'} \right\}.
\]

Notice that in the special case \( m = 3 \), we have

\[
\partial_{\mathrm{mc}} \mathcal{D}_{3,\sigma} = \emptyset, \quad \partial \mathcal{D}_{3,\sigma} = \partial_{\mathrm{sc}} \mathcal{D}_{3,\sigma},
\]

i.e. there are no multiple collisions when we consider only three particles.
**Definition 4.1.** Let \( m \geq 3 \) and \( Z_m \in \partial_{sc} D_{m,\sigma} \). Then there is a unique triplet \((i, j, k)\) such that \( Z_m \in \Sigma_{ijk} \). In this case we will say that \( Z_m \) is an \((i; j, k)\) simple collision and we will write

\[
\Sigma_{ijk}^{sc} := \{Z_m = (X_m, V_m) \in \partial_{sc} D_{m,\sigma} : Z_m \text{ is } (i; j, k) \text{ simple collision}\}.
\]

(4.8)

**Remark 4.1.** Notice that

\[
\Sigma_{ijk}^{sc} \cap \Sigma_{i'j'k'}^{sc} = \emptyset, \quad \forall (i, j, k) \neq (i', j', k') \in I_m,
\]

and \( \partial_{sc} D_{m,\sigma} \) decomposes to:

\[
\partial_{sc} D_{m,\sigma} = \bigcup_{(i,j,k) \in I_m} \Sigma_{ijk}^{sc}.
\]

For the purposes of defining a global flow, throughout this chapter we use the following notation:

**Definition 4.2.** Let \((i, j, k)\) \( \in I_m \) and \( Z_m \in \Sigma_{ijk}^{sc} \). We introduce

\[
(\tilde{\omega}_1, \tilde{\omega}_2) := \frac{1}{\sqrt{2\sigma}} (x_j - x_i, x_k - x_i) \in S_1^{2d-1}.
\]

(4.9)

Therefore, each \((i; j, k)\) simple collision naturally induces impact directions \((\tilde{\omega}_1, \tilde{\omega}_2) \in S_1^{2d-1}\), and a collisional transformation \( T_{\tilde{\omega}_1, \tilde{\omega}_2} \).

We also give the following definition:

**Definition 4.3.** Let \( m \geq 3 \), \((i, j, k)\) \( \in I_m \) and \( Z_m = (X_m, V_m) \in \Sigma_{ijk}^{sc} \). We denote

\[
Z_m^* = (X_m, V_m^*),
\]
where

\[ V_m^* = (v_1, \ldots, v_{i-1}, v_i^*, v_{i+1}, \ldots, v_{j-1}, v_j^*, v_{j+1}, \ldots, v_{k-1}, v_k^*, v_{k+1}, \ldots, v_m), \]

and

\[ (v_i^*, v_j^*, v_k^*) = T_{\tilde{\omega}_1, \tilde{\omega}_2}(v_i, v_j, v_k), \quad (\tilde{\omega}_1, \tilde{\omega}_2) \in S_1^{2d-1} \] are given by (4.9).

### 4.2 Classification of simple collisions

It will be useful to classify simple collisions in order to eliminate collisions which graze under time evolution. For this purpose, we introduce the following language:

**Definition 4.4.** Let \( m \geq 3 \), \((i, j, k)\) \( \in \mathcal{I}_m \) and \( Z_m \in \Sigma_{ijk}^{sc} \). The configuration \( Z_m \) is called:

- pre-collisional when
  \[ b(\tilde{\omega}_1, \tilde{\omega}_2, v_j - v_i, v_k - v_i) < 0, \]
- post-collisional when
  \[ b(\tilde{\omega}_1, \tilde{\omega}_2, v_j - v_i, v_k - v_i) > 0, \]
- grazing when
  \[ b(\tilde{\omega}_1, \tilde{\omega}_2, v_j - v_i, v_k - v_i) = 0, \]

where \((\tilde{\omega}_1, \tilde{\omega}_2) \in S_1^{2d-1}\) is given by (4.9) and \( b \) is given by (3.5).
Remark 4.2. Let \( m \geq 3 \), \( (i,j,k) \in I_m \) and \( Z_m \in \Sigma_{ijk} \). Using (3.14), we obtain the following:

(i) \( Z_m \) is pre-collisional iff \( Z_m^* \) is post-collisional.

(ii) \( Z_m \) is post-collisional iff \( Z_m^* \) is pre-collisional.

(iii) \( Z_m = Z_m^* \) iff \( Z_m \) is grazing.

We refine the phase space according to:

\[
D_{m,\sigma}^* = \hat{D}_{m,\sigma} \cup \partial_{sc,ng} D_{m,\sigma},
\]

where

\[
\partial_{sc,ng} D_{m,\sigma} = \{ Z_m = (X_m, V_m) \in \partial_{sc} D_{m,\sigma} : Z_m \text{ is non-grazing} \},
\]

is the part of the boundary consisting of simple, non-grazing collisions. Notice that \( D_{m,\sigma}^* \) is a full measure subset of \( D_{m,\sigma} \) and \( \partial_{sc,ng} D_{m,\sigma} \) is a full surface measure subset of \( \partial D_{m,\sigma} \), since its complement constitutes of lower dimension submanifolds of \( \partial D_{m,\sigma} \).

4.3 Construction of the local flow

Heuristically speaking a configuration will evolve under the free flow as long as it is in \( \hat{D}_{m,\sigma} \), and will transform velocities under the collisional transformation whenever it reaches the boundary in a simple, non-grazing collision. However, in order to establish a well-defined dynamics, we need to exclude configurations which run into “pathological” trajectories under time evolution, meaning multiple collisions,
grazing collisions or infinitely many collisions in finite time. In this section we show that each \( Z_m \in D^*_{m,\sigma} \) follows a well-defined trajectory for short time.

Next Lemma defines the flow for any initial configuration \( Z_m \in D^*_{m,\sigma} \) up to the time of the first collision.

**Lemma 4.4.** Let \( m \geq 3 \) and \( Z_m = (X_m, V_m) \in D^*_{m,\sigma} \). Then there is a time \( \tau^1_{Z_m} \in (0, \infty] \) such that defining \( Z_m(\cdot) : (0, \tau^1_{Z_m}] \to \mathbb{R}^{2dm} \) by:

\[
Z_m(t) := \begin{cases} 
(X_m + tV_m, V_m), & \text{if } Z_m \text{ is non-collisional or post-collisional}, \\
(X_m + tV^*_m, V^*_m), & \text{if } Z_m \text{ is pre-collisional},
\end{cases}
\]

the following hold:

(i) \( Z_m(t) \in \hat{D}_{m,\sigma}, \forall t \in (0, \tau^1_{Z_m}) \),

(ii) if \( \tau^1_{Z_m} < \infty \), then \( Z_m(\tau^1_{Z_m}) \in \partial D_{m,\sigma} \),

(iii) If \( Z_m \in \Sigma^{sc}_{ijk} \) for some \( (i, j, k) \in I_m \), and \( \tau^1_{Z_m} < \infty \), then \( Z_m(\tau^1_{Z_m}) \notin \Sigma_{ijk} \).

The time \( \tau^1_{Z_m} \) is called the first (forward) collision time of \( Z_m \).

**Remark 4.3.** One can similarly construct the first backwards collision time of \( Z_m \) for the backwards in time evolution of \( Z_m \), which will belong to \([−\infty, 0)\). We investigate the properties of the first forward collision time, and similar results hold for the first backwards collision time as well.

**Proof.** Let us make the convention \( \inf \emptyset = +\infty \). We define

\[
\tau^1_{Z_m} = \begin{cases} 
\inf \{ t > 0 : X_m + tV_m \in \partial D_{m,\sigma} \}, & \text{if } Z_m \text{ is post-collisional}, \\
\inf \{ t > 0 : X_m + tV^*_m \in \partial D_{m,\sigma} \}, & \text{if } Z_m \text{ is pre-collisional}.
\end{cases}
\] (4.12)
• Assume that \( Z_m \in \hat{D}_{m,\sigma} \). Since \( \hat{D}_{m,\sigma} \) is open and the free flow is continuous, we obtain \( \tau_{Z_m}^1 > 0 \), and claims (i)-(ii) follow immediately from (4.12). Part (iii) is not applicable in this case.

• Assume now that \( Z_m \in \partial_{sc,ng} D_{m,\sigma} \), hence \( Z_m \) is a simple non-grazing collision. Therefore we may distinguish the following cases:

  - \( Z_m \) is an \((i; j, k)\) post-collisional configuration: For any \( t > 0 \), we have

    \[
    d^2(x_i + tv_i; x_j + tv_j, x_k + tv_k) \quad \text{(4.13)}
    \]

    \[
    = |x_i - x_j + (v_i - v_j)t|^2 + |x_i - x_k + (v_i - v_k)t|^2 \\
    = |x_i - x_j|^2 + |x_i - x_k|^2 + t^2 (|v_i - v_j|^2 + |v_i - v_k|^2) + \\
    + 2t (\langle x_i - x_j, v_i - v_j \rangle + \langle x_i - x_k, v_i - v_k \rangle) \\
    \geq 2\sigma^2 + 2tb(x_j - x_i, x_k - x_i, v_j - v_i, v_k - v_i) \\
    > 2\sigma^2, \quad \text{(4.14)}
    \]

    since \( b(\tilde{\omega}_1, \tilde{\omega}_2, v_j - v_i, v_k - v_i) > 0 \). This inequality and the fact that \( Z_m \) is simple collision imply that \( \tau_{Z_m}^1 > 0 \), and claim (i) holds. Claim (ii) follows from (4.12) and claim (iii) follows from (4.14).

  - \( Z_m \) is an \((i; j, k)\) pre-collisional configuration: We use the same argument for \( Z_m^* \) which is \((i; j, k)\) post-collisional, by Remark 4.2.

The result is proved. \( \square \)

Let us make an elementary, but crucial remark which will turn of fundamental importance when extending the flow globally in time.

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Remark 4.4. For configurations with $\tau_{Z_m}^1 = \infty$ the flow is globally defined as the free flow. In the case where $\tau_{Z_m}^1 < \infty$ and $Z_m(\tau_{Z_m}^1) \in \partial_{sc,ng} \mathcal{D}_{m,\sigma}$, we may apply Lemma 4.4 once more, considering $Z_m(\tau_{Z_m}^1)$ as initial point, and extend the flow up to the second collision time:

$$\tau_{Z_m}^2 := \tau_{Z_m}(\tau_{Z_m}^1).$$

Moreover, if $\tau_{Z_m} < \infty$ and $Z_m(\tau_{Z_m}^1) \in \Sigma^{sc}_{ijk}$ for some $(i, j, k) \in I_m$, part (iii) of Lemma 4.4 implies that:

$$Z_m(\tau_{Z_m}^2) \notin \Sigma_{ijk}.$$

### 4.5 Extension to a global interaction flow

In this section, we extract a null set from $\mathcal{D}_{m,\sigma}^*$ such that the flow is globally defined for positive times on the complement. For this purpose, we first truncate positions and velocities considering two parameters $1 < R < \rho$, and then truncate time with a small parameter $\delta$ in the scaling:

$$0 < \delta R << \sigma << 1 < R < \rho.$$ (4.15)

Let us note that similar results can be obtained for negative times as well.

Throughout this section, we consider parameters satisfying the scaling (4.15). Recall that given $r > 0$ we denote the $dm$-ball of radius $r > 0$, centered at the origin, as:

$$B_{r}^{dm} = \{ x \in \mathbb{R}^{dm} : |x| \leq r \}. $$ (4.16)

We first assume initial positions are in $B_{\rho}^{dm}$ and initial velocities in $B_{R}^{dm}$. For
$m \geq 3$, we decompose $D_{m,\sigma}^* \cap (B_{\rho}^{dm} \times B_R^{dm})$ in the following subsets:

$I_{\text{free}} := \{ Z_m = (X_m, V_m) \in D_{m,\sigma}^* \cap (B_{\rho}^{dm} \times B_R^{dm}) : \tau_{Z_m}^1 > \delta \}$,

$I_\delta := \{ Z_m = (X_m, V_m) \in D_{m,\sigma}^* \cap (B_{\rho}^{dm} \times B_R^{dm}) : \tau_{Z_m}^1 = \delta \}$,

$I_{\text{sc,ng}}^1 := \{ Z_m = (X_m, V_m) \in D_{m,\sigma}^* \cap (B_{\rho}^{dm} \times B_R^{dm}) : \tau_{Z_m}^1 < \delta, Z_m(\tau_{Z_m}^1) \in \partial_{\text{sc,ng}} D_{m,\sigma}, \text{ and } \tau_{Z_m}^2 > \delta \}$,

$I_{\text{sc,ng}}^2 := \{ Z_m = (X_m, V_m) \in D_{m,\sigma}^* \cap (B_{\rho}^{dm} \times B_R^{dm}) : \tau_{Z_m}^1 < \delta, Z_m(\tau_{Z_m}^1) \in \partial_{\text{sc,ng}} D_{m,\sigma}, \text{ but } Z_m(\tau_{Z_m}^1) \text{ is grazing} \}$,

$I_{\text{mc}}^1 := \{ Z_m = (X_m, V_m) \in D_{m,\sigma}^* \cap (B_{\rho}^{dm} \times B_R^{dm}) : \tau_{Z_m}^1 < \delta, Z_m(\tau_{Z_m}^1) \in \partial_{\text{mc}} D_{m,\sigma} \}$,

$I_{\text{sc,ng}}^2 := \{ Z_m = (X_m, V_m) \in D_{m,\sigma}^* \cap (B_{\rho}^{dm} \times B_R^{dm}) : \tau_{Z_m}^1 < \delta, Z_m(\tau_{Z_m}^1) \in \partial_{\text{sc,ng}} D_{m,\sigma}, \text{ but } \tau_{Z_m}^2 \leq \delta \}$.  \( (4.17) \)

Notice that for $Z_m \in I_{\text{free}} \cup I_{\text{sc,ng}}^1$, thanks to Lemma 4.4, the flow is well defined up to time $\delta$, and there occurs at most one simple non-grazing collision in $(0, \delta)$. We aim to estimate the measure of $I_\delta \cup I_{\text{sc,ng}}^1 \cup I_{\text{mc}}^1 \cup I_{\text{sc,ng}}^2$, up to the truncation parameters.
4.5.1 Covering arguments

In this part of the section, we make a shell-like covering of the set $I_{mc}^1 \cup I_{sc,ng}^2$ in a way that we can estimate the measure of the coverings. In particular, Lemma 4.6 is a crucial tool which serves as the inductive step to eliminate pathological initial configurations which lead to multiple collisions or infinitely many collisions in a finite time. Shell covering is used in the binary case as well, see Alexander [1], however combinatorics of the ternary case makes this covering argument much more delicate than the one needed for the binary case. Let us note that similar coverings can be obtained for the backwards in time flow as well.

Lemma 4.6. For $m \geq 4$, the following inclusion holds:

$$I_{mc}^1 \cup I_{sc,ng}^2 \subseteq \bigcup_{(i,j,k) \neq (i',j',k') \in J_m} (U_{ijk} \cap U_{i'j'k'}),$$

where, given $(i,j,k) \in J_m$, we denote

$$U_{ijk} = \left\{ Z_m = (X_m, V_m) \in B_{\rho}^{dm} \times B_{R}^{dm} : 2\sigma^2 \leq d^2(x_i; x_j, x_k) \leq (\sqrt{2}\sigma + 4\delta R)^2 \right\}.$$

(4.18)

For $m = 3$, there holds:

$$I_{mc}^1 = I_{sc,ng}^2 = \emptyset.$$

Proof. For $m = 3$, we have seen that $\partial_{mc} D_{3,\sigma} = \emptyset$, hence $I_{mc}^1 = \emptyset$. Also, since $m = 3$, we trivially obtain $J_3 = \{(1, 2, 3)\}$, hence Remark 4.4 implies that $\tau_{Z_m}^2 = \infty$ i.e. there is no other collision in the future, so $I_{sc,ng}^2 = \emptyset$. 29
Therefore,
\[ I_{mc}^1 = I_{sc,ng}^2 = \emptyset. \]

Assume now that \( m \geq 4 \). We first assume that either \( Z_m \in \hat{D}_{m,\sigma} \) or \( Z_m \) is post-collisional. Therefore, up to time \( \tau_{1m}^1 \), we have free flow i.e.
\[ Z_m(t) = (X_m + tV_m, V_m), \quad \forall t \in (0, \tau_{1m}^1]. \]

We distinguish the following cases:

- \( Z_m \in I_{mc}^1 \): We have \( \tau_{1m}^1 \leq \delta \) and \( Z_m(\tau_{1m}^1) \in \partial_{mc} D_{m,\sigma} \). So there are \((i, j, k) \neq (i', j', k') \in I_m\) such that
  \[ d^2\left( x_i(\tau_{1m}^1) ; x_j(\tau_{1m}^1), x_k(\tau_{1m}^1) \right) = 2\sigma^2, \quad (4.19) \]
  \[ d^2\left( x_i'(\tau_{1m}^1) ; x_j'(\tau_{1m}^1), x_k'(\tau_{1m}^1) \right) = 2\sigma^2. \quad (4.20) \]

Since there is free motion up to \( \tau_{1m}^1 \), triangle inequality implies
\[ |x_i - x_j| \leq |x_i(\tau_{1m}^1) - x_j(\tau_{1m}^1)| + \delta|v_i - v_j| \leq |x_i(\tau_{1m}^1) - x_j(\tau_{1m}^1)| + 2\delta R. \quad (4.21) \]

Since there is collision at \( \tau_{1m}^1 \), we have
\[ |x_i(\tau_{1m}^1) - x_j(\tau_{1m}^1)|^2 + |x_i(\tau_{1m}^1) - x_k(\tau_{1m}^1)|^2 = 2\sigma^2 \Rightarrow |x_i(\tau_{1m}^1) - x_j(\tau_{1m}^1)| \leq \sqrt{2}\sigma. \quad (4.22) \]

Combining (4.21) - (4.22), we obtain
\[ |x_i - x_j|^2 \leq |x_i(\tau_{1m}^1) - x_j(\tau_{1m}^1)|^2 + 4\sqrt{2}\sigma \delta R + 4\delta^2 R^2. \quad (4.23) \]

Using the same argument for the pair \((i, k)\), adding, and recalling the fact that there is \((i; j, k)\) simple collision at \( \tau_{1m}^1 \), we obtain
\[ 2\sigma^2 \leq d^2(x_i; x_j, x_k) \leq 2\sigma^2 + 8\sqrt{2}\sigma R \delta + 8\delta R^2 \leq 2\sigma^2 + 8\sqrt{2}\sigma R \delta + 16\delta R^2 \]
\[ = (\sqrt{2}\sigma + 4\delta R)^2 \quad (4.24) \]
\[ \Rightarrow Z_m \in U_{ijk}, \]
where the lower inequality holds trivially since $Z_m \in D_{m,\sigma}$.

Applying the same argument for $(i',j',k')$ we get $Z_m \in U_{i'j'k'}$, therefore

$$I^1_{mc} \subseteq \bigcup_{(i,j,k) \neq (i',j',k') \in I_m} (U_{ijk} \cap U_{i'j'k'}). \quad (4.25)$$

- $Z_m \in I^2_{sc,ng}$: Assuming that $Z_m(\tau^1_{Z_m})$ is an $(i,j,k)$ non-grazing collision, Remark 4.4 guarantees that $Z_m(\tau^2_{Z_m}) \notin \Sigma_{ijk}$. So $Z_m(\tau^2_{Z_m}) \in \Sigma_{i'j'k'}$ for some $(i',j',k') \neq (i,j,k)$.

Clearly all particles perform free motion in $(0,\tau^1_{Z_m}]$, so the same argument we used to obtain (4.24) yields

$$2\sigma^2 \leq d^2(x_i; x_j, x_k) \leq (\sqrt{2}\sigma + 4\delta R)^2 \Rightarrow Z_m \in U_{ijk}. \quad (4.26)$$

Moreover, particles keep performing free motion in $[\tau^1_{Z_m}, \tau^2_{Z_m})$ except particles $i,j,k$ whose velocities instantaneously transform because of the collision at $\tau^1_{Z_m}$.

Recall we wish to prove as well:

$$Z_m \in U_{i'j'k'} \iff 2\sigma^2 \leq d^2(x_i'; x_{j'}, x_{k'}) \leq (\sqrt{2}\sigma + 4\delta R)^2. \quad (4.27)$$

The lower inequality trivially holds because of the phase space so it suffices to prove the upper inequality. Notice that it is impossible to have $\{i',j',k'\} = \{i,j,k\}$, since $(i',j',k') \neq (i,j,k) \in I_m$ by assumption. Therefore, it suffices to distinguish the following cases:

(I) $i', j', k' \notin \{i,j,k\}$: Since particles $(i',j',k')$ perform free motion up to $\tau^2_{Z_m}$, a similar argument to the one we used to obtain (4.24) yields $Z_m \in U_{i'j'k'}$. The
only difference is that we apply the argument up to time $\tau_{Z_m}^2$ instead of $\tau_{Z_m}^1$. Claim (4.27) is proved.

(II) There is at least one recollision i.e. at least one of $i', j', k'$ belongs to $\{i, j, k\}$ but no more than two. The argument is similar to (I), the only difference being that velocities of the recolliding particles transform at $\tau_{Z_m}^1$. Since the argument is similar for all cases, let us provide the proof in detail only for one case, for instance $(i', j', k') = (i, k, k')$, for some $k' > k$.

Let us denote
\[
(v_i^*(\tau_{Z_m}^1), v_j^*(\tau_{Z_m}^1), v_k^*(\tau_{Z_m}^1)) = T_{\tilde{\omega}_1(\tau_{Z_m}^1), \tilde{\omega}_2(\tau_{Z_m}^1)}(v_i(\tau_{Z_m}^1), v_j(\tau_{Z_m}^1), v_k(\tau_{Z_m}^1))
\]
where
\[
(\tilde{\omega}_1(\tau_{Z_m}^1), \tilde{\omega}_2(\tau_{Z_m}^1)) := \frac{1}{\sqrt{2\epsilon}}(x_j(\tau_{Z_m}^1) - x_i(\tau_{Z_m}^1), x_k(\tau_{Z_m}^1) - x_i(\tau_{Z_m}^1)) \in S_{ij}^{d-1},
\]
since $Z_{m}(\tau_{Z_m}^1) \in \Sigma_{ijk}^{sc}$. The fact that $V_m \in B_{R}^{dm}$, conservation of energy by the free flow and conservation of energy by the collision (3.11) imply
\[
v_i^*(\tau_{Z_m}^1), v_j^*(\tau_{Z_m}^1), v_k^*(\tau_{Z_m}^1) \in B_{R}^{d}.
\]
(4.28)

For the pair $(i, k)$, we have
\[
x_i(\tau_{Z_m}^2) = x_i(\tau_{Z_m}^1) + (\tau_{Z_m}^2 - \tau_{Z_m}^1)v_i^*(\tau_{Z_m}^1) = x_i + \tau_{Z_m}^1v_i + (\tau_{Z_m}^2 - \tau_{Z_m}^1)v_i^*(\tau_{Z_m}^1),
x_k(\tau_{Z_m}^2) = x_k(\tau_{Z_m}^1) + (\tau_{Z_m}^2 - \tau_{Z_m}^1)v_k^*(\tau_{Z_m}^1) = x_k + \tau_{Z_m}^1v_k + (\tau_{Z_m}^2 - \tau_{Z_m}^1)v_k^*(\tau_{Z_m}^1),
\]

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so

\[ x_i - x_k = x_i(\tau_{Z_m}^2) - x_k(\tau_{Z_m}^2) - \tau_{Z_m}^1(v_i - v_k) - (\tau_{Z_m}^2 - \tau_{Z_m}^1) (v_i^* (\tau_{Z_m}^1) - v_k^* (\tau_{Z_m}^1)). \]

Therefore, triangle inequality implies

\[
|x_i - x_k| \leq |x_i(\tau_{Z_m}^2) - x_k(\tau_{Z_m}^2)| + \tau_{Z_m}^1 |v_i - v_k| + (\tau_{Z_m}^2 - \tau_{Z_m}^1) |v_i^* (\tau_{Z_m}^1) - v_k^* (\tau_{Z_m}^1)| \\
\leq |x_i(\tau_{Z_m}^2) - x_k(\tau_{Z_m}^2)| + 2\tau_{Z_m}^1 R + 2(\tau_{Z_m}^2 - \tau_{Z_m}^1) R \\
= |x_i(\tau_{Z_m}^2) - x_k(\tau_{Z_m}^2)| + 2\tau_{Z_m}^2 R \\
\leq |x_i(\tau_{Z_m}^2) - x_k(\tau_{Z_m}^2)| + 2\delta R, \tag{4.31}
\]

where to obtain (4.30) we use triangle inequality and (4.28), and to obtain (4.31) we use the assumption \( \tau_{Z_m}^2 \leq \delta \). For the pair \((i, k')\), we proceed similarly, the only difference being that only particle \(i\) collides at \(\tau_{Z_m}^1\). We obtain

\[ x_i - x_{k'} = x_i(\tau_{Z_m}^2) - x_{k'}(\tau_{Z_m}^2) - \tau_{Z_m}^1 (v_i - v_{k'}) - (\tau_{Z_m}^2 - \tau_{Z_m}^1)(v_i^* (\tau_{Z_m}^1) - v_{k'}). \]

So triangle inequality and Remark \(4.28\) imply

\[
|x_i - x_{k'}| \leq |x_i(\tau_{Z_m}^2) - x_{k'}(\tau_{Z_m}^2)| + \tau_{Z_m}^1 |v_i - v_{k'}| + (\tau_{Z_m}^2 - \tau_{Z_m}^1) |v_i^* (\tau_{Z_m}^1) - v_{k'}| \\
\leq |x_i(\tau_{Z_m}^2) - x_{k'}(\tau_{Z_m}^2)| + 2\tau_{Z_m}^1 R + 2(\tau_{Z_m}^2 - \tau_{Z_m}^1) R \\
= |x_i(\tau_{Z_m}^2) - x_{k'}(\tau_{Z_m}^2)| + 2\tau_{Z_m}^2 R \\
\leq |x_i(\tau_{Z_m}^2) - x_{k'}(\tau_{Z_m}^2)| + 2\delta R,
\]

By an argument similar to (4.24), inequality (4.27) follows. In all other cases (4.27) follows in a similar way.
Combining (4.26), (4.27), we obtain the result.

Assume now that $Z_m$ is pre-collisional. Then, by Remark 4.2, $Z_m^*$ is post-collisional. Conservation of energy (3.11) yields $Z_m^* \in B_{\rho}^{dm} \times B_{R}^{dm}$, so following the same arguments as in the post-collisional case, we obtain the required result. The proof is complete.

4.6.1 Measure estimates

Now we wish to estimate the measure of $I_{sc,g}^1 \cup I_{mc}^1 \cup I_{sc,ng}^2$ in order to show that outside of a small measure set we have a well defined flow up to small time $\delta$. To estimate the measure of $I_{mc}^1 \cup I_{sc,ng}^2$, we will strongly rely on the shell-like covering made in Lemma 4.6. Let us note that the loss of symmetry induced by the ternary collisions makes the measure estimates significantly harder than in the binary case. In particular, in the binary case, the covering is achieved by actual spherical shells, while in the ternary case spherical symmetry is lost, therefore new treatment is needed to obtain the necessary estimates. Let us note that similar estimates can be obtained for the backwards in time flow as well.

First we estimate the measure of $I_{sc,g}^1$:

**Lemma 4.7.** Let $m \geq 3$. Then the sets $I_{\delta}^1$, $I_{sc,g}^1$ is of zero Lebesgue measure.

**Proof.** Let us first prove the claim for $I_{sc,g}^1$. Notice that

$$I_{sc,g}^1 \subseteq \bigcup_{(i,j,k) \in \mathcal{I}_m} M_{ijk},$$

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where
\[ M_{ijk} = \{ Z_m \in D^{*,\sigma}_{m,\rho} \cap (B^{dm}_{\rho} \times B^{dm}_{R}) : Z_m(\tau^1_{Z_m}) \text{ is an } (i; j, k) \text{ grazing collision} \}. \]

We claim that for each \((i, j, k) \in \mathcal{I}_m\), the set \(M_{ijk}\) is of zero measure. Indeed, consider \((i, j, k) \in \mathcal{I}_m\) and \(Z_m \in M_{ijk}\). Since \(Z_m(\tau^1_{Z_m})\) is an \((i; j, k)\) simple collision, we have
\[
|x_i - x_j + \tau^1_{Z_m}(v_i - v_j)|^2 + |x_i - x_k + \tau^1_{Z_m}(v_i - v_k)|^2 = 2\sigma^2.
\]
But \(Z_m(\tau^1_{Z_m})\) is assumed to be grazing collision so
\[
\langle x_j - x_i + \tau^1_{Z_m}(v_j - v_i), v_j - v_i \rangle + \langle x_k - x_i + \tau^1_{Z_m}(v_k - v_i), v_k - v_i \rangle = 0
\]
Using (4.33) to eliminate \(\tau^1_{Z_m}\) from (4.32), we obtain that \(M_{ijk}\) is embedded in an 1-codimension submanifold of \(D^{*,\sigma}_{m,\rho} \cap (B^{dm}_{\rho} \times B^{dm}_{R})\), so it is of zero Lebesgue measure.

By sub-additivity, \(I_{sc,g}^1\) is of zero Lebesgue measure as well. To prove the claim for \(I_s^1\), we follow a similar argument after appropriately covering it.

We observe that Lemmas 4.6-4.7 imply that it suffices to estimate the measure of \(U_{ijk} \cap U_{i'j'k'}\), for all \((i, j, k) \neq (i', j', k') \in \mathcal{I}_m\).

For this purpose, let us introduce some notation. Given \(m \geq 3\) and \((i, j, k) \in \mathcal{I}_m\), we define the following sets:

- Given \((x_j, x_k) \in \mathbb{R}^{2d}\), we define
  \[
  S_i(x_j, x_k) = \{ x_i \in \mathbb{R}^d : (x_i, x_j, x_k) \in U_{ijk} \}. \tag{4.34}
  \]
- Given \((x_i, x_k) \in \mathbb{R}^{2d}\), we define
  \[
  S_j(x_i, x_k) = \{ x_j \in \mathbb{R}^d : (x_i, x_j, x_k) \in U_{ijk} \}. \tag{4.35}
  \]
Given \((x_i, x_j) \in \mathbb{R}^{2d}\), we define
\[
S_k(x_i, x_j) = \left\{ x_k \in \mathbb{R}^d : (x_i, x_j, x_k) \in U_{ijk} \right\}.
\]

Lemma 4.8. Let \(m \geq 3\) and \((i, j, k) \in I_m\). Then the following estimates hold:
\[
|S_i(x_j, x_k)|_d \leq C_{d,R} \sigma_0 \delta, \quad \forall (x_j, x_k) \in \mathbb{R}^{2d}, \tag{4.37}
\]
\[
|S_j(x_i, x_k)|_d \leq C_{d,R} \sigma_0 \delta, \quad \forall (x_i, x_k) \in \mathbb{R}^{2d}, \tag{4.38}
\]
\[
|S_k(x_i, x_j)|_d \leq C_{d,R} \sigma_0 \delta, \quad \forall (x_i, x_j) \in \mathbb{R}^{2d}. \tag{4.39}
\]

Proof. For convenience, let us write \(\sigma_0 = \sqrt{2} \sigma, \delta_0 = 4 \delta R\). Scaling (4.15) implies
\[
0 < \delta_0 << \sigma_0 < 1. \tag{4.40}
\]

The proof of (4.38)-(4.39): By symmetry it suffices to prove (4.39). Consider \((x_i, x_j) \in \mathbb{R}^{2d}\), and let us write \(\alpha = |x_i - x_j|\). Recalling (4.36), we have
\[
S_k(x_i, x_j) = \left\{ x_k \in \mathbb{R}^d : \sigma_0^2 - \alpha^2 \leq |x_i - x_k|^2 \leq (\sigma_0 + \delta_0)^2 - \alpha^2 \right\}.
\]

We distinguish the following cases:

- \(\alpha > \sigma_0\): We have
  \[
  (\sigma_0 + \delta_0) - \alpha^2 < (\sigma_0 + \delta_0)^2 - \alpha^2 = \delta_0(2\sigma_0 + \delta_0) < \delta_0,
  \]
  since \(0 < \delta_0 << \sigma_0 << 1\). Thus
  \[
  S_k(x_i, x_j) \subseteq \left\{ x_k \in \mathbb{R}^d : |x_i - x_k| \leq \sqrt{\delta_0} \right\},
  \]
so

\[ |S_k(x_i, x_j)|_d \lesssim \delta_0^{d/2} \leq \delta_0 = 4R\delta, \]  \hspace{1cm} (4.41)

since \( \delta_0 < 1 \) and \( d \geq 2 \).

- \( \alpha \leq \sigma_0 \): Recalling (4.36), we have

\[ S_k(x_i, x_j) = \left\{ x_k \in \mathbb{R}^d : \sqrt{\sigma_0^2 - \alpha^2} \leq |x_i - x_k| \leq \sqrt{(\sigma_0 + \delta_0)^2 - \alpha^2} \right\}. \]

Thus \( S_k(x_i, x_j) \) is a spherical shell in \( \mathbb{R}^d \), of inner radius \( \sqrt{\sigma_0^2 - \alpha^2} \) and outer radius \( \sqrt{(\sigma_0 + \delta_0)^2 - \alpha^2} \). Therefore

\[ |S_k(x_i, x_j)|_d \simeq \left( \sqrt{(\sigma_0 + \delta_0)^2 - \alpha^2} \right)^d - \left( \sqrt{\sigma_0^2 - \alpha^2} \right)^d \hspace{1cm} (4.42) \]

\[ = \frac{(\sigma_0 + \delta_0)^2 - \sigma_0^2}{\sqrt{(\sigma_0 + \delta_0)^2 - \alpha^2} + \sqrt{\sigma_0^2 - \alpha^2}} \sum_{m=0}^{d-1} \left( \sqrt{(\sigma_0 + \delta_0)^2 - \alpha^2} \right)^{d-1-m} \left( \sqrt{\sigma_0^2 - \alpha^2} \right)^m \]

\[ = \frac{\delta_0 (2\sigma_0 + \delta_0)}{\sqrt{(\sigma_0 + \delta_0)^2 - \alpha^2} + \sqrt{\sigma_0^2 - \alpha^2}} \sum_{m=0}^{d-1} \left( \sqrt{(\sigma_0 + \delta_0)^2 - \alpha^2} \right)^{d-1-m} \left( \sqrt{\sigma_0^2 - \alpha^2} \right)^m \]

\[ \leq \frac{\delta_0}{\sqrt{(\sigma_0 + \delta_0)^2 - \alpha^2} + \sqrt{\sigma_0^2 - \alpha^2}} \left( \sqrt{(\sigma_0 + \delta_0)^2 - \alpha^2} + (d - 1) \sqrt{\sigma_0^2 - \alpha^2} \right) \]

\[ \leq d\delta_0 = 4dR\delta, \hspace{1cm} (4.43) \]

where to obtain (4.43) we use the fact that \( 0 < \delta_0 << \sigma_0 << 1 \).

Combining (4.41)-(4.44), we obtain (4.39).
The proof of (4.37): Consider \((x_j, x_k) \in \mathbb{R}^d\). Completing the square, one can see that
\[
S_i(x_j, x_k) = \left\{ x_i \in \mathbb{R}^d : \sigma_0^2 - \alpha^2 \leq \left| x_i - \frac{x_j + x_k}{2} \right| \leq (\sigma + \delta_0)^2 - \alpha^2 \right\},
\]
where
\[
\sigma_0 = \sigma, \quad \delta_0 = \frac{4\delta R}{\sqrt{2}}, \quad \alpha = \frac{1}{2} \sqrt{2(|x_j|^2 + |x_k|^2) - |x_j + x_k|^2}.
\]
Scaling (4.15) implies
\[
0 < \delta_0 << \sigma_0 << 1.
\]
The estimate follows by an argument identical to the proof of (4.38)-(4.39).

The proof is complete. \(\blacksquare\)

Lemma 4.9. Let \(m \geq 3\), \(1 < R < \rho\) and \(0 < \delta R < \sigma << 1\). Then the following estimate holds:
\[
|I^1_\delta \cup I^1_{sc,g} \cup I^1_{mc} \cup I^2_{sc,ng}|_{2d_m} \leq C_{m,d,R}\rho^{d(m-2)}\delta^2.
\]

Proof. For \(m = 3\), the result comes trivially from Lemmas 4.6-4.7.

Assume \(m \geq 4\). Lemmas 4.6-4.7 and sub-additivity imply it suffices to uniformly estimate the measure of \(U_{ijk} \cap U_{i'j'k'}\), for all \((i, j, k) \neq (i', j', k') \in \mathcal{J}_m\).

Consider \((i, j, k) \neq (i', j', k') \in \mathcal{J}_m\), and recall notation from (4.34)-(4.36). We will strongly rely on Lemma 4.8.

We consider the following cases:
(I) $i', j', k' \notin \{i, j, k\}$: Fubini’s Theorem and (4.39) imply

\[
|U_{ijk} \cap U_{i'j'k'}|_{2dm} \lesssim R^{dm} \rho^{d(m-6)} \int_{B^d_{\rho}} 1_{S_k(x_i, x_j) \cap S_{k'}(x_{i'}, x_{j'})} \, dx_i \, dx_j \, dx_k \, dx_{i'} \, dx_{j'} \, dx_{k'} \\
\leq R^{dm} \rho^{d(m-6)} \times \left( \int_{B^d_{\rho} \times B^d_{\rho} \times \mathbb{R}^d} 1_{S_k(x_i, x_j)} \, dx_k \, dx_j \, dx_i \right) \left( \int_{B^d_{\rho} \times B^d_{\rho} \times \mathbb{R}^d} 1_{S_{k'}(x_{i'}, x_{j'})} \, dx_{k'} \, dx_{j'} \, dx_i \right) \\
\leq C_{d, R} \rho^{d(m-2)} \delta^2.
\]

(II) Exactly one of $i', j', k'$ belongs to $\{i, j, k\}$: Without loss of generality, we consider the case $i' = i, j' \neq j, k \neq k'$, and all other cases occurring can be treated similarly using the corresponding estimate from (4.37)-(4.39). Fubini’s Theorem and (4.39) imply

\[
|U_{ijk} \cap U_{i'j'k'}|_{2dm} \lesssim R^{dm} \rho^{d(m-5)} \int_{B^d_{\rho}} 1_{S_k(x_i, x_j) \cap S_{k'}(x_{i'}, x_{j'})} \, dx_i \, dx_j \, dx_k \, dx_{i'} \, dx_{j'} \, dx_{k'} \\
\leq R^{dm} \rho^{d(m-5)} \times \left( \int_{B^d_{\rho}} \left( \int_{B^d_{\rho} \times \mathbb{R}^d} 1_{S_k(x_i, x_j)} \, dx_k \, dx_j \right) \left( \int_{B^d_{\rho} \times \mathbb{R}^d} 1_{S_{k'}(x_{i'}, x_{j'})} \, dx_{k'} \, dx_{j'} \right) \, dx_i \right) \\
\leq C_{m, R} \rho^{d(m-2)} \delta^2.
\]

(III) Exactly two of $i', j', k'$ belong to $\{i, j, k\}$: Without loss of generality, we consider the case $i' = i, j' = j, k \neq k'$, and all other cases occurring can be treated similarly using the corresponding estimate from (4.37)-(4.39). Fubini’s Theo-
rem and (4.39) imply
\[ |U_{ijk} \cap U_{ijk'}|_{2dm} \leq R^{dm} \rho^{d(m-4)} \int_{B_p^d \times B_p^d \times B_p^d} 1_{S_k(x_i,x_j) \cap S_{k'}(x_i,x_j')} \, dx_i \, dx_j \, dx_k \, dx_k'. \]
\[ \leq R^{dm} \rho^{d(m-4)} \int_{B_p^d \times B_p^d} \left( \int_{\mathbb{R}^d} 1_{S_k(x_i,x_j)} \, dx_k \right) \left( \int_{\mathbb{R}^d} 1_{S_{k'}(x_i,x_j)} \, dx_k' \right) \, dx_j \, dx_i \]
\[ \leq C_{m,R} \rho^{d(m-2)} \delta^2. \]

By similar arguments we obtain the same estimate for all the combinations \((i, j, k) \neq (i', j', k') \in \mathcal{I}_m\). Combining these estimates, and using Lemmas 4.7, 4.6 we obtain the required estimate.

**Remark 4.5.** The coverings and estimates needed to produce a global in time flow in the phase space \(\tilde{D}_{m,\sigma}\) (see (2.7)) are very similar with the only exception being the case where an \((i; j, k)\) interaction is succeeded by a \((j; i, k)\) interaction or a \((k; i, j)\) interaction in short time \(\delta\). This case is clearly not recorded in the phase space we consider. To treat this case, one can estimate the measure of the intersection of two annular regions of width \(\delta\) and show that this measure is at most of order \(\delta^{1+\epsilon}\), where \(\epsilon > 0\). As we will see, such an estimate will be sufficient to generate a global flow.

### 4.9.1 The global interaction flow

We inductively use Lemma 4.9 to define a global flow which preserves energy for almost all configuration. For this purpose, given \(Z_m = (X_m, V_m) \in \mathbb{R}^{2dm}\), we define its kinetic energy as:
\[ E_m(Z_m) := \frac{1}{2} \sum_{i=1}^{m} |v_i|^2 \quad (4.45) \]

For convenience, let us define the \(m\)-particle free flow which is nothing more than the rectilinear motion of particles in time.
Definition 4.5. Let \( m \in \mathbb{N} \). We define the \( m \)-particle free flow as the family of measure-preserving maps \( (\Phi_t^m)_{t \in \mathbb{R}} : \mathbb{R}^{2dm} \to \mathbb{R}^{2dm} \), given by
\[
\Phi_t^m Z_m = \Phi_m^t(X_m, V_m) := (X_m + tV_m, V_m).
\] (4.46)
The map \( \Phi_t^m \) is clearly a measure-preserving diffeomorphism of \( \mathbb{R}^{2dm} \), for any \( t \in \mathbb{R} \).

We now state the Existence Theorem of the \( m \)-particle \( \sigma \)-interaction zone flow.

Theorem 4.9.1 (Existence of the interaction flow). Let \( m \in \mathbb{N} \) and \( 0 < \sigma << 1 \).

There exists a full measure \( G_\delta \)-subset \( \Gamma \subseteq D_{m,\sigma}^* \) and a measure-preserving family of diffeomorphisms \( (\Psi_t^m)_{t \in \mathbb{R}} : \Gamma \to \Gamma \) such that
\[
\Psi_t^{s}Z_m = (\Psi_t^m \circ \Psi_s^m)(Z_m) = (\Psi_s^m \circ \Psi_t^m)(Z_m), \quad \forall Z_m \in \Gamma, \quad \forall t, s \in \mathbb{R}, \quad (4.47)
\]
\[
E_m(\Psi_t^m Z_m) = E_m(Z_m), \quad \forall Z_m \in \Gamma, \quad \forall t \in \mathbb{R}, \quad (4.48)
\]
\[
\text{For } m \geq 3 : \quad \Psi_t^m Z_m^* = \Psi_t^m Z_m, \quad \sigma - a.e. \text{ on } \Gamma \cap \partial \text{sc,ng } D_{m,\sigma}, \quad \forall t \in \mathbb{R}, \quad (4.49)
\]
where \( \sigma \) is the surface measure induced on \( \partial D_m \) by the Lebesgue measure. This family of maps is called the \( m \)-particle \( \sigma \)-interaction zone flow. For \( m = 1, 2 \), the flow coincides with the free flow.

Remark 4.6. While the inductive step of proof is inspired by [33], we emphasize that we had to carefully address all the rest of the properties of the flow. In particular, since this is the first study which rigorously treats ternary collisions, we carefully present the initial step of the construction, including showing that the flow is measure-preserving, as well as the proof of (4.49) which will be of fundamental importance for the derivation of the Liouville equation.
Proof. For $m = 1, 2$ the flow is trivially defined as the free flow:

$$\Psi^t_m Z_m \equiv \Phi^t_m Z_m,$$

where $\Phi^t_m$ is the free flow given by (4.46). It is clearly measure-preserving, and satisfies (4.47)-(4.48).

Assume now that $m \geq 3$. We will construct a full measure subset of the phase space where the forward in time flow can be well-defined.

Fix $t > 0$, $R > 1$, and consider $0 < \delta R << \sigma$ such that $k = t/\delta \in \mathbb{N}$.

**Initial step:**

(a) **Construction of the flow:** Using the sets defined in (4.17), we define

$$I^{+}_0(t, R, \delta) := I_0^1 \cup I_{s,c,g}^1 \cup I_{m,c}^1 \cup I_{s,c,n}^2 \subseteq B^{dm}_R \times B^{dm}_R. \quad (4.50)$$

Moreover, applying Lemma 4.9 with $\rho = R$, the measure of $I^{+}_0(t, R, \delta)$ can be estimated as:

$$|I^{+}_0(t, R, \delta)|_{dm} \leq C_{m,d,R} R^{d(m-2)} \delta^2.$$

For convenience, let us write

$$G = \left[ (B^{dm}_R \times B^{dm}_R) \cap D^*_m \right] \setminus I^{+}_0(t, R, \delta).$$

By (4.50), (4.17), each $Z_m \in G$ faces at most one simple non-grazing collision in $(0, \delta)$.

(b) **$G$ is open and the flow is a measure preserving diffeomorphism:** Let us first prove that $G$ is open. Fix any $Z_{m,0} \in G$. By smoothness of the pre-collisional and post-collisional conditions and the first collision time, we conclude the following:
• If $\tau_{Z_{m,0}}^1 = \infty$, there is an open neighborhood $U_{Z_{m,0}}$ of $Z_{m,0}$ such that each $Z_m \in U_{Z_{m,0}}$ faces no collisions in the future.

• If $\tau_{Z_{m,0}}^1 \in (0, \delta)$, there is an open neighborhood $U_{Z_{m,0}}$ of $Z_{m,0}$ such that any $Z_m \in U_{Z_{m,0}}$ faces exactly the same collision in $(0, \delta)$, as $Z_{m,0}$ does.

In other words, each $Z_m \in U_{Z_{m,0}}$ has the same collision history in $(0, \delta)$ as $Z_{m,0}$. This proves that $U_{Z_{m,0}} \subseteq G$, thus $G$ is open.

To prove that the flow is a measure-preserving diffeomorphism, we will use the fact that the free flow is a diffeomorphism of Jacobian one, and the fact that for any $(i, j, k) \in I_m$, the map

$$T_{ij}^{jk} : Z_m \in \Sigma_{ij}^{sc} \rightarrow Z_m^* \in \Sigma_{ij}^{sc}$$

is by (3.15) a diffeomorphism as well. Moreover, recalling notation from (4.9), (3.3), partial differentiation implies

$$| \text{Jac} T_{ij}^{jk}(Z_m) | = | \det T_{\tilde{\omega}_1, \tilde{\omega}_2} | = 1, \quad \forall Z_m \in \Sigma_{ij}^{sc}, \quad (4.52)$$

where $(\tilde{\omega}_1, \tilde{\omega}_2)$ are given by (4.9), and to obtain (4.52) we use (3.16). Fix $Z_{m,0} \in G$. We distinguish the following cases:

• $\tau_{Z_{m,0}}^1 = \infty$: Due to same collision history, there is an open neighborhood $U_{Z_{m,0}} \subseteq G$ of $Z_{m,0}$ such that

$$\Psi_{m}^{\delta} Z_m = \Phi_{m}^{\delta} Z_m, \forall Z_m \in U_{Z_{m,0}}.$$

Therefore, $\Psi_{m}^{\delta}$ is a diffeomorphism of $U_{Z_{m,0}}$, and $\text{Jac} \Psi_{m}^{\delta} Z_{m,0} = 1.$
• $0 < \tau_{Z_{m,0}}^1 < \delta$: Due to same collision history, there is an open neighborhood $U_{Z_{m,0}} \subseteq G$ of $Z_{m,0}$ such that any $Z_m \in U_{Z_{m,0}}$ faces exactly the same collision in $(0, \delta)$, as $Z_{m,0}$ does. Let us assume that $Z_{m,0}(\tau_{Z_{m,0}}^1) \in \Sigma_{ij}^{sc}$, for some $(i, j, k) \in \mathcal{I}_N$. Then we have

$$\psi^\delta_m(Z_m) = (\Phi^{\delta - \tau_{Z_{m,0}}^1}_m \circ T_{ij}^m \circ \Phi^{\tau_{Z_{m,0}}^1}_m)(Z_m), \quad Z_m \in U_{Z_{m,0}}.$$  

Since the free flow, the first collision time map $\tau^1$ and the map $T_{ij}^m$ are smooth, we obtain that $\psi^\delta_m$ is differentiable in $U_{Z_{m,0}}$ and $|\text{Jac}\psi^\delta_m| = 1$. Since $T_{ij}^m$ is an involution, we clearly obtain that $\psi^\delta_m$ is a diffeomorphism of $U_{Z_{m,0}}$.

Since $Z_{m,0} \in G$ is arbitrary, we conclude that $\psi^\delta_m$ is a measure-preserving diffeomorphism of $G$.

**Induction up to $k$:** Since the collisional transformation preserves velocities magnitude, triangle inequality implies that for any $Z_m \in \left( (B_{\rho}^{dm} \times B_{\rho}^{dm}) \cap \mathcal{D}_{m,\sigma}^* \right) \setminus \mathcal{I}_{0}^+ (R, \delta)$, we have

$$\psi^\delta_m Z_m \in B_{R(1+\delta)}^{dm} \times B_{R}^{dm}.$$  

Since the flow up to $\delta$ is shown to be measure-preserving, applying Lemma 4.9 for $\rho = R(1+ \delta)$, we may find a closed subset

$$\mathcal{I}_{1}^+ (R, \delta) \subseteq B_{R}^{dm} \times B_{R}^{dm},$$  

with

$$|\mathcal{I}_{1}^+ (R, \delta)|_{2dm} \leq C_{m,d,R} \rho^{d(m-2)}(1+\delta)^{d(m-2)} \delta^2,$$  

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such that each initial configuration in $[(B_R^{dm} \times B_R^{dm}) \cap \mathcal{D}^\ast_{m,\sigma}] \setminus [I^+_0 (R, \delta) \cup I^+_1 (R, \delta)]$ faces at most one simple non-grazing collision in $[0, \delta]$, and at most one simple non-grazing collision in $[\delta, 2\delta]$. With a similar argument, we may see that the flow generated in $[\delta, 2\delta]$ is a measure-preserving diffeomorphism.

Repeating this process $k := t/\delta$ times we may construct a closed subset

$$I^+_\delta (t, R) := \bigcup_{j=0}^{k-1} I^+_j (R, \delta) \subseteq B_R^{dm} \times B_R^{dm},$$

with

$$|I^+_\delta (t, R)|_{2dm} \leq C_{m, d, R} R^{d(m-2)} \delta^2 \sum_{j=0}^{k-1} (1 + j\delta)^{d(m-2)} \leq C_{m, d, R} R^{d(m-2)} \delta^2 k (1 + t)^{d(m-2)} = C_{m, d, R, t} \delta,$$

such that any $Z_m \in [(B_R^{dm} \times B_R^{dm}) \cap \mathcal{D}^\ast_{m,\sigma}] \setminus I^+_\delta (t, R)$ evolves up to $t$, and faces at most one simple non-grazing collision in each time interval $[j\delta, (j+1)\delta]$, $j = 0, ..., k-1$.

**Global extension:** The flow generated is measure-preserving and independent of $\delta$. Therefore, defining the closed set

$$I^+ (t, R) = \bigcap_{0 < \delta R < \sigma} \bigcap_{t/\delta \in \mathbb{N}} I^+_\delta (t, R) \subseteq B_R^{dm} \times B_R^{dm},$$

we get

$$|I^+ (t, R)|_{2dm} = 0,$$

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and the flow up to $t$ is well-defined for any $Z_m \in \left[ (B_{R}^{dm} \times B_{R}^{dm}) \cap D^*_{m,\sigma} \right] \setminus I^+(t,R)$ meaning $Z_m$ faces finitely many non-grazing simple collisions up to time $t$. Moreover, the flow up to $t$ is a measure-preserving diffeomorphism.

Considering sequences $R_n \to \infty$, $t_n \to \infty$, and defining the $F_\sigma$ set

$$I^+ = \bigcup_{n=1}^{\infty} I^+(t_n, R_n),$$

we obtain

$$|I^+|_{2dm} = 0.$$  

Therefore, the set

$$\Gamma^+ = D^*_{m,\sigma} \setminus I^+,$$

is a full measure $G_\delta$-subset of $D_{m,\sigma}$. For any $t > 0$ and $Z_m \in \Gamma^+$, we define $\Psi^t_m Z_m$ to be the trajectory followed by $Z_m$ up to time $t$. Clearly, for $t > 0$, the map $\Psi^t_m : \Gamma^+ \to \Gamma^+$ is measure-preserving diffeomorphism, it satisfies (4.47) and

$$\Psi^t_m(Z_m) = \Psi^t_m(Z^*_m), \quad \forall Z_m \in \Gamma^+ \cap \partial_{sc,ng} D_{m,\sigma}.$$

For negative times, since similar estimates hold for the backwards in time flow, we may construct a full measure subset $\Gamma^-$, with complement $I^-$, and the flow $\Psi^t_m : \Gamma^- \to \Gamma^-$ in similar way. For $t = 0$, we trivially define $\Psi^0_m$ as the identity map.

We define $\Gamma = \Gamma^+ \cap \Gamma^-$ and $I = I^+ \cup I^-$. Clearly $\Gamma$ is a full measure $G_\delta$-subset of the phase space and $(\Psi^t_m)_{t \in \mathbb{R}} : \Gamma \to \Gamma$ is a family measure-preserving diffeomorphisms satisfying (4.47) and (4.49) on $\Gamma \cap \partial_{sc,ng} D_{m,\sigma}$. Conservation of energy (4.48) is easily obtained in $\Gamma$, since energy is preserved in any of the finitely many, simple, non-grazing collisions occurring up to time $t$. 

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The proof of (4.49): In order to prove that (4.49) holds a.e. on $\Gamma \cap \partial_{sc,ng} \mathcal{D}_{m,\sigma}$, it suffices to prove $I \cap \partial_{sc,ng} \mathcal{D}_{m,\sigma}$ is $\sigma$-null subset of $\partial_{sc,ng} \mathcal{D}_{m,\sigma}$. We denote $\lambda$ to be the Lebesgue measure on $\mathbb{R}^{2dm}$ and $\sigma$ the surface measure induced by $\lambda$ on $\partial_{sc,ng} \mathcal{D}_{m,\sigma}$. Given a time $t_0 > 0$ and any measurable $A \subseteq \partial_{sc,ng} \mathcal{D}_{m,\sigma}$, Fubini’s Theorem implies

$$\lambda \left( \{ \Psi_{t_0}^t y : (y,t) \in A \times (0,t_0) \} \right) = \int_0^{t_0} \sigma \left( \{ \Psi_{t_0}^t y : y \in A \} \right) \, dt = \sigma(A)t_0,$$

since the flow is measure preserving. In particular taking $A = I \cap \partial_{sc,ng} \mathcal{D}_{m,\sigma}$, and assuming $\sigma(A) > 0$, we get that

$$\lambda \left( \{ \Psi_{t_0}^t y : (y,t) \in A \times (0,t_0) \} \right) > 0,$$

which is a contradiction since the flow is a.e. defined. Therefore, $\sigma(A) = 0$, and (4.49) is proved. The proof is complete. \qed

Remark 4.7. We have seen that the flow can be defined only a.e. in $\mathcal{D}_{m,\sigma}$. However to simplify the notation, without loss of generality, we may assume that the flow is well defined on the whole phase space $\mathcal{D}_{m,\epsilon}$.

4.10 The Liouville equation and flow operators

In this section we formally derive the $m$-particle of $\sigma$-interaction zone Liouville equation, for $m \geq 3$, and then define the $m$-particle $\sigma$ interaction zone flow operator and the $m$-particle free flow operator.

Consider $m \geq 3$ and let $P_0$ be a Borel probability measure on $\mathbb{R}^{2dm}$, absolutely continuous with respect to the Lebesgue measure, with a probability density $f_{m,0}$ satisfying the following properties:
• $f_{m,0}$ is supported in $\mathcal{D}_{m,\sigma}$ i.e.

$$\text{supp } f_{m,0} := \{Z_m \in \mathbb{R}^{2dm} : f_{m,0}(Z_m) \neq 0\} \subseteq \mathcal{D}_{m,\sigma}. \quad (4.53)$$

• $f_{m,0}$ is symmetric i.e. for any permutation $p_m$ of the $m$-particles, there holds:

$$f_{m,0}(Z_{p_m}) = f_{m,0}(Z_m), \quad \forall Z_m \in \mathbb{R}^{2dm}. \quad (4.54)$$

Let us note that $P_0$ expresses the initial distribution in space and velocities of the $m$-particles. We are interested in the evolution of this measure under the interaction flow. For this purpose, given $t \geq 0$ we define $P_t$ to be the push-forward of $P_0$ under the interaction flow i.e.

$$P_t(A) = P_0 \left( \Psi_m^{-t}(A) \right), \quad A \subseteq \mathbb{R}^{2dm} \text{ Borel measurable.}$$

Conservation of measure under the interaction flow implies that $P_t$ is absolutely continuous with probability density given by

$$f_m(t, Z_m) = \begin{cases} f_{m,0} \circ \Psi_m^{-t}, & \text{a.e. in } \mathcal{D}_{m,\sigma}, \\ 0, & \text{a.e. in } \mathbb{R}^{2dm} \setminus \mathcal{D}_{m,\sigma}. \end{cases} \quad (4.55)$$

Clearly $f_m(t, Z_m)$ is symmetric and supported in $\mathcal{D}_{m,\sigma}$, for all $t \geq 0$. Moreover, by definition we have

$$f_m(0, Z_m) = f_{m,0} \circ \Psi_m^0(Z_m) = f_{m,0}(Z_m), \quad \text{in } \mathcal{D}_{m,\sigma}. \quad (4.56)$$

Additionally, since $m \geq 3$, (4.49) implies

$$f_m(t, Z_m^*) = f_{m,0} \circ \Psi_m^{-t}(Z_m^*)$$

$$= f_{m,0} \circ \Psi_m^{-t}(Z_m)$$

$$= f_m(t, Z_m), \quad \sigma - \text{a.e. on } \partial_{sc,ng} \mathcal{D}_{m,\sigma}, \quad \forall t \geq 0.$$
Thus,
\[ f_m(t, Z_m^*) = f_m(t, Z_m), \quad \sigma - \text{a.e. on } \partial_{\text{sc,ng}} D_{m,\sigma}, \quad \forall t \geq 0. \quad (4.57) \]

Finally, recall from (4.55) that
\[ f_m(t, Z_m) = f_{m,0} \circ \Psi_m^{-t}(Z_m), \quad \text{a.e. in } D_{m,\sigma}, \quad \forall t \geq 0. \quad (4.58) \]

Combining (4.56), (4.57) and (4.58), we obtain that \( f_m \) satisfies the \( m \)-particle Liouville equation in mild form
\[
\begin{cases}
    f_m(t, Z_m) = f_{m,0}(\Psi_m^{-t}(Z_m)), & \text{a.e. in } D_{m,\sigma}, \quad \forall t \geq 0, \\
    f_m(t, Z_m^*) = f_m(t, Z_m), & \sigma - \text{a.e. on } \partial_{\text{sc,ng}} D_{m,\sigma}, \quad \forall t \geq 0, \\
    f_m(0, Z_m) = f_{m,0}(Z_m), & \text{in } D_{m,\sigma}.
\end{cases}
\quad (4.59)
\]

Formally assuming that \( f_m \) is smooth, the chain rule yields that \( f_m \) needs to satisfy the \( m \)-particle Liouville equation in \( D_{m,\sigma} \):
\[
\begin{cases}
    \partial_t f_m + \sum_{i=1}^m v_i \cdot \nabla_{x_i} f_m = 0, & (t, Z_m) \in (0, \infty) \times \overset{\curvearrowright}{D}_{m,\sigma}, \\
    f_m(t, Z_m^*) = f_m(t, Z_m), & (t, Z_m) \in [0, \infty) \times \partial_{\text{sc}} D_{m,\sigma}, \\
    f_m(0, Z_m) = f_{m,0}(Z_m), & Z_m \in \overset{\curvearrowright}{D}_{m,\sigma}.
\end{cases}
\quad (4.60)
\]

Now, we introduce some notation defining the \( m \)-particle \( \sigma \)-interaction zone flow operator and the \( m \)-particle free flow operator. For convenience, let us denote
\[ C^0(D_{m,\sigma}) := \{ g_m \in C^0(\mathbb{R}^{2dm}) : \text{supp } g_m \subseteq D_{m,\epsilon} \}. \quad (4.61) \]
Definition 4.6. For \( t \in \mathbb{R} \), we define \( m \)-particle \( \sigma \)-interaction zone flow operator
\[
T^t_m : C^0(\mathcal{D}_{m,\sigma}) \to C^0(\mathcal{D}_{m,\sigma})
\]
as:
\[
T^t_m g_m(Z_m) = \begin{cases} 
  g_m(\Psi_m^{-t}Z_m), & \text{if } Z_m \in \mathcal{D}_{m,\sigma}, \\
  0, & \text{if } Z_m \notin \mathcal{D}_{m,\sigma}.
\end{cases}
\] (4.62)

Remark 4.8. Given an initial probability density \( f_{m,0} \), satisfying (4.53)-(4.54), the function
\[
f_m(t, Z_m) = T^t_m f_{m,0}(Z_m)
\]
is formally the unique solution to the Liouville equation (4.60) with initial data \( f_{m,0} \).

We also define the \( m \)-particle free flow operator.

Definition 4.7. For \( t \in \mathbb{R} \) and \( m \in \mathbb{N} \), we define the \( m \)-particle free flow operator
\[
S^t_m : C^0(\mathbb{R}^{2dm}) \to C^0(\mathbb{R}^{2dm})
\]
as:
\[
S^t_m g_m(Z_m) = g_m(\Phi_m^{-t}Z_m) = g_m(X_m - tV_m, V_m).
\] (4.63)
Chapter 5

BBGKY hierarchy, Boltzmann hierarchy and the ternary Boltzmann equation

In this chapter we consider \( N \)-particles of \( \epsilon \)-interaction zone, where \( N \geq 3 \) and \( 0 < \epsilon << 1 \). We integrate the \( N \)-particle Liouville’s equation to formally obtain a linear hierarchy of integro-differential equations satisfied by the marginals of its solution (BBGKY hierarchy). We then formally derive the limiting hierarchy (Boltzmann hierarchy) occurring under the appropriate scaling and formally show it reduces to a nonlinear integro-differential equation (the new ternary Boltzmann equation) for chaotic initial data. Then, we investigate important properties of the ternary Boltzmann equation, which as expected are related to analogous properties of the Boltzmann equation.

5.1 The BBGKY hierarchy

Consider \( N \)-particles of interaction zone \( 0 < \epsilon << 1 \), where \( N \geq 3 \). For \( s \in \mathbb{N} \), we define the \( s \)-marginal of a symmetric probability density \( f_N \), supported in \( D_{N,\epsilon} \), as
where for $Z_s = (X_s, V_s) \in \mathbb{R}^{2d s}$, we write $Z_N = (X_s, x_{s+1}, \ldots, x_N, V_s, v_{s+1}, \ldots, v_N)$. It is straightforward that, for all $1 \leq s \leq N$, the marginals $f_N^{(s)}$ are symmetric probability densities, supported in $\mathcal{D}_{s, \epsilon}$ and

$$f_N^{(s)}(Z_s) = \int_{\mathbb{R}^{2d(N-s)}} f_N(Z_N) \, dx_{s+1} \ldots dx_N \, dv_{s+1} \ldots dv_N, \quad 1 \leq s < N,$$

$$f_N^{(s)}(Z_s) = f_N, \quad s = N,$$

$$0, \quad s > N,$$

$$f_N^{(s)}(Z_s) = \int_{\mathbb{R}^{2d}} f_N^{(s+1)}(X_N, V_N) \, dx_{s+1} \, dv_{s+1}, \quad \forall 1 \leq s \leq N - 1.$$

Assume now that $f_N$ is formally the solution to the $N$-particle Liouville equation (4.60) with initial data $f_{N,0}$. We seek to formally find a hierarchy of equations satisfied by the marginals of $f_N$. The answer is obvious for $s \geq N$ since by definition

$$f_N^{(s)} = 0, \quad \text{for } s > N,$$

and

$$f_N^{(N)} = f_N$$

We observe that $\partial \mathcal{D}_{N, \epsilon}$ is equivalent up to surface measure zero to $\Sigma^X \times \mathbb{R}^{dN}$ where

$$\Sigma^X := \bigcup_{(i,j,k) \in J_N} \Sigma_{i,j,k}^{sc,X},$$

and $\Sigma_{i,j,k}^{sc,X}$ are the projections in space of $\Sigma_{i,j,k}^{sc,X}$, given in (4.8). Notice that (5.4) is a pairwise disjoint union.

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The main part of the hierarchy will come after integrating by parts the Liouville equation. Consider $1 \leq s \leq N - 1$. The boundary and initial conditions can be easily recovered integrating Liouville’s equation boundary and initial conditions respectively i.e.

$$
\begin{cases}
  f_N^{(s)}(t, Z_s^*) = f_N^{(s)}(t, Z_s), & (t, Z_s) \in [0, \infty) \times \partial_{sc} D_{s,\epsilon}, \quad s \geq 3, \\
  f_N^{(s)}(0, Z_s) = f_N^{(s)}(0, Z_s), & Z_s \in \mathcal{D}_{s,\epsilon}.
\end{cases}
$$

(5.5)

Notice that for $s = 1, 2$ there is no boundary condition, since $D_{s,\epsilon} = \mathbb{R}^{2d_s}$ by (4.4).

Consider now a smooth test function $\phi_s$ compactly supported in $(0, \infty) \times \mathcal{D}_{s,\epsilon}$ such that whenever $(i, j, k) \in \mathcal{J}_N$ with $j \leq s$, the following holds:

$$
\phi_s(t, p_s Z_N^*) = \phi_s(t, p_s Z_N) = \phi_s(t, Z_s), \quad \forall (t, Z_N) \in (0, \infty) \times \Sigma_{s,ijk}^{sc},
$$

(5.6)

where $p_s : \mathbb{R}^{2d_N} \to \mathbb{R}^{2d_s}$ denotes the natural projection in space and velocities, given by

$$
p_s(Z_N) = Z_s, \quad \forall Z_N \in \mathbb{R}^{2d_N}.
$$

Multiplying the Liouville equation by $\phi_s$, and integrating over $(0, \infty) \times \mathcal{D}_{N,\epsilon}$, we obtain its weak form

$$
\int_{(0,\infty) \times \mathcal{D}_{N,\epsilon}} \left( \partial_t f_N(t, Z_N) + \sum_{i=1}^{N} v_i \nabla_{x_i} f_N(t, Z_N) \right) \phi_s(t, Z_s) \, dX_N \, dV_N \, dt = 0.
$$

(5.7)

For the time derivative in (5.7), we use Fubini’s Theorem and integration by parts.
in time to obtain

\[
\int_{(0, \infty) \times \mathcal{D}_{s, \epsilon}} \partial_t f_N(t, Z_N) \phi_s(t, Z_s) \, dX_N \, dV_N \, dt
\]

\[= - \int_{\mathcal{D}_{s, \epsilon}} \int_0^\infty f_N(t, Z_N) \partial_t \phi_s(t, Z_s) \, dt \, dX_N \, dV_N \quad (5.9)\]

\[= - \int_0^\infty \int_{\mathbb{R}^{2dN}} f_N(t, Z_N) \partial_t \phi_s(t, Z_s) \, dX_N \, dV_N \, dt \quad (5.10)\]

\[= - \int_0^\infty \int_{\mathbb{R}^{2dN}} f_N^{(s)}(t, Z_s) \partial_t \phi_s(t, Z_s) \, dX_s \, dV_s \, dt \quad (5.11)\]

\[= - \int_{(0, \infty) \times \mathcal{D}_{s, \epsilon}} \partial_t f_N^{(s)}(t, Z_s) \phi_s(t, Z_s) \, dX_s \, dV_s \, dt, \quad (5.12)\]

where to obtain (5.9), we integrate by parts in time and use the fact that \( \phi_s \) is compactly supported in \((0, \infty) \times \mathcal{D}_{s, \epsilon}\), to obtain (5.10) we use the fact that \( f_N \) is supported in \((0, \infty) \times \mathcal{D}_{N, \epsilon}\), to obtain (5.11) we use (5.1), and to obtain (5.12) we integrate by parts over time, and use again the fact that \( \phi_s \) is compactly supported in \((0, \infty) \times \mathcal{D}_{s, \epsilon}\).

For the material derivative term in (5.7), the Divergence Theorem implies that for any \( t > 0 \), we have

\[
\int_{\mathcal{D}_{N, \epsilon}} \sum_{i=1}^N v_i \nabla_{x_i} f_N(t, Z_N) \phi_s(t, Z_s) \, dX_N \, dV_N
\]

\[= \int_{\mathcal{D}_{N, \epsilon}} \text{div}_{X_N} \left[ f_N(t, Z_N) V_N \right] \phi_s(t, Z_s) \, dX_N \, dV_N
\]

\[= A_1 + A_2, \quad (5.14)\]

where

\[A_1 := - \int_{\mathcal{D}_{N, \epsilon}} V_N \cdot \nabla_{X_N} \phi_s(t, Z_s) f_N(t, Z_N) \, dX_N \, dV_N,\]
\( A_2 := \int_{\Sigma^X \times \mathbb{R}^d} \hat{n}(X_N) \cdot V_N f_N(t, Z_N) \phi_s(t, Z_s) \ dV_N \ d\sigma, \)

and \( \Sigma^X \) is given by (5.4), \( \hat{n}(X_N) \) is the outwards normal vector on \( \Sigma^X \) at \( X_N \in \Sigma^X \), \( d\sigma \) is the surface measure on \( \Sigma^X \). Moreover

\[
\int_{D_{N,\epsilon}} V_N \cdot \nabla X_N \phi_s(t, Z_s) f_N(t, Z_N) \ dX_N \ dV_N = \int_{\mathbb{R}^{2dN}} V_s \cdot \nabla X_s \phi_s(t, Z_s) f_N(t, Z_N) \ dX_N \ dV_N
\]

(5.15)

\[
= \int_{\mathbb{R}^{2dN}} V_s \cdot \nabla X_s \phi_s(t, Z_s) f_N^{(s)}(t, Z_s) \ dX_s \ dV_s
\]

(5.16)

\[
= - \int_{\mathbb{R}^{2dN}} \text{div}_{X_s} \left[ f_N^{(s)}(t, Z_s) V_s \right] \phi_s(t, Z_s) \ dX_s \ dV_s
\]

(5.17)

\[
= - \int_{D_{N,\epsilon}} \sum_{i=1}^s v_i \nabla x_i f_N^{(s)}(t, Z_s) \phi_s(t, Z_s) \ dX_s \ dV_s,
\]

(5.18)

where to obtain (5.16) we use the fact that \( f_N \) is supported in \( D_{N,\epsilon} \), to obtain (5.17) we use (5.1), and to obtain (5.18) we use again the Divergence Theorem and the fact that \( \phi_s \) is compactly supported in \((0, \infty) \times D_{s,\epsilon}\). Combining (5.7), (5.12), (5.14), (5.19), and recalling the space boundary decomposition (5.4), we obtain

\[
\left( t \right) \in \bigcup_{(i,j,k) \in I_N} \left( 0, \infty \right) \times \Sigma_{i,j,k} \times \mathbb{R}^d \quad 0 < t < \infty \quad \hat{n}_{ijk} \right) \cdot V_N f_N(t, Z_N) \phi_s(t, Z_s) \ dV_N \ d\sigma_{ijk} \ dt = \]

\[
= \sum_{(i,j,k) \in I_N} \int_{\left( 0, \infty \right) \times \Sigma_{i,j,k} \times \mathbb{R}^d} \hat{n}_{ijk}(X_N) \cdot V_N f_N(t, Z_N) \phi_s(t, Z_s) \ dV_N \ d\sigma_{ijk} \ dt
\]

(5.19)

where for \( (i, j, k) \in I_N \) and \( t > 0 \), we denote

\[
C_{ijk}(t) = - \int_{\Sigma_{i,j,k} \times \mathbb{R}^d} \hat{n}_{ijk}(X_N) \cdot V_N f_N(t, Z_N) \phi_s(t, Z_s) \ dV_N \ d\sigma_{ijk},
\]

(5.20)
and \( \mathbf{n}_{ijk}(X_N) \) is the outwards normal vector on \( \Sigma_{ijk}^{sc,X} \) at \( X_N \in \Sigma_{ijk}^{sc,X} \), \( d\sigma_{ijk} \) is the surface measure on \( \Sigma_{ijk}^{sc,X} \). We easily calculate

\[
- \mathbf{n}_{ijk}(X_N) \cdot V_N = (\sqrt{2})^{-1} \frac{\langle \frac{x_j-x_i}{\sqrt{2} \epsilon}, v_j - v_i \rangle + \langle \frac{x_k-x_i}{\sqrt{2} \epsilon}, v_k - v_i \rangle}{\sqrt{1 + \langle \frac{x_j-x_i}{\sqrt{2} \epsilon}, \frac{x_k-x_i}{\sqrt{2} \epsilon} \rangle}}.
\]

(5.22)

Notice that since we are integrating over \( \Sigma_{ijk}^{sc,X} \), we have

\[
\left( \frac{x_j-x_i}{\sqrt{2 \epsilon}}, \frac{x_k-x_i}{\sqrt{2 \epsilon}} \right) \in S_{1}^{2d-1}.
\]

Making the change of variables \((v_i, v_j, v_k) \rightarrow (v_i^*, v_j^*, v_k^*)\), under the collisional transformation induced by \( \left( \frac{x_j-x_i}{\sqrt{2 \epsilon}}, \frac{x_k-x_i}{\sqrt{2 \epsilon}} \right) \), using (5.22), Proposition 3.1 parts (iv), (v) and the boundary condition of (4.60), we obtain

\[
C_{ijk}(t) = (\sqrt{2})^{-1} \int_{\mathbb{R}^{2dN} \times \Sigma_{ijk}^{sc,X}} \frac{\langle \frac{x_j-x_i}{\sqrt{2 \epsilon}}, v_j^* - v_i^* \rangle + \langle \frac{x_k-x_i}{\sqrt{2 \epsilon}}, v_k^* - v_i^* \rangle}{\sqrt{1 + \langle \frac{x_j-x_i}{\sqrt{2 \epsilon}}, \frac{x_k-x_i}{\sqrt{2 \epsilon}} \rangle}} f_N(t, Z_N^*) \phi_s(t, \pi_s Z_N^*) dV_N d\sigma_{ijk}
\]

\[
= -(\sqrt{2})^{-1} \int_{\mathbb{R}^{2dN} \times \Sigma_{ijk}^{sc,X}} \frac{\langle \frac{x_j-x_i}{\sqrt{2 \epsilon}}, v_j - v_i \rangle + \langle \frac{x_k-x_i}{\sqrt{2 \epsilon}}, v_k - v_i \rangle}{\sqrt{1 + \langle \frac{x_j-x_i}{\sqrt{2 \epsilon}}, \frac{x_k-x_i}{\sqrt{2 \epsilon}} \rangle}} f_N(t, Z_N^*) \phi_s(t, \pi_s Z_N^*) dV_N d\sigma_{ijk}.
\]

(5.24)

Equations (5.21)-(5.24) and the test function condition (5.6) imply

\[
C_{ijk}(t) = 0, \quad \forall (i, j, k) \notin \tilde{\mathcal{I}}_N, \quad \forall t > 0,
\]

(5.25)

where

\[
\tilde{\mathcal{I}}_N := \{(i, j, k) \in \mathcal{I}_N : 1 \leq i \leq s < j < k \leq N\}.
\]

(5.26)
Notice we immediately take that the \((N-1)\)-marginal satisfies the \((N-1)\)-Liouville equation i.e.

\[
\begin{aligned}
\partial_t f_{N}^{(N-1)} + \sum_{i=1}^{N} u_i \nabla_{x_i} f_{N}^{(N-1)} = 0, & \quad (t, Z_{N-1}) \in (0, \infty) \times \hat{\mathcal{D}}_{N-1, \epsilon}, \\
\end{aligned}
\]

\(f_{N}^{(N-1)}(t, Z_{N-1}) = f_{N}^{(N-1)}(t, Z_{N-1}), & \quad (t, Z_{N-1}) \in [0, \infty) \times \partial_{sc} \mathcal{D}_{N-1, \epsilon},
\]

\(f_{N}^{(N-1)}(0, Z_{N-1}) = f_{N,0}^{(N-1)}(Z_{N-1}), & \quad Z_{N-1} \in \hat{\mathcal{D}}_{N-1, \epsilon}.
\]

This fact comes in agreement with physical intuition. Indeed, since we have three particles interaction we do not expect to be able recover \(f_{N}^{(N-1)}\) from \(f_{N}^{(N)}\).

For \(1 \leq s \leq N - 2\) and \((i,j,k) \in \tilde{J}_N\), Fubini’s Theorem implies that the \((dN-1)\)-surface measure on \(\Sigma_{ijk}^{sc,X}\) can be written as

\[
d\sigma_{ijk}(X_N) = dS_{x_i}(x_j, x_k) \prod_{\ell=1, \ell \neq j,k}^{N} dx_{\ell},
\]

where, given \(x_i \in \mathbb{R}^d\), \(dS_{x_i}\) is the \((2d-1)\)-surface measure on the \((2d-1)\)-sphere of center \((x_i, x_i) \in \mathbb{R}^{2d}\) and radius \(\sqrt{2\epsilon}\):

\[
S_{x_i} = \{ (x_j, x_k) \in \mathbb{R}^{2d} : |x_j - x_i|^2 + |x_k - x_i|^2 = 2\epsilon^2 \}.
\]

The decomposition (5.28) and the symmetry assumption on \(f_N\) yield that

\[
C_{ijk}(t) = C_{i,s+1,s+2}(t), & \quad \forall (i,j,k) \in \tilde{J}_N, \quad \forall t > 0.
\]
Therefore, recalling (5.25)-(5.26), (5.30) yield

\[ \sum_{(i,j,k)\in J_N} C_{ijk}(t) = \sum_{(i,j,k)\in \tilde{J}_N} C_{ijk}(t) \]

\[ = \sum_{i=1}^{s} \sum_{j=s+1}^{N-1} \sum_{k=j+1}^{N} C_{i,s+1,s+2}(t) \]

\[ = \sum_{i=1}^{s} \sum_{j=s+1}^{N-1} (N-j)C_{i,s+1,s+2}(t) \]

\[ = (1 + 2 + \ldots + N - s - 1) \sum_{i=1}^{s} C_{i,s+1,s+2}(t) \]

\[ = \frac{1}{2} (N-s)(N-s-1) \sum_{i=1}^{s} C_{i,s+1,s+2}(t), \quad \forall t > 0. \] (5.31)

Hence, thanks to (5.21)-(5.22), (5.31), (5.20) can be rewritten as

\[ \int_{(0,\infty)\times D_{s,\epsilon}} \left( \partial_t f_N^{(s)}(t,Z_s) + \sum_{i=1}^{s} v_i \nabla_x f_N^{(s)}(t,Z_s) \right) \phi_s(t,Z_s) \, dX_s \, dV_s \, dt \]

\[ = \frac{1}{2} (N-s)(N-s-1) \sum_{i=1}^{s} \int_{0}^{\infty} C_{i,s+1,s+2}(t) \, dt. \] (5.32)

Let us fix \( i \in \{1,\ldots,s\} \). Performing the change of variables

\[ \begin{cases} 
\omega_1 = \frac{x_{s+1} - x_i}{\sqrt{2\epsilon}}, \\
\omega_2 = \frac{x_{s+2} - x_i}{\sqrt{2\epsilon}},
\end{cases} \]

and recalling the notation from (3.5)

\[ b(\omega_1,\omega_2, v_{s+1} - v_i, v_{s+2} - v_i) = \langle \omega_1, v_{s+1} - v_i \rangle + \langle \omega_2, v_{s+2} - v_i \rangle, \]

we obtain thanks to (5.21)-(5.22), (5.28), (5.1), the fact that \( \text{supp} \, f_N^{(s+2)} \subseteq D_{s+2,\epsilon} \).
that
\[
\int_0^\infty C_{i,s+1,s+2}(t) \, dt =
\int_{(0,\infty) \times D_{s,\epsilon}} 2^{d-1} \epsilon^{2d-1} \int_{\mathbb{R}^{2d-1} \times \mathbb{R}^d} \frac{b(\omega_1, \omega_2, v_{s+1} - v_i, v_{s+2} - v_i)}{\sqrt{1 + \langle \omega_1, \omega_2 \rangle}}
\times f_N^{(s+2)}(t, X_s, x_i + \sqrt{2\epsilon} \omega_1, x_i + \sqrt{2\epsilon} \omega_2, V_s, v_{s+1}, v_{s+2}) \, d\omega_1 \, d\omega_2 \, dv_{s+1} \, dv_{s+2} \, dX_s \, dV_s \, dt.
\] (5.33)

Splitting the cross-section to positive and negative parts, followed by an application of the relevant boundary condition to the positive part ((5.5), or the boundary condition of (4.60) if \(s = N - 2\), and use of the substitution \((\omega_1, \omega_2) \to (-\omega_1, -\omega_2)\) for the negative part, the right hand side of (5.33) becomes:

\[
\int_{(0,\infty) \times D_{s,\epsilon}} 2^{d-1} \epsilon^{2d-1} \int_{\mathbb{R}^{2d-1} \times \mathbb{R}^d} b_+(\omega_1, \omega_2, v_{s+1} - v_i, v_{s+2} - v_i)
\times \left( f_N^{(s+2)}(t, Z_{s+2,\epsilon}^i) - f_N^{(s+2)}(t, Z_{s+2,\epsilon}^i) \right) \, d\omega_1 \, d\omega_2 \, dv_{s+1} \, dv_{s+2} \, dX_s \, dV_s \, dt,
\] (5.34)

where given \(i \in \{1, \ldots, s\}\), we denote
\[
Z_{s+2,\epsilon}^i = (x_1, \ldots, x_i, \ldots, x_s, x_i - \sqrt{2\epsilon} \omega_1, x_i - \sqrt{2\epsilon} \omega_2, v_1, \ldots v_{i-1}, v_i, v_{i+1}, \ldots, v_s, v_{s+1}, v_{s+2}),
\]
\[
Z_{s+2,\epsilon}^* = (x_1, \ldots, x_i, \ldots, x_s, x_i + \sqrt{2\epsilon} \omega_1, x_i + \sqrt{2\epsilon} \omega_2, v_1, \ldots v_{i-1}, v_i^*, v_{i+1}, \ldots, v_s, v_{s+1}^*, v_{s+2}).
\] (5.35)

Finally, we combine (5.2)-(5.3), (5.27), (5.32)-(5.34), to formally obtain the BBGKY hierarchy for \(s \in \mathbb{N}\):

\[
\begin{aligned}
\partial_t f_N^{(s)} + \sum_{i=1}^s v_i \nabla_{x_i} f_N^{(s)} &= \mathcal{C}^{N}_{s,s+2} f_N^{(s+2)}, \quad (t, Z_s) \in (0, \infty) \times \mathcal{D}_{s,\epsilon}, \\
f_N^{(s)}(t, Z_s^*) = f_N^{(s)}(t, Z_s), \quad (t, Z_s) \in [0, \infty) \times \partial_{sc} \mathcal{D}_{s,\epsilon}, \text{ whenever } s \geq 3, \\
f_N^{(s)}(0, Z_s) = f_N^{(s)}(0, Z_s), \quad Z_s \in \mathcal{D}_{s,\epsilon},
\end{aligned}
\] (5.36)
where
\[ \mathcal{E}_{s,s+2}^N = \mathcal{E}_{s,s+2}^{N,+} - \mathcal{E}_{s,s+2}^{N,-}. \]  
(5.37)

For \(1 \leq s \leq N - 2\) we denote
\[ \mathcal{E}_{s,s+2}^{N,+} f_N^{(s+2)}(t, Z_s) = A_{N,\epsilon,s} \sum_{i=1}^{s} \int_{\mathbb{R}^{2d-1} \times \mathbb{R}^{2d}} \frac{b_+}{\sqrt{1 + \langle \omega_1, \omega_2 \rangle}} \times f_N^{(s+2)}(t, Z_{s+2,i}^*) \, d\omega_1 \, d\omega_2 \, dv_{s+1} \, dv_{s+2}, \]  
(5.38)

\[ \mathcal{E}_{s,s+2}^{N,-} f_N^{(s+2)}(t, Z_s) = A_{N,\epsilon,s} \sum_{i=1}^{s} \int_{\mathbb{R}^{2d-1} \times \mathbb{R}^{2d}} \frac{b_+}{\sqrt{1 + \langle \omega_1, \omega_2 \rangle}} \times f_N^{(s+2)}(t, Z_{s+2,i}^*) \, d\omega_1 \, d\omega_2 \, dv_{s+1} \, dv_{s+2}, \]  
(5.39)

where we use the notation
\[ A_{N,\epsilon,s} = 2^{d-2}(N - s)(N - s - 1)\epsilon^{2d-1}, \]

\[ b_+ = \max\{b, 0\}, \]

\[ b = b(\omega_1, \omega_2, v_{s+1} - v_i, v_{s+2} - v_i), \]

\[ Z_{s+2,i} = (x_1, \ldots, x_i, \ldots, x_s, x_i - \sqrt{2\epsilon}\omega_1, x_i - \sqrt{2\epsilon}\omega_2, v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_s, v_{s+1}, v_{s+2}), \]

\[ Z_{s+2,i}^* = (x_1, \ldots, x_i, \ldots, x_s, x_i + \sqrt{2\epsilon}\omega_1, x_i + \sqrt{2\epsilon}\omega_2, v_1, \ldots, v_{i-1}, v_i^*, v_{i+1}, \ldots, v_s, v_{s+1}, v_{s+2}^*). \]  
(5.40)

For \(s \geq N - 1\) we trivially define
\[ \mathcal{E}_{s,s+2}^N \equiv 0. \]  
(5.41)

Duhamel’s formula implies that the BBGKY hierarchy can be written in mild form as follows
\[ f_N^{(s)}(t, Z_s) = T_s f_N^{(s)}(Z_0) + \int_0^t T_{t-\tau} \mathcal{E}_{s,s+2} f_N^{(s+2)}(\tau, Z_s) \, d\tau, \quad s \in \mathbb{N}, \]  
(5.42)

where \(T_s\) is the \(s\)-particle \(\epsilon\)-interaction zone flow operator given in \([4.62]\).
5.2 The Boltzmann hierarchy

We will now derive the Boltzmann hierarchy as the formal limit of the BBGKY hierarchy as $N \to \infty$ and $\epsilon \to 0^+$ under the scaling

$$N \epsilon^{d-1/2} \simeq 1. \tag{5.43}$$

This scaling guarantees that for a fixed $s \in \mathbb{N}$, we have

$$A_{N,\epsilon,s} \to 1, \quad \text{as } N \to \infty \text{ and } \epsilon \to 0^+ \text{ in the scaling (5.43).}$$

Formally taking the limit under the scaling imposed we may define the following collisional operator:

$$C_{\infty,s,s}^{\infty} f^{(s+2)}(t, Z_s) = \sum_{i=1}^{s} \int_{(S_1^{2d-1} \times \mathbb{R}^{2d})} \frac{b_+}{\sqrt{1 + \langle \omega_1, \omega_2 \rangle}} f^{(s+2)}(t, Z_{s,i}^{*}) \times d\omega_1 d\omega_2 dv_{s+1} dv_{s+2}, \tag{5.44}$$

$$C_{\infty,s,s}^{-\infty} f^{(s+2)}(t, Z_s) = \sum_{i=1}^{s} \int_{(S_1^{2d-1} \times \mathbb{R}^{2d})} \frac{b_+}{\sqrt{1 + \langle \omega_1, \omega_2 \rangle}} \times f^{(s+2)}(t, Z_{s,i}^{*}) \times d\omega_1 d\omega_2 dv_{s+1} dv_{s+2}, \tag{5.45}$$

and

$$b_+ = \max\{b, 0\},$$

$$b = b(\omega_1, \omega_2, v_{s+1} - v_i, v_{s+2}, v_i),$$

$$Z_{s+2}^i = (x_1, \ldots, x_i, \ldots, x_{s}, x_i, x_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_s, v_{s+1}, v_{s+2}), \tag{5.46}$$

$$Z_{s+2}^{i*} = (x_1, \ldots, x_i, \ldots, x_{s}, x_i, x_1, \ldots, v_{i-1}, v_i^*, v_{i+1}, \ldots, v_s, v_{s+1}^*, v_{s+2}^*).$$
Now we are ready to introduce the Boltzmann hierarchy. More precisely, given an initial probability density $f_0$, the Boltzmann hierarchy for $s \in \mathbb{N}$ is given by:

$$\begin{align*}
\frac{\partial}{\partial t} f^{(s)} + \sum_{i=1}^{s} v_i \nabla_x f^{(s)} &= C_{s,s+2}^\infty f^{(s+2)}, \quad (t, Z_s) \in (0, \infty) \times \mathbb{R}^{2d}, \\
f^{(s)}(0, Z_s) &= f^{(s)}_0(Z_s), \quad \forall Z_s \in \mathbb{R}^{2d}.
\end{align*}$$

(5.47)

Duhamel's formula implies that the Boltzmann hierarchy can be written in mild form as follows

$$f^{(s)}(t, Z_s) = S^t_s f^{(s)}_0(Z_s) + \int_0^t S^{t-\tau}_s C_{s,s+2}^\infty f^{(s+2)}(\tau, Z_s) \, d\tau, \quad s \in \mathbb{N}. \quad (5.48)$$

where $S^t_s$ denotes $s$-particle free flow operator given in (4.63).

### 5.3 The ternary Boltzmann equation

In most applications, particles are initially independently distributed. This translates to factorized Boltzmann hierarchy initial data i.e.

$$f^{(s)}_0(Z_s) = f^{\otimes s}_0(Z_s) = \prod_{i=1}^{s} f_0(x_i, v_i), \quad s \in \mathbb{N}, \quad (5.49)$$

where $f_0 : \mathbb{R}^{2d} \to \mathbb{R}$ is a given function. One can easily verify that the ansatz

$$f^{(s)}(t, Z_s) = f^{\otimes s}(t, Z_s) = \prod_{i=1}^{s} f(t, x_i, v_i), \quad s \in \mathbb{N}, \quad (5.50)$$

solves the Boltzmann hierarchy with initial data given by (5.49), if $f : [0, \infty) \times \mathbb{R}^{2d} \to \mathbb{R}$ satisfies the following nonlinear integro-differential equation:

$$\begin{align*}
\frac{\partial}{\partial t} f + v \cdot \nabla_x f &= Q_3(f, f, f), \quad (t, x, v) \in (0, \infty) \times \mathbb{R}^{2d}, \\
f(0, x, v) &= f_0(x, v), \quad (x, v) \in \mathbb{R}^{2d}, \quad (5.51)
\end{align*}$$
which we call the ternary Boltzmann equation. The collisional operator $Q_3$ is given by

$$Q_3(f, f, f)(t, x, v) = \int_{S^{2d-1} \times \mathbb{R}^{2d}} b_+ \frac{b}{\sqrt{1 + \langle \omega_1, \omega_2 \rangle}} (f^* f_1^* f_2^* - f f_1 f_2) \, d\omega_1 \, d\omega_2 \, dv_1 \, dv_2,$$

(5.52)

where

$$b_+ = \max\{b, 0\},$$

$$b = b(\omega_1, \omega_2, v_{s+1} - v_i, v_{s+2} - v_i),$$

$$f^* = f(t, x, v^*), \quad f = f(t, x, v),$$

(5.53)

$$f_1^* = f(t, x, v_1^*), \quad f_1 = f(t, x, v_1),$$

$$f_2^* = f(t, x, v_2^*), \quad f = f(t, x, v_2).$$

Duhamel’s formula implies the ternary Boltzmann equation can be written in mild form as follows

$$f(t, x, v) = S_1^t f_0(x, v) + \int_0^t S_1^{t-\tau} Q_3(f, f, f)(\tau, x, v) \, d\tau,$$

(5.54)

where

$$S_1^t g(x, v) = g(x - tv, v), \quad \forall (t, x, v) \in [0, \infty) \times \mathbb{R}^{2d}, \quad g : \mathbb{R}^{2d} \to \mathbb{R}.$$

**Remark 5.1.** We will see in Chapter [4] that both mild Boltzmann hierarchy and mild ternary Boltzmann equation are well-posed in appropriate functional spaces. Therefore, the tensorized product of the mild solution to the ternary Boltzmann equation will give the unique mild solution to the Boltzmann hierarchy.

Since the operator $Q_3$ tracks only central collision particles, asymmetry can be observed in its weak formulation, therefore entropy production and conservation...
laws are not expected. However, if one considers a symmetrized operator $\tilde{Q}_3$ where both central and adjacent collision particles are tracked, we can recover the entropy dissipation and the conservation laws of the classical Boltzmann equation (see [17]).

5.4 The symmetrized ternary Boltzmann equation

As pointed out in Remark 2.1, starting from the phase space (2.7), one can derive a symmetrized version of the operator $Q_3$, where the tracked particle faces interactions both as a central and an adjacent particle. In particular, after deriving the corresponding BBGKY and Boltzmann hierarchies in the scaling (5.43), one arrives at the symmetrized ternary Boltzmann equation:

$$
\begin{aligned}
\partial_t f + v \cdot \nabla f &= \tilde{Q}_3(f, f, f), \quad (t, x, v) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d, \\
f(0, x, v) &= f_0(x, v), \quad \mathbb{R}^d \times \mathbb{R}^d,
\end{aligned}
$$

(5.55)

where

$$
\tilde{Q}_3(f, f, f) = Q_3(f, f, f) + 2Q'_3(f, f, f),
$$

(5.56)

$Q_3(f, f, f)$ is given by (5.52)-(5.53) and

$$
Q'_3(f, f, f) = \int_{\mathbb{R}^{2d-1} \times \mathbb{R}^d} \frac{b_+ (\omega_1, \omega_2, v - v_1, v_2 - v_1)}{\sqrt{1 + \langle \omega_1, \omega_2 \rangle}} (f^{1*} f^{1*} f^{2*} - f f f_2) d\omega_1 d\omega_2 dv_1 dv_2,
$$

(5.57)

$$
\begin{aligned}
f^{1*} &:= f(v^{1*}), \quad v^{1*} := v - c_{\omega_1, \omega_2, v_1, v_2} \omega_1, \\
f^{1*} &:= f(v^{1*}), \quad v^{1*} := v_1 + c_{\omega_1, \omega_2, v_1, v_2} (\omega_1 + \omega_2), \\
f^{2*} &:= f(v^{2*}), \quad v^{2*} := v_2 - c_{\omega_1, \omega_2, v_2, v_1, v_2} \omega_2, \\
c_{\omega_1, \omega_2, v_1, v_2} &= \frac{\langle \omega_1, v - v_1 \rangle + \langle \omega_2, v_2 - v_1 \rangle}{1 + \langle \omega_1, \omega_2 \rangle}.
\end{aligned}
$$

(5.58)

It turns out that the symmetrized operator $\tilde{Q}_3$ satisfies analogous statistical and entropy production properties as the classical Boltzmann equation operator.
We include these calculations for the sake of completeness since this is the first time this type of ternary equations appears in the literature. In the following, most calculations are formal, assuming that the functions involved are continuous and sufficiently decreasing in velocities for the integrals involved to make sense.

5.4.1 Weak form and the Boltzmann inequality

First, one can derive a weak formulation for the operator \( \tilde{Q}_3 \) which implies entropy dissipation as in the classical case.

**Proposition 5.5.** Consider a function \( f : [0, \infty) \times \mathbb{R}^2d \rightarrow \mathbb{R} \). Then the following hold:

(i) For any test function \( \phi : [0, \infty) \times \mathbb{R}^2d \rightarrow \mathbb{R} \), the following weak identity holds:

\[
\int_{\mathbb{R}^d} \tilde{Q}_3(f, f, f) \phi \, dv = \frac{1}{2} \int_{\mathbb{S}^{2d-1} \times \mathbb{R}^d} \frac{b_+(\omega_1, \omega_2, v_1 - v, v_2 - v)}{\sqrt{1 + \langle \omega_1, \omega_2 \rangle}} (f^* f_1^* f_2^* - f f_1 f_2) \times (\phi + \phi_1 + \phi_2 - \phi^* - \phi_1^* - \phi_2^*) \, d\omega_1 \, d\omega_2 \, dv_1 \, dv_2 \, dv, \tag{5.59}
\]

where we use notation from (5.53).

(ii) In case \( f > 0 \), the following analogue of the Boltzmann inequality (see [17]) holds:

\[
\tilde{D}(f) := \int_{\mathbb{R}^d} \tilde{Q}_3(f, f, f) \ln f \, dv \leq 0, \tag{5.60}
\]

i.e. the so-defined entropy dissipation \( \tilde{D}(f) \) is negative.
Proof. (i) Let us write

\[ A = \frac{\int_{S^2_{d-1}} b_+ (\omega_1, \omega_2, v_1 - v, v_2 - v)}{\sqrt{1 + \langle \omega_1, \omega_2 \rangle}} (f^* f_1 f_2^* - f f_1 f_2) \phi \, d\omega_1 \, d\omega_2 \, dv_1 \, dv_2 \, dv \]

\[ A^* = \frac{\int_{S^2_{d-1}} b_+ (\omega_1, \omega_2, v_1 - v, v_2 - v)}{\sqrt{1 + \langle \omega_1, \omega_2 \rangle}} (f^* f_1 f_2^* - f f_1 f_2) \phi^* \, d\omega_1 \, d\omega_2 \, dv_1 \, dv_2 \, dv, \]

and for \( i \in \{1, 2\} \)

\[ A_i = \frac{\int_{S^2_{d-1}} b_+ (\omega_1, \omega_2, v_1 - v, v_2 - v)}{\sqrt{1 + \langle \omega_1, \omega_2 \rangle}} (f^* f_1 f_2^* - f f_1 f_2) \phi_i \, d\omega_1 \, d\omega_2 \, dv_1 \, dv_2 \, dv \]

\[ A_i^* = \frac{\int_{S^2_{d-1}} b_+ (\omega_1, \omega_2, v_1 - v, v_2 - v)}{\sqrt{1 + \langle \omega_1, \omega_2 \rangle}} (f^* f_1 f_2^* - f f_1 f_2) \phi_i^* \, d\omega_1 \, d\omega_2 \, dv_1 \, dv_2 \, dv. \]

We proceed as follows:

- Making the involutionary substitution \((v^*, v_1^*, v_2^*) \rightarrow (v, v_1, v_2)\) and using (3.14), we see that
  \[ A^* = -A, \quad A_i^* = -A_i, \text{ for } i \in \{1, 2\}. \]  \hspace{1cm} (5.61)

- Making the substitution \((\omega_1, \omega_2, v_1, v_2) \rightarrow (\omega_2, \omega_1, v_2, v_1)\), we obtain
  \[ A_2 = A_1. \]  \hspace{1cm} (5.62)

- Making the substitution \((v, v_1) \rightarrow (v_1, v)\) we see that
  \[ A_1 = \int_{\mathbb{R}^d} Q'(f, f, f) \phi \, dv. \]  \hspace{1cm} (5.63)

Therefore using (5.61)-(5.63), we conclude

\[ 2 \int_{\mathbb{R}^d} \tilde{Q}_3(f, f, f) \phi \, dv = 2 \int_{\mathbb{R}^d} \left( Q_3(f, f, f) + 2Q'_3(f, f, f) \right) \phi \, dv = 2A + 4A_1 = 2(A + A_1 + A_2) = A + A_1 + A_2 - A^* - A_1^* - A_2^*, \]
and (5.59) follows.

(ii) Considering \( f > 0 \), and choosing \( \phi = \ln f \) in (5.59), we obtain (5.60) as follows:

\[
\tilde{D}(f) = \int_{\mathbb{R}^d} \tilde{Q}_3(f, f, f) \ln f \, dv
= \int_{\mathbb{R}^d} \frac{b_+(\omega_1, \omega_2, v_1 - v, v_2 - v)}{\sqrt{1 + \langle \omega_1, \omega_2 \rangle}} \left( f^* f_1^* f_2^* - f f_1 f_2 \right) \ln \frac{f f_1 f_2}{f^* f_1^* f_2^*} \, d\omega_1 \, d\omega_2 \, dv_1 \, dv_2 \, dv
\leq 0,
\]

where we used the elementary inequality

\[
(x - y) \ln \frac{y}{x} \leq 0, \quad \forall x, y > 0.
\] (5.64)

\[ \Box \]

Remark 5.2. In fact, using (3.12) and a similar argument, we can see that Proposition 5.5 is valid for the more general class of operators with cross-sections of the form

\[
B(\omega_1, \omega_2, \nu_1, \nu_2) = \Phi(|\tilde{\nu}|) \tilde{b}(\nu, \omega, \langle \omega_1, \omega_2 \rangle),
\] (5.65)

where \( \Phi : [0, \infty) \to [0, \infty) \), \( \tilde{b} : [-1, 1] \times [-\frac{1}{2}, \frac{1}{2}] \to [0, \infty) \) is even with respect to the first argument and we use the notation

\[
\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, \quad \nu = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}, \quad |\tilde{\nu}| := \sqrt{|\nu_1|^2 + |\nu_2|^2 + |\nu_1 - \nu_2|^2}.
\]

5.5.1 Collision invariants and local conservation laws

Now, we aim to investigate the ternary collision invariants which together with the Boltzmann inequality (5.60) will show that the operator \( \tilde{Q}_3 \) vanishes only for Maxwellian distributions. We first give the collision invariants definition:
Definition 5.1. A continuous function $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is called collision invariant if

$$
\phi(v_1^*) + \phi(v_2^*) + \phi(v_3^*) = \phi(v_1) + \phi(v_2) + \phi(v_3), \quad \forall (\omega_1, \omega_2, v_1, v_2, v_3) \in \mathbb{S}_1^{2d-1} \times \mathbb{R}^d.
$$

We will show that all collision invariants are of the form

$$
\phi(v) = \alpha + \langle b, v \rangle + c|v|^2, \quad v \in \mathbb{R}^d,
$$

for some $\alpha, c \in \mathbb{R}$ and $b \in \mathbb{R}^d$.

We will rely on the following well-known Lemma, for the proof see [16], p. 75.

Lemma 5.6. The following hold:

(i) Let $n \in \mathbb{N}$, and assume $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous function satisfying:

$$
\phi(v_1 + v_2) = \phi(v_1) + \phi(v_2), \quad \forall v_1, v_2 \in \mathbb{R}^n,
$$

Then

$$
\phi(v) = \langle b, v \rangle,
$$

for some $b \in \mathbb{R}^n$.

(ii) Assume $\phi : [0, \infty) \rightarrow \mathbb{R}$ is a continuous function satisfying:

$$
\phi(r + s) = \phi(r) + \phi(s), \quad \forall r, s \geq 0.
$$

Then

$$
\phi(r) = cr,
$$

for some constant $c \in \mathbb{R}$. 

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In order to find the collision invariants we will use the following weaker form of Lemma 5.6 for odd functions:

**Lemma 5.7.** Let $n \in \mathbb{N}$, and assume $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ is an odd continuous function satisfying:

$$\phi(v_1 + v_2) = \phi(v_1) + \phi(v_2), \quad \forall v_1 \perp v_2 \in \mathbb{R}^n,$$  \hspace{1cm} (5.68)

Then

$$\phi(v) = \langle b, v \rangle,$$

for some $b \in \mathbb{R}^n$.

**Proof.** Our proof is based on an argument developed in [16]. By part (i) of Lemma 5.6, it suffices to show that $\phi$ satisfies (5.67). Notice that since $\phi$ is odd we have $\phi(0) = 0$. Consider $v_1, v_2 \in \mathbb{R}^n$. By (5.68), we may assume without loss of generality that $\langle v_1, v_2 \rangle \neq 0$. Consider $\xi \in \mathbb{R}^n$ with

$$\langle \xi, v_1 \rangle = \langle \xi, v_2 \rangle = 0, \quad |\xi|^2 = |\langle v_1, v_2 \rangle| > 0.$$  \hspace{1cm} (5.69)

Applying (5.68) for the the pairs of perpendicular vectors $(v_1, \xi)$, $(v_2, \xi)$, we obtain

$$\phi(v_1 + \xi) = \phi(v_1) + \phi(\xi),$$  \hspace{1cm} (5.70)

$$\phi(v_2 + \xi) = \phi(v_2) + \phi(\xi) = \phi(v_2) + \phi(-\xi),$$  \hspace{1cm} (5.71)

$$\phi(v_2 - \xi) = \phi(v_2) + \phi(-\xi) = \phi(v_2) - \phi(\xi),$$  \hspace{1cm} (5.72)

where to obtain (5.72) we use the fact that $\phi$ is odd. We distinguish the following cases:
(i) \( \langle v_1, v_2 \rangle > 0 \): Condition (5.69) implies \( v_1 + \xi \perp v_2 - \xi \). Indeed we have

\[
\langle v_1 + \xi, v_2 - \xi \rangle = \langle v_1, v_2 \rangle - \langle v_1, \xi \rangle + \langle \xi, v_2 \rangle - |\xi|^2 = 0.
\]

Then implies

\[
\phi(v_1 + v_2) = \phi((v_1 + \xi) + (v_2 - \xi)) = \phi(v_1 + \xi) + \phi(v_2 - \xi) = \phi(v_1) + \phi(\xi) + \phi(v_2) - \phi(\xi)
\]

(5.74)

where to obtain (5.73), we apply (5.68) for the perpendicular vectors \( v_1 + \xi, v_2 - \xi \), and to obtain (5.74) we use (5.70), (5.72). The claim is proved for this case.

(ii): \( \langle v_1, v_2 \rangle < 0 \). Condition (5.69) implies \( v_1 + \xi \perp v_2 + \xi \). Indeed we have

\[
\langle v_1 + \xi, v_2 + \xi \rangle = \langle v_1, v_2 \rangle + \langle v_1, \xi \rangle + \langle \xi, v_2 \rangle + |\xi|^2 = 0.
\]

Then we have

\[
\phi(v_1 + v_2) + \phi(2\xi) = \phi((v_1 + v_2) + 2\xi) = \phi((v_1 + \xi) + (v_2 + \xi)) = \phi(v_1 + \xi) + \phi(v_2 + \xi)
\]

(5.77)

where to obtain (5.75), we apply (5.68) for the perpendicular vectors \( v_1 + v_2, 2\xi \), to obtain (5.76), we apply (5.68) for the perpendicular vectors \( v_1 + \xi, v_2 + \xi \), and
to obtain (5.77) we use (5.70)-(5.71). But notice that what was proven in case (i) implies
\[ \phi(2x) = 2\phi(x), \quad \forall x \in \mathbb{R}^n, \]
hence
\[ \phi(2\xi) = 2\phi(\xi). \] (5.78)
Then the claim follows from (5.75)-(5.77) and (5.78).

The result is proved. \(\square\)

We are now able to find all the collision invariants.

**Proposition 5.8.** All collision invariant functions are of the form:
\[ \phi = \alpha + \langle b, v \rangle + c|v|^2, \] (5.79)
for some \(\alpha, c \in \mathbb{R}\) and \(b \in \mathbb{R}^d\). Conversely, every function of the form (5.79) is collision invariant.

**Proof.** Consider first a function of the form (5.79). Then collision invariance directly follows from the conservation of momentum and energy of an interaction and linearity.

To prove the other direction, assume \(\phi\) is a collision invariant function. We claim first that the function \(\phi(v_1) + \phi(v_2) + \phi(v_3)\) is constant whenever \(v_1 + v_2 + v_3\) and \(|v_1|^2 + |v_2|^2 + |v_3|^2\) are constant. Indeed, consider \(v_1', v_2', v_3' \in \mathbb{R}^d\) such that
\[ v_1 + v_2 + v_3 = v_1' + v_2' + v_3', \]
\[ |v_1|^2 + |v_2|^2 + |v_3|^2 = |v_1'|^2 + |v_2'|^2 + |v_3'|^2. \]
Then by Proposition 3.2 there exists \((\omega_1, \omega_2) \in S_1^{2d-1}\), such that
\[(v'_1, v'_2, v'_3) = T_{\omega_1, \omega_2}(v_1, v_2, v_3) = (v^*_1, v^*_2, v^*_3)\).

By the collision invariance of \(\phi\) we get,
\[
\phi(v_1) + \phi(v_2) + \phi(v_3) = \phi(v^*_1) + \phi(v^*_2) + \phi(v^*_3) = \phi(v'_1) + \phi(v'_2) + \phi(v'_3),
\]
which proves the claim. In other words, since \(\phi\) is assumed continuous, we may write
\[
\phi(v_1) + \phi(v_2) + \phi(v_3) = \Phi\left(|v_1|^2 + |v_2|^2 + |v_3|^2, v_1 + v_2 + v_3\right), \tag{5.80}
\]
for some continuous function \(\Phi : [0, +\infty) \times \mathbb{R}^d \to \mathbb{R}\).

We proceed the proof in the following steps:

(i) \(\phi\) is even, \(\phi(0) = 0\): Given \(v \in \mathbb{R}^d\), we choose \(v_1 = v, v_2 = -v\) and \(v_3 = 0\) in (5.80) to obtain
\[
\phi(v) = \frac{1}{2} \Phi(2|v|^2, 0), \quad \forall v \in \mathbb{R}^d.
\]
Therefore, there is continuous function \(\psi : [0, +\infty) \to \mathbb{R}\), with
\[
\phi(v) = \psi(|v|^2), \quad \forall v \in \mathbb{R}^d. \tag{5.81}
\]
Setting \(v_3 = 0\) in equation (5.80), and using (5.81), we obtain
\[
\Phi\left(|v_1|^2 + |v_2|^2, v_1 + v_2\right) = \psi\left(|v_1|^2\right) + \psi\left(|v_2|^2\right), \quad \forall v_1, v_2 \in \mathbb{R}^d. \tag{5.82}
\]
Setting \(v_1 = v, v_2 = 0\) in (5.82), and using the fact that \(\psi(0) = \phi(0) = 0\), we obtain
\[
\Phi(|v|^2, v) = \psi(|v|^2), \quad \forall v \in \mathbb{R}^d. \tag{5.83}
\]
Considering \( v_1, v_2 \in \mathbb{R}^d \) with \( v_1 \perp v_2 \), we have

\[
\psi(|v_1 + v_2|^2) = \Phi(|v_1 + v_2|^2, v_1 + v_2)
\]

(5.84)

\[
= \Phi(|v_1|^2 + |v_2|^2, v_1 + v_2)
\]

(5.85)

\[
= \psi(|v_1|^2) + \psi(|v_2|^2);
\]

(5.86)

where to obtain (5.84) we use (5.83), to obtain (5.85) we use the assumption \( v_1 \perp v_2 \), and to obtain (5.86) we use (5.82). We have shown that

\[
\psi(|v_1|^2 + |v_2|^2) = \psi(|v_1|^2) + \psi(|v_2|^2), \quad \forall v_1 \perp v_2.
\]

(5.87)

We choose unit vectors \( n_1, n_2 \in \mathbb{R}^d \) with \( n_1 \perp n_2 \). Then (5.87) implies

\[
\psi(r+s) = \psi(|r^{1/2}n_1|^2 + |s^{1/2}n_2|^2) = \psi(|r^{1/2}n_1|^2) + \psi(|s^{1/2}n_2|^2) = \psi(r) + \psi(s), \quad \forall r, s \geq 0.
\]

By part (ii) of Lemma 5.7 we conclude

\[
\psi(r) = cr, \quad \text{for some } c \in \mathbb{R},
\]

thus

\[
\phi(v) = \psi(|v|^2) = c|v|^2,
\]

and the result follows for the even case.

(ii) \( \phi \) is odd: Assume \( \phi \) is odd, and consider \( v_1, v_2 \in \mathbb{R}^d \) with \( v_1 \perp v_2 \). Equation (5.80), for \( v_3 = 0 \), and the identity

\[
|v_1|^2 + |v_2|^2 = \langle v_1 + v_2, v_1 + v_2 \rangle, \quad \text{since } v_1 \perp v_2,
\]

imply there is a continuous function \( h : \mathbb{R}^d \to \mathbb{R} \) such that

\[
\phi(v_1) + \phi(v_2) = h(v_1 + v_2), \quad \forall v_1 \perp v_2 \in \mathbb{R}^d.
\]

(5.88)
We apply (5.88) with one vector being \( v_1 + v_2 \) and the other being 0 to obtain:

\[
\phi(v_1 + v_2) = \phi(v_1 + v_2) + \phi(0) = h(v_1 + v_2). \tag{5.89}
\]

Equations (5.88)-(5.89) imply

\[
\phi(v_1 + v_2) = \phi(v_1) + \phi(v_2), \quad \forall v_1 \perp v_2 \in \mathbb{R}^d.
\]

Hence, Lemma 5.7 yields

\[
\phi(v) = \langle b, v \rangle,
\]

for some \( b \in \mathbb{R}^d \), and the result follows for the odd case.

(iii) \( \phi(0) = 0 \): We split \( \phi \) to an even and an odd part \( \phi = \phi_e + \phi_o \), where

\[
\phi_e(v) = \frac{1}{2} \left( \phi(v) + \phi(-v) \right), \quad \phi_e(0) = 0,
\]

\[
\phi_o(v) = \frac{1}{2} \left( \phi(v) - \phi(-v) \right).
\]

By linearity of the collisional transformation, we notice that \( \phi_e \) and \( \phi_o \) are collision invariant as well so cases (i)-(ii) yield

\[
\phi_e(v) = c|v|^2, \quad \text{for some } c \in \mathbb{R},
\]

\[
\phi_o(v) = \langle b, v \rangle, \quad \text{for some } b \in \mathbb{R}^n.
\]

Therefore,

\[
\phi(v) = \langle b, v \rangle + c|v|^2.
\]

(iv) General case: Let us write \( \phi(0) = \alpha \). Then the function \( \psi(v) := \phi(v) - \alpha \) is collision invariant and \( \psi(0) = 0 \), so part (iii) implies the result.
Recalling $\mathcal{D}$ from (5.60), let us solve the functional equations:

$$\tilde{\mathcal{D}}(f) = 0,$$  \hspace{1cm} (5.90)

and

$$\tilde{Q}_3(f, f, f) = 0,$$  \hspace{1cm} (5.91)

for continuous and sufficiently decreasing in velocities $f > 0$.

**Proposition 5.9.** Consider a positive, decreasing in velocities function $f : [0, \infty) \times \mathbb{R}^d \rightarrow (0, \infty)$. Then the following hold:

(i) $\tilde{\mathcal{D}}(f) = 0$ iff $f$ is Maxwellian i.e. $f$ is of the form

$$f = \frac{R}{(2\pi T)^{d/2}} e^{-\frac{|v-U|^2}{2T}}.$$

for some continuous functions $R, T : [0, \infty) \times \mathbb{R}^d \rightarrow (0, \infty)$ and $U : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$.

(ii) $\tilde{Q}_3(f, f, f) = 0$ iff $f$ is a Maxwellian.

**Proof.** Let us first prove (i). By (5.59), we have

$$\tilde{\mathcal{D}}(f) = \int_{S_{d-1} \times \mathbb{R}^d} \frac{b_+}{1 + \langle \omega_1, \omega_2 \rangle} (f^* f_1^* f_2^* - f f_1 f_2) \ln \frac{f f_1 f_2}{f^* f_1 f_2} dv_1 dv_2 dv_1 dv dv.

Using the elementary fact that given $x, y > 0$, there holds

$$(x - y) \ln \frac{y}{x} \leq 0, \text{ with equality holding iff } x = y,$$

we obtain that

$$\tilde{\mathcal{D}}(f) = 0 \iff f^* f_1^* f_2^* = f f_1 f_2,$$  \hspace{1cm} (5.92)
or after taking logarithms

\[ \tilde{\mathcal{D}}(f) = 0 \Leftrightarrow \ln f^* + \ln f_1^* + \ln f_2^* = \ln f + \ln f_1 + \ln f_2. \]  

(5.93)

Assume \( \tilde{\mathcal{D}}(f) = 0 \). Then, (5.93) implies that \( \ln f \) is collision invariant, hence Proposition 5.8 yields

\[ \ln f = \alpha + \langle b, v \rangle + c|v|^2, \]

for some continuous functions \( \alpha, c : [0, \infty) \times \mathbb{R}^d \to \mathbb{R} \) and \( b : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \) or equivalently

\[ f = \frac{R}{(2\pi T)^d/2} e^{-|v-u|^2/2T}, \]

(5.94)

for some continuous functions \( R, T : [0, \infty) \times \mathbb{R}^d \to \mathbb{R} \) and \( U : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \). Since \( f \) is assumed to be positive and decreasing in velocities, we obtain that \( R, T > 0 \). Therefore \( f \) is Maxwellian. Assuming now that \( f \) is Maxwellian, conservation of momentum (3.10) and conservation of energy (3.11) imply that \( \ln f \) is collision invariant, and the claim follows again from (5.93).

Let us now prove (ii). We start by observing

\[ \tilde{Q}_3(f, f, f) = 0 \Rightarrow \tilde{\mathcal{D}}(f) = 0 \Rightarrow f \text{ is Maxwellian}, \]

(5.95)

where to obtain the second implication we use part (i). To prove the other direction, let us assume that \( f \) is Maxwellian. Hence \( \ln f \) is collision invariant thus \( f^* f_1^* f_2^* = f f_1 f_2 \). Therefore, \( \tilde{Q}_3(f, f, f) = 0 \).

Multiplying (5.55) with the collision invariants and integrating in velocities, one obtains, at least formally, the following local conservation laws:
Proposition 5.10. Let $f$ be a solution to the symmetrized ternary Boltzmann equation (5.55). Then the following local conservation laws hold:

(i) Local conservation of mass:

$$
\partial_t \int_{\mathbb{R}^d} f \, dv + \nabla_x \cdot \int_{\mathbb{R}^d} fv \, dv = 0.
$$

(ii) Local conservation of momentum:

$$
\partial_t \int_{\mathbb{R}^d} vf \, dv + \nabla_x \cdot \int_{\mathbb{R}^d} fv \otimes v \, dv = 0.
$$

(iii) Local conservation of energy:

$$
\partial_t \int_{\mathbb{R}^d} |v|^2 f \, dv + \nabla_x \cdot \int_{\mathbb{R}^d} |v|^2 fv \, dv = 0.
$$

In particular if $f$ is assumed to be Maxwellian with macroscopic variables $R, U, T$, then one can show that $R, U, T$ constitute the Euler compressible system.

An interesting question would be the study of the hydrodynamic limits of the symmetrized ternary Boltzmann equation (5.51). See [57] for an extensive study of the hydrodynamic limits of the binary Boltzmann equation.

Remark 5.3. By Remark 5.2, one can see that Proposition 5.9 and Proposition 5.10 are valid for the more general class of operators with cross-sections of the form (5.65).

5.10.1 The $\mathcal{H}$-Theorem

Let $f$ be a positive and decreasing in velocities solution of (5.55), and let us define the entropy functional

$$
\mathcal{H}(f) := \int_{\mathbb{R}^d} f \ln f \, dv.
$$

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We calculate

\[
(\partial_t + v \cdot \nabla_x)\mathcal{H} = \int_{\mathbb{R}^d} (1 + \ln f)\partial_t f \, dv + \int_{\mathbb{R}^d} (1 + \ln f) v \cdot \nabla_x f \, dv
\]
\[
= \int_{\mathbb{R}^d} (\partial_t f + v \cdot \nabla_x f) \, dv + \int_{\mathbb{R}^d} \tilde{Q}_3(f, f, f) \ln f \, dv \quad (5.100)
\]
\[
= \tilde{\mathcal{D}}(f) \quad (5.101)
\]
\[
\leq 0, \quad (5.102)
\]

where to obtain (5.100) we use the fact that \( f \) solves (5.55), to obtain (5.101) we use the local conservation of mass (5.96), and to obtain (5.102) we use the Boltzmann inequality (5.60). Part (i) of Proposition 5.9 yields that equality holds in (5.102) iff \( f \) is Maxwellian. This is an analogue of the famous \( \mathcal{H} \)-Theorem, see [17] for the binary case, stating that the entropy is a Lyapunov functional for the Boltzmann equation and that equilibrium is achieved only by Maxwellian distributions.

In the special space-homogeneous case, where \( f = f(t, v) \), we obtain:

\[
\frac{\partial \mathcal{H}(f)}{\partial t} = \frac{\partial}{\partial t} \int_{\mathbb{R}^d} f \ln f \, dv \leq 0,
\]

and

\[
\frac{\partial \mathcal{H}(f)}{\partial t} = 0 \text{ iff } f \text{ is Maxwellian.}
\]

In other words the entropy is a non-increasing quantity in time. In particular the entropy is constant in time, i.e. equilibrium is achieved, iff \( f \) is Maxwellian.

Remark 5.4. By Remark 5.2, one sees that the \( \mathcal{H} \)-theorem is valid for the more general class of operators with cross-sections of the form (5.65).
Chapter 6

Local well-posedness

In this chapter, we prove local well-posedness for the BBGKY hierarchy and Boltzmann hierarchy and the ternary Boltzmann equation. As expected, these well-posedness proofs are closely related, and they rely on defining appropriate functional spaces and establishing appropriate a-priori bounds. The spaces that we introduce for the BBGKY hierarchy and Boltzmann hierarchy are inspired by the spaces used in [33, 65].

Given $s \in \mathbb{N}$ and $\epsilon > 0$, recall from (4.61) the notation

$$ C_0^0(D_{s,\epsilon}) = \{ g_s : \mathbb{R}^d \rightarrow \mathbb{R} \text{ such that } g_s|_{D_{s,\epsilon}} \text{ is continuous and } g_s|_{\mathbb{R}^{2d_1} \setminus D_{s,\epsilon}} = 0 \}. \quad (6.1) $$

Also, recall that "$\lesssim$" and "$\simeq$" denote inequality and equality respectively, up to constant $C_d > 0$ (see (1.13)-(1.12)).

6.1 LWP for the BBGKY hierarchy

Consider $(N, \epsilon)$ in the scaling (5.43). For $1 \leq s \leq N$ and $\beta > 0$ we define the Banach spaces

$$ X_{N,\beta,s} := \{ g_{N,s} \in C_0^0(D_{s,\epsilon}) : |g_{N,s}|_{N,\beta,s} < \infty \}, $$
with norm

\[ |g_{N,s}|_{N,\beta,s} = \sup_{Z_s \in \mathbb{R}^{2ds}} |g_{N,s}(Z_s)| e^{\beta E_s(Z_s)}, \]

where \( E_s(Z_s) \) is the \( s \)-particle kinetic energy of the given by (4.45).

For \( s > N \) we trivially define

\[ X_{N,\beta,s} := \{0\}. \]

**Remark 6.1.** Given \( t \in \mathbb{R} \) and \( s \in \mathbb{N} \), conservation of energy under the interaction flow (4.48) implies that the \( s \)-particle \( \epsilon \)-interaction zone flow operator \( T^t_s : X_{N,\beta,s} \to X_{N,\beta,s} \), given in (4.62), is an isometry i.e.

\[ |T^t_s g_{N,s}|_{N,\beta,s} = |g_{N,s}|_{N,\beta,s}, \quad \forall g_{N,s} \in X_{N,\beta,s}. \]

Indeed, consider \( g_{N,s} \in X_{N,\beta,s} \) and \( Z_s \in \mathbb{R}^{2ds} \). If \( Z_s \notin D_{s,\epsilon} \), the result is trivial since \( g_{N,s} \) is supported in \( D_{s,\epsilon} \). Assume \( Z_s \in D_{s,\epsilon} \). Then Theorem 4.9.1 yields

\[
e^{\beta E_s(Z_s)} |T^t_s g_{N,s}| = e^{\beta E_s(Z_s)} |(g_{N,s} \circ \Psi^{-t}_s)(Z_s)|
\]

\[
= e^{\beta E_t(Z_s)} |g_{N,s}(\Psi_{s}^{-t} Z_s)|
\]

\[
\leq |g_{N,s}|_{N,s,\beta},
\]

so

\[ |T^t_s g_{N,s}|_{N,s,\beta} \leq |g_{N,s}|_{N,s,\beta}. \]

The other side of the inequality comes similarly using the fact that \( Z_s = \Psi^{-t}_s(\Psi^t_s Z_s) \).

Consider \( \mu \in \mathbb{R} \). We define the Banach space

\[ X_{N,\beta,\mu} := \{G_N = (g_{N,s})_{s \in \mathbb{N}} : g_{N,s} \in X_{N,\beta,s}, \quad \forall s \in \mathbb{N} \text{ and } \|G_N\|_{N,\beta,\mu} < \infty \}, \]

with norm

\[ \|G_N\|_{N,\beta,\mu} = \sup_{s \in \mathbb{N}} e^{\mu s} |g_{N,s}|_{N,\beta,s} = \max_{s \in \{1, \ldots, N\}} e^{\mu s} |g_{N,s}|_{N,\beta,s}. \]
Remark 6.2. Given $t \in \mathbb{R}$, Remark 6.1 implies that the map $T^t : X_{N,\beta,\mu} \to X_{N,\beta,\mu}$ given by

$$T^t G_N = (T^t_s g_{N,s})_{s \in \mathbb{N}},$$

is an isometry i.e.

$$\|T^t G_N\|_{N,\beta,\mu} = \|G_N\|_{N,\beta,\mu}, \quad \forall G_N \in X_{N,\beta,\mu}.$$

Finally, given $T > 0$, $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ and $\beta, \mu : [0, T] \to \mathbb{R}$ decreasing functions of time with $\beta(0) = \beta_0$, $\beta(T) > 0$, $\mu(0) = \mu_0$, we define the Banach space

$$X_{N,\beta,\mu} := L^\infty ([0, T], X_{N,\beta(t),\mu(t)}),$$

with norm

$$||| G_N |||_{N,\beta,\mu} = \sup_{t \in [0, T]} \| G_N(t) \|_{N,\beta(t),\mu(t)}.$$

Proposition 6.2. Let $T > 0$, $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ and $\beta, \mu : [0, T] \to \mathbb{R}$ decreasing functions with $\beta_0 = \beta(0)$, $\beta(T) > 0$, $\mu_0 = \mu(0)$. Then for any $G_N = (g_{N,s})_{s \in \mathbb{N}} \in X_{N,\beta_0,\mu_0}$, the following estimates hold:

(i) $||| G_N |||_{N,\beta,\mu} \leq \| G_N \|_{N,\beta_0,\mu_0}$.

(ii) $\left| \left| \left| \int_0^t T^\tau G_N \, d\tau \right| \right| \right|_{N,\beta,\mu} \leq T \| G_N \|_{N,\beta_0,\mu_0}$.

Proof. Let $G_N = (g_{N,s})_{s \in \mathbb{N}} \in X_{N,\beta_0,\mu_0}$. To prove (i), let us fix $t \in [0, T]$ and $Z_s \in \mathbb{R}^{2d_N}$. The fact that $\beta, \mu$ are decreasing implies

$$e^{s\mu(t) + \beta(t)E_s(Z_s)} |g_{N,s}(Z_s)| \leq e^{s\mu_0 + \beta_0 E_s(Z_s)} |g_{N,s}(Z_s)| \leq \| G_N \|_{N,\beta_0,\mu_0},$$

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and (i) is proved.

To prove (ii), fix again $t \in [0, T]$, $s \in \mathbb{N}$ and $Z_s \in \mathbb{R}^{2d}$. Then the fact that $\beta, \mu$ are decreasing and Remark 6.2 yield

\[
e^{s\mu(t)+\beta(t)E_s(Z_s)} \left| \int_0^t T_s g_{N,s}(Z_s) \, d\tau \right| \leq e^{s\mu_0+\beta_0E_s(Z_s)} \int_0^t \left| T_s g_{N,s}(Z_s) \right| \, d\tau \\
\leq \int_0^t \left\| T^\tau G_N \right\|_{N,\beta_0,\mu_0} \, d\tau \\
\leq T \left\| G_N \right\|_{N,\beta_0,\mu_0},
\]

and claim (ii) follows. \qed

We now prove an important continuity estimate about the collisional operator $C^N_{m,m+2}$ for given $m \in \mathbb{N}$.

**Lemma 6.3.** Let $m \in \mathbb{N}$, $\beta > 0$ and $g_{N,m+2} \in X_{N,m+2,\beta}$. Then, for any $Z_m \in \mathcal{D}_{m,\epsilon}$, the following continuity estimate holds:

\[
\left| C^N_{m,m+2} g_{N,m+2}(Z_m) \right| \lesssim \beta^{-d} \left( m \beta^{-1/2} + \sum_{i=1}^m |v_i| \right) e^{-\beta E_m(Z_m)} |g_{N,m+2}|_{N,\beta,m+2}.
\]

**Proof.** Let $g_{N,m+2} \in X_{N,m+2,\beta}$ and $Z_m = (X_m, V_m) \in \mathbb{N}$. If $m \geq N-1$ both sides vanish, so we may assume that $m \leq N-2$. Notice that conservation of energy (3.11) implies

\[
E_{m+2}(Z_{m+2,i}^*) = E_{m+2}(Z_{m+2,i}), \quad \forall i = 1, \ldots, m.
\]

Moreover, for any $(\omega_1, \omega_2, v_1, v_2, v_3) \in S^{2d-1}_1 \times \mathbb{R}^d$, (3.8) and Cauchy-Schwartz in-
equality yield
\[
\frac{b_+(\omega_1, \omega_2, v_2 - v_1, v_3 - v_1)}{\sqrt{1 + \langle \omega_1, \omega_2 \rangle}} \leq \sqrt{2} \left( |\omega_1| |v_2 - v_1| + |\omega_2| |v_3 - v_1| \right)
\leq 2\sqrt{2} (|v_1| + |v_2| + |v_3|) \quad (6.5)
\leq 3 (|v_1| + |v_2| + |v_3|) .
\]

Therefore, using (6.4), the definition of the norm and the scaling \( N \epsilon^{d-1/2} \sim 1 \), we get
\[
\left| \mathcal{G}^N_{m,m+2} g_{N,m+2}(Z_m) \right| \leq 2e^{-\beta E_m(Z_m)} |g_{N,m+2}|_{N,\beta,m+2}
\times \sum_{i=1}^m \int_{\mathbb{R}^d} \frac{b_+(\omega_1, \omega_2, v_{m+1} - v_i, v_{m+2} - v_i)}{\sqrt{1 + \langle \omega_1, \omega_2 \rangle}} e^{-\frac{\beta}{2} (|v_{m+1}|^2 + |v_{m+2}|^2)} d\omega_1 d\omega_2 dv_{m+1} dv_{m+2}
\leq e^{-\beta E_m(Z_m)} |g_{N,m+2}|_{N,\beta,m+2}
\times \sum_{i=1}^m \int_{\mathbb{R}^d} (|v_i| + |v_{m+1}| + |v_{m+2}|) e^{-\frac{\beta}{2} (|v_{m+1}|^2 + |v_{m+2}|^2)} dv_{m+1} dv_{m+2}.
\]

Using Fubini’s theorem and the elementary integrals
\[
\int_0^\infty e^{-\frac{\beta}{2} x^2} dx \simeq \beta^{-1/2}, \quad \int_0^\infty x e^{-\frac{\beta}{2} x^2} dx \simeq \beta^{-1}, \quad (6.6)
\]
we obtain the required estimated.

Now we define a mild solution of the BBGKY hierarchy in the scaling \( (5.43) \) as follows:

**Definition 6.1.** Consider \( T > 0, \beta_0 > 0, \mu_0 \in \mathbb{R} \) and the decreasing functions \( \beta, \mu : [0,T] \to \mathbb{R} \) with \( \beta(0) = \beta_0, \beta(T) > 0, \mu(0) = \mu_0 \). Consider also initial data \( G_{N,0} = (g_{N,s,0}) \in X_{N,\beta_0,\mu_0} \). A map \( G_N = (g_{N,s})_{s \in \mathbb{N}} \in X_{N,\beta,\mu} \) is a mild solution of
the BBGKY hierarchy in \([0, T]\) with initial data \(G_{N,0}\), if it satisfies:

\[
G_N(t) = \mathcal{T}t G_{N,0} + \int_0^t \mathcal{T}^{t-\tau} \mathcal{E}_N G_N(\tau) \, d\tau,
\]

where

\[
\mathcal{E}_N G_N = \left(e_{N,s,s+2}^N g_{N,s+2}\right)_{s \in \mathbb{N}}.
\]

and \(\mathcal{T}\) is given by (6.3).

We will address the well-posedness of the BBGKY hierarchy by a fixed point argument. For this purpose, we state the following important estimate:

**Lemma 6.4.** Let \(\beta_0 > 0\), \(\mu_0 \in \mathbb{R}\), \(T > 0\) and \(\lambda \in (0, \beta_0/T)\). Consider the functions \(\beta_\lambda, \mu_\lambda : [0, T] \to \mathbb{R}\) given by

\[
\beta_\lambda(t) = \beta_0 - \lambda t, \\
\mu_\lambda(t) = \mu_0 - \lambda t.
\]

Then for any \(\mathcal{F}(t) \subseteq [0, t]\) measurable, \(s \in \mathbb{N}\) and \(G_N = (g_{N,s})_{s \in \mathbb{N}} \in X_{N,\beta_\lambda,\mu_\lambda}\) the following bound holds:

\[
e^{s\mu_\lambda(t)} \left| \int_{\mathcal{T}(t)} \mathcal{T}^{t-\tau} \mathcal{E}_s \mathcal{E}_N g_{N,s+2} G_N(\tau) \, d\tau \right|_{N,\beta_\lambda(t),s} \leq C(d, \beta_0, \mu_0, T, \lambda) \|\|G_N\|\|_{N,\beta_\lambda,\mu_\lambda},
\]

where

\[
C(d, \beta_0, \mu_0, T, \lambda) \simeq \lambda^{-1} e^{-2\mu_\lambda(T)} \beta_\lambda(T)^{-d} \left(1 + \beta_\lambda(T)^{-1/2}\right).
\]

In other words

\[
\left| \int_{\mathcal{T}(t)} \mathcal{T}^{t-\tau} \mathcal{E}_N G_N(\tau) \, d\tau \right|_{N,\beta_\lambda,\mu_\lambda} \leq C(d, \beta_0, \mu_0, T, \lambda) \|\|G_N\|\|_{N,\beta_\lambda,\mu_\lambda}.
\]
Proof. Fix \( t \in [0, T] \), \( \mathcal{F}(t) \subseteq [0, t] \) measurable and \( s \in \mathbb{N} \). Fix \( Z_s \in \mathbb{R}^{2d_s} \) and assume without loss of generality that \( Z_s \in \mathcal{D}_{s,e} \). For \( \tau \in [0, t] \), let us write
\[
Z_s^{t-\tau} = (X_s^{t-\tau}, V_s^{t-\tau}) = \Psi_s^{t-\tau} Z_s.
\] (6.9)

Conservation of energy (4.48) implies that
\[
E_s(Z_s^{t-\tau}) = E_s(Z_s),
\] (6.10)

hence combing the definition of the interaction flow operator (4.62) and (6.10), we obtain:
\[
e^{\beta(t)E_s(Z_s)} \left| \mathcal{F}(t) \right| T_s^{t-\tau} e_s^{N_s, s+2} g_{N_s, s+2}(\tau) d\tau \leq \int_{\mathcal{F}(t)} e^{\beta(t)E_s(Z_s)} \left| e_s^{N_s, s+2} g_{s+2}(\tau, Z_s^{t-\tau}) \right| d\tau.
\] (6.11)

But Lemma 6.3 and (6.10) imply
\[
e^{\beta(t)E_s(Z_s)} \left| e_s^{N_s, s+2} g_{N_s, s+2}(\tau, Z_s^{t-\tau}) \right| \lesssim \beta(t) e^{\lambda(t)E_s(Z_s)}
\] (6.12)

By the definition of the norms we get
\[
|g_{N_s, s+2}(\tau)|_{N, \beta(t), s+2} \leq e^{-(s+2)\mu(t)} ||G_N(\tau)||_{N, \beta(t), \mu(t)}
\] (6.13)

Since \( \beta, \mu \) are decreasing, (6.11)-(6.13) imply
\[
e^{\beta(t)E_s(Z_s)} \left| \mathcal{F}(t) \right| T_s^{t-\tau} e_s^{N_s, s+2} g_{N_s, s+2}(\tau, Z_s) \right|_{N, \beta(t), s} \lesssim ||G_N||_{N, \beta(t), \mu(t)} e^{-2\mu(T)}
\] (6.14)

\[\times \beta(t)^{-d} \int_{\mathcal{F}(t)} Q(\tau, t, Z_s) d\tau,
\]
where, for $\tau \leq t$, we denote
\[
Q(\tau, t, Z_s) = \left( s \beta (T)^{-1/2} + \sum_{i=1}^{s} |v_i(t-\tau)| \right) e^{\lambda (s + E_s(Z_s))(\tau - t)}.
\]
(6.14)

But by Cauchy-Schwartz inequality, we obtain
\[
\sum_{i=1}^{s} |v_i(t-\tau)| \leq \left( \sum_{i=1}^{s} |v_i(t-\tau)|^2 \right)^{1/2} \leq \frac{s}{2} + E_s(Z_s t - \tau) < s + E_s(Z_s),
\]
by (6.10). Therefore,
\[
\int_{\mathcal{T}(t)} Q(\tau, t, Z_s) d\tau \leq \left( s \beta (T)^{-1/2} + s + E_s(Z_s) \right) \int_{0}^{t} e^{\lambda (s + E_s(Z_s))(\tau - t)} d\tau \\
\leq \frac{s \beta (T)^{-1/2} + s + E_s(Z_s)}{\lambda (s + E_s(Z_s))} \\
\leq \frac{1 + \beta(T)^{-1/2}}{\lambda}.
\]
Thus, we get
\[
e^{-\mu_\lambda(t)} \left| \int_{\mathcal{T}(t)} T_{s-t}^{\tau} \frac{\partial}{\partial \tau} e^{N_{s,s+2,2}(\tau)} d\tau \right| \leq C(d, \beta_0, \mu_0, T, \lambda) \| G_N \|_{N, \beta, \mu},
\]
where
\[
C(d, \beta_0, \mu_0, T, \lambda) \simeq \lambda^{-1} e^{-2\mu_\lambda(T)} \beta(T)^{-d} \left( 1 + \beta(T)^{-1/2} \right).
\]
The result is proved.

Choosing $\lambda = \beta_0/2T$, Lemma 6.4 implies well-posedness of the BBGKY hierarchy up to short time.

**Theorem 6.4.1** (LWP for the BBGKY hierarchy). Let $\beta_0 > 0$ and $\mu_0 \in \mathbb{R}$. Then there is $T = T(d, \beta_0, \mu_0) > 0$ such that for any initial datum $F_{N,0} = (f_{N,0}^{(s)})_{s \in \mathbb{N}} \in$
there is unique mild solution $F_N \in X_{N,\beta,\mu}$ to the BBGKY hierarchy in $[0,T]$ for the functions $\beta, \mu : [0,T] \to \mathbb{R}$ given by

$$
\beta(t) = \beta_0 - \frac{\beta_0}{2T} t,
$$

$$
\mu(t) = \mu_0 - \frac{\beta_0}{2T} t.
$$

(6.15)

The solution $F_N$ satisfies the bound:

$$
|||F_N|||_{N,\beta,\mu} \leq 2 \|F_{N,0}\|_{N,\beta,\mu_0}.
$$

(6.16)

Moreover, for any $\mathcal{F}(t) \subseteq [0,t]$ measurable, the following bounds hold:

$$
|||\int_{\mathcal{F}(t)} \mathcal{T}^{t-\tau} C_N G_N(\tau) \, d\tau|||_{N,\beta,\mu} \leq \frac{1}{8} |||G_N|||_{N,\beta,\mu}, \quad \forall G_N \in X_{N,\beta,\mu}.
$$

(6.17)

The time $T$ is explicitly given by:

$$
T = T(d, \beta_0, \mu_0) \simeq \frac{\beta_0^{d+1} e^{2\mu_0 - \beta_0}}{2^{d+4} \left( 1 + \sqrt{\frac{2}{\beta_0}} \right)}.
$$

(6.18)

Proof. Consider $F_{N,0} \in X_{N,\beta_0,\mu_0}$. Let us define the operator $\mathcal{L} : X_{N,\beta,\mu} \to X_{N,\beta,\mu}$, given by

$$
\mathcal{L}G_N(t) = \mathcal{T}^t F_{N,0} + \int_0^t \mathcal{T}^{t-\tau} C_N G_N(\tau) \, d\tau, \quad t \in [0,T).
$$

Choosing

$$
T \simeq \frac{\beta_0^{d+1} e^{2\mu_0 - \beta_0}}{2^{d+4} \left( 1 + \sqrt{\frac{2}{\beta_0}} \right)},
$$

equation (6.8) implies that

$$
C(d, \beta_0, \mu_0, T, \beta_0/2T) = \frac{1}{8}.
$$

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Then, for any \( t \in [0, T] \) and \( \mathcal{F}(t) \subseteq [0, t] \) measurable, Lemma 6.4 implies
\[
\left\| \int_{\mathcal{F}(t)} \mathcal{T}^{t-\tau} C_N G_N(\tau) \, d\tau \right\|_{N,\beta,\mu} \leq \frac{1}{8} \| G_N \|_{N,\beta,\mu}, \quad \forall G_N \in X_{N,\beta,\mu}.
\]
(6.19)
In the special case where \( \mathcal{F}(t) = [0, t] \), (6.19) implies \( \mathcal{L} \) is a contraction, thus it has a unique fixed point \( F_N \in X_{N,\beta,\mu} \). Clearly \( F_N \) is the unique mild solution to the BBGKY hierarchy in \([0, T]\) with initial datum \( F_{N,0} \) and satisfies (6.17). Moreover, estimate (6.17), Remark 6.2 and part (i) of Proposition 6.2 imply
\[
\| F_N \|_{N,\beta(t),\mu(t)} \leq \| \mathcal{T}^t F_{N,0} \|_{N,\beta(t),\mu(t)} + \left\| \int_0^t \mathcal{T}^{t-\tau} C_N F_N(\tau) \, d\tau \right\|_{N,\beta(t),\mu(t)}
\]
\[
= \| F_{N,0} \|_{N,\beta(t),\mu(t)} + \left\| \int_0^t \mathcal{T}^{t-\tau} C_N F_N(\tau) \, d\tau \right\|_{N,\beta(t),\mu(t)}
\]
\[
\leq \| F_{N,0} \|_{N,\beta_0,\mu_0} + \frac{1}{8} \| F_N \|_{N,\beta,\mu},
\]
which yields
\[
\| F_N \|_{N,\beta,\mu} \leq 2 \| F_{N,0} \|_{N,\beta_0,\mu_0}.
\]

6.5 LWP for the Boltzmann hierarchy

Well-posedness for the Boltzmann hierarchy is identical, so we will just state the corresponding results. Given \( \beta > 0 \) and \( s \in \mathbb{N} \) we define the Banach spaces
\[
X_{\infty,\beta,s} := \left\{ g_s \in C^0(\mathbb{R}^{2ds}) : |g_s|_{\infty,\beta,s} < \infty \right\},
\]
with norm
\[
|g_s|_{\infty,\beta,s} = \sup_{Z_s \in \mathbb{R}^{2ds}} |g_s(Z_s)| e^{\beta E_s(Z_s)}.
\]
Remark 6.3. Given $t \in \mathbb{R}$ and $s \in \mathbb{N}$, it is clear that the $s$-particle free flow operator

$$S^t_s : X_{\infty, \beta, s} \rightarrow X_{\infty, \beta, s}$$

is an isometry i.e.

$$|S^t_s g_s|_{\infty, \beta, s} = |g_s|_{\infty, \beta, s}, \quad \forall g_s \in X_{\infty, \beta, s}.$$

Consider as well $\mu \in \mathbb{R}$. We define the Banach space

$$X_{\infty, \beta, \mu} := \{ G = (g_s)_{s \in \mathbb{N}} : \|G\|_{\infty, \beta, \mu} < \infty \},$$

with norm

$$\|G\|_{\infty, \beta, \mu} := \sup_{s \in \mathbb{N}} e^{\mu s} |g_s|_{\infty, \beta, s}.$$

Remark 6.4. Given $t \in \mathbb{R}$, it is clear that the map $S^t : X_{\infty, \beta, \mu} \rightarrow X_{\infty, \beta, \mu}$ given by

$$S^t G = (S^t_s g_s)_{s \in \mathbb{N}}, \quad (6.20)$$

is an isometry i.e.

$$\|S^t G\|_{\infty, \beta, \mu} = \|G\|_{\infty, \beta, \mu}, \quad \forall G \in X_{\infty, \beta, \mu}.$$

Finally, for $T > 0$, $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ and $\beta, \mu : [0, T] \rightarrow \mathbb{R}$ decreasing functions of time with $\beta(T) > 0$ we define the Banach space

$$X_{\infty, \beta, \mu} = L^\infty([0, T], X_{\infty, \beta(t), \mu(t)}),$$

with norm

$$|||G||| = \sup_{t \in [0, T]} \|G(t)\|_{\infty, \beta(t), \mu(t)}.$$

Similarly to Proposition 6.2 we get the following:
Proposition 6.6. Let $T > 0$, $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ and $\beta, \mu : [0, T] \to \mathbb{R}$ decreasing functions with $\beta_0 = \beta(0)$, $\beta(T) > 0$ $\mu_0 = \mu(0)$. Then for any $G = (g_s)_{s \in \mathbb{N}} \in X_{\infty, \beta_0, \mu_0}$, the following estimates hold:

(i) $|||G|||_{\infty, \beta, \mu} \leq ||G||_{\infty, \beta_0, \mu_0}$.

(ii) $||\int_0^t S^\tau G d\tau||_{\infty, \beta, \mu} \leq T ||G||_{\infty, \beta_0, \mu_0}$.

Similarly to Lemma 6.3, we get the following estimate:

Lemma 6.7. Let $m \in \mathbb{N}$, $\beta > 0$ and $g_{m+2} \in X_{\infty, m+2, \beta}$. The following continuity estimate holds:

$|C_{m, m+2}^\infty g_{m+2}(Z_m)| \lesssim \beta^{-d} \left( m\beta^{-1/2} + \sum_{i=1}^m |v_i| \right) e^{\beta E_m(Z_m)} |g_{m+2}|_{\infty, \beta, m+2}, \quad \forall Z_m \in \mathbb{R}^{2dm}$.

Now we define a mild solution of the Boltzmann hierarchy as follows.

Definition 6.2. Consider $T > 0$, $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ and the decreasing functions $\beta, \mu : [0, T] \to \mathbb{R}$ with $\beta(0) = \beta_0$, $\beta(T) > 0$, $\mu(0) = \mu_0$. Consider also initial data $G_0 = (g_s, 0) \in X_{\infty, \beta_0, \mu_0}$. A map $G = (g_s)_{s \in \mathbb{N}} \in X_{\infty, \beta, \mu}$ is a mild solution of the Boltzmann hierarchy in $[0, T]$, with initial data $G_0$, if it satisfies:

$G(t) = S^t G_0 + \int_0^t S^{t-\tau} C_{\infty} G(\tau) d\tau,$

where

$C_{\infty} G = \left( C_{s, s+2}^\infty g_{s+2} \right)_{s \in \mathbb{N}}$, and

$S^t$ is given by (6.20).
We proceed identically as in the BBGKY hierarchy case, so we omit the proofs. In particular, we have the analogue of Lemma 6.4

\textbf{Lemma 6.8.} Let $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$, $T > 0$ and $\lambda \in (0, \beta_0/T)$. Consider the functions $\beta_\lambda, \mu_\lambda : [0, T] \to \mathbb{R}$ given by (6.7). Then for any $\mathcal{F}(t) \subseteq [0, t]$ measurable, $s \in \mathbb{N}$ and $G = (g_s)_{s \in \mathbb{N}} \in X_{\infty, \beta_\lambda, \mu_\lambda}$, the following bound holds:

\[
e^{s \mu_\lambda(t)} \left| \int_{\mathcal{F}(t)} S_s^{t-\tau} \mathcal{E}_\infty^{s, s+2} g_{s+2}(\tau) d\tau \right|_{\infty, \beta_\lambda(t), s} \leq C(d, \beta_0, \mu_0, T, \lambda) \|||G|||_{\infty, \beta_\lambda, \mu_\lambda},
\]

where

\[
C(d, \beta_0, \mu_0, T, \lambda) \simeq \lambda^{-1} e^{-2\mu(T)\beta_\lambda(T)^{-d}} \left( 1 + \beta_\lambda(T)^{-1/2} \right).
\]

In other words

\[
\left| \int_{\mathcal{F}(t)} S^{t-\tau} \mathcal{E}_\infty G(\tau) d\tau \right|_{N, \beta_\lambda, \mu_\lambda} \leq C(d, \beta_0, \mu_0, T, \lambda) |||G|||_{\infty, \beta_\lambda, \mu_\lambda}.
\]

Choosing $\lambda = \beta_0/2T$, Lemma 6.8 directly implies well-posedness of the Boltzmann hierarchy up to short time.

\textbf{Theorem 6.8.1} (LWP for the Boltzmann hierarchy). Let $\beta_0 > 0$ and $\mu_0 \in \mathbb{R}$. Then there is $T = T(d, \beta_0, \mu_0) > 0$ such that for any initial datum $F_0 = (f_0^{(s)})_{s \in \mathbb{N}} \in X_{\infty, \beta_0, \mu_0}$ there is unique mild solution $F \in X_{\infty, \beta, \mu}$ to the Boltzmann hierarchy in $[0, T]$ for the functions $\beta, \mu : [0, T] \to \mathbb{R}$ given by (6.15).

Moreover, for any $\mathcal{F}(t) \subseteq [0, t]$ measurable, the following estimates hold:

\[
\left| \int_{\mathcal{F}(t)} S^{t-\tau} C_\infty G(\tau) d\tau \right|_{\infty, \beta, \mu} \leq \frac{1}{8} |||G|||_{\infty, \beta, \mu}, \quad \forall G \in X_{\infty, \beta, \mu}, \quad (6.21)
\]

\[
|||F|||_{\infty, \beta, \mu} \leq 2 \|||F_0|||_{\infty, \beta_0, \mu_0}, \quad (6.22)
\]

and the time $T$ is explicitly given by (6.18).
6.9 LWP for the ternary Boltzmann equation and propagation of chaos

In this section, we show local well-posedness for the ternary Boltzmann equation and that, for chaotic initial data, their tensorized product produces the unique mild solution of the Boltzmann hierarchy. Therefore uniqueness implies that the mild solution to the Boltzmann hierarchy remains factorized under time evolution, hence chaos is propagated in time.

For \( \beta > 0 \) let us define the Banach space

\[
X_{\beta,\mu} := \left\{ g \in C^0(\mathbb{R}^{2d}) : |g|_{\beta,\mu} < \infty \right\},
\]

with norm

\[
|g|_{\beta,\mu} = \sup_{(x,v) \in \mathbb{R}^{2d}} |g(x,v)| e^{\mu + \frac{\beta}{2}|v|^2}.
\]

Notice that for any \( t \in [0,T] \) and \( g \in X_{\beta,\mu} \), we have

\[
|S_t g|_{\beta,\mu} = |g|_{\beta,\mu},
\]

i.e. \( S_t^d : X_{\beta,\mu} \to X_{\beta,\mu} \) is an isometry.

Consider \( \beta_0 > 0, \mu_0 \in \mathbb{R}, T > 0 \) and \( \beta, \mu : [0,T] \to \mathbb{R} \) decreasing functions of time with \( \beta(0) = \beta_0, \beta(T) > 0 \) and \( \mu(0) = \mu_0 \).

Let us define the Banach space

\[
X_{\beta,\mu} := L^\infty([0,T], X_{\beta(t),\mu(t)});
\]

with norm

\[
\|g\|_{\beta,\mu} = \sup_{t \in [0,T]} |g(t)|_{\beta(t),\mu(t)}.
\]
Remark 6.5. Let $T > 0$, $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ and $\beta, \mu : [0, T] \to \mathbb{R}$ decreasing functions with $\beta_0 = \beta(0)$, $\beta(T) > 0$, $\mu_0 = \mu(0)$. Then for any $g \in X_{\beta_0, \mu_0}$, the following estimate holds:

$$\|g\|_{\beta, \mu} \leq |g|_{\beta_0, \mu_0}.$$  

Proof. Since $\beta, \mu$ are decreasing, for any $t \in [0, T]$, we obtain

$$e^{\mu(t) + \beta(t)|v|^2}|g(x, v)| \leq e^{\mu_0 + \beta_0|v|^2}|g(x, v)|,$$

and the claim follows. \[
\]

Similarly to Lemmas 6.3, 6.7, we obtain the following continuity estimate on the ternary collisional operator $Q_3$:

Lemma 6.10. Let $\beta > 0$, $\mu \in \mathbb{R}$. Then for any $g, h \in X_{\beta, \mu}$ and $(x, v) \in \mathbb{R}^2$, the following nonlinear continuity estimate holds:

$$|Q_3(g, g, g)(x, v) - Q_3(h, h, h)(x, v)| \lesssim \beta^{-d} \left(\beta^{-1/2} + |v|\right) e^{-3\mu - \frac{d}{2}|v|^2} \left(|g|_{\beta, \mu} + |h|_{\beta, \mu}\right)^2 \times |g - h|_{\beta, \mu},$$

and

$$|Q_3(g, g, g)(x, v)| \lesssim \beta^{-d} \left(\beta^{-1/2} + |v|\right) e^{-3\mu - \frac{d}{2}|v|^2} |g|^3_{\beta, \mu}.$$  

Proof. Let $g, h \in X_{\beta, \mu}$ and $(x, v) \in \mathbb{R}^2$. Then, estimate (6.5) yields

$$|Q_3(g, g, g)(x, v) - Q_3(h, h, h)(x, v)| \leq 3 \int_{S^d_{1} \times \mathbb{R}^d} (|v| + |v_1| + |v_2|) \times (|g^* g_1 g_2^* - h^* h_1^* h_2^*| + |g g_1 g_2 - h h_1 h_2|) \, d\omega_1 \, d\omega_2 \, dv_1 \, dv_2,$$

But triangle inequality implies

$$|g g_1 g_2 - h h_1 h_2| \leq |g_1| \cdot |g_2| \cdot |g - h| + |h| \cdot |g_2| \cdot |g_1 - h_1| + |h| \cdot |h_1| \cdot |g_2 - h_2|,$$

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and similarly
\[ |g^* g_1^* g_2^* - h^* h_1^* h_2^*| \leq |g_1^*| \cdot |g_2^*| \cdot |g^* - h^*| + |h^*| \cdot |g_2^*| \cdot |g_1^* - h_1^*| + |h^*| \cdot |h_1^*| \cdot |g_2^* - h_2^*|. \]

Therefore, performing the change of variables \((v^*, v_1^*, v_2^*) \rightarrow (v, v_1, v_2)\) to the postcollisional term and using conservation of energy (in the same spirit as in the proof of Lemma 6.3), the definition of the norm implies
\[ |Q^3(g, g, g)(x, v) - Q^3(h, h, h)(x, v)| \lesssim e^{-3\mu - \frac{\beta}{2}|v|^2} (|g|_{\beta, \mu} + |h|_{\beta, \mu})^2 |g - h|_{\beta, \mu} \times \int_{\mathbb{R}^{2d}} (|v| + |v_1| + |v_2|) e^{-\frac{\beta}{2}|v_1|^2} e^{-\frac{\beta}{2}|v_2|^2} dv_1 dv_2, \]
and the result follows exactly as in the proof of Lemma 6.3.

We define mild solutions to the ternary Boltzmann equation as follows:

**Definition 6.3.** Consider \( T > 0, \beta_0 > 0, \mu_0 \in \mathbb{R} \) and \( \beta, \mu : [0, T] \rightarrow \mathbb{R} \) decreasing functions of time, with \( \beta(0) = \beta_0, \beta(T) > 0, \mu(0) = \mu_0 \). Consider also initial data \( g_0 \in X_{\beta_0, \mu_0} \). A map \( g \in X_{\beta, \mu} \) is a mild solution to the ternary Boltzmann equation in \([0, T]\), with initial data \( g_0 \in X_{\beta_0, \mu_0} \), if it satisfies

\[ g(t) = S^t_1 g_0 + \int_0^t S^{t-\tau}_1 Q^3(g, g, g)(\tau) d\tau. \quad (6.24) \]

where \( S^t_1 \) denotes the free flow of 1-particle given in (4.63).

A similar proof to Lemma 6.4 gives the following:

**Lemma 6.11.** Let \( \beta_0 > 0, \mu_0 \in \mathbb{R}, T > 0 \) and \( \lambda \in (0, \beta_0/T) \). Consider the functions \( \beta_\lambda, \mu_\lambda : [0, T] \rightarrow \mathbb{R} \) given by (6.7). Then for any \( h, g \in X_{\beta_\lambda, \mu_\lambda} \) the following bounds hold:

\[ \left\| \int_0^t S^{t-\tau}_1 Q^3(g - h, g - h, g - h)(\tau) d\tau \right\|_{\beta_\lambda, \mu_\lambda} \leq C(d, \beta_0, \mu_0, T, \lambda) (|g|_{\beta_\lambda, \mu_\lambda} + |h|_{\beta_\lambda, \mu_\lambda})^2 |g - h|_{\beta_\lambda, \mu_\lambda}, \]

where \( S^t_1 \) denotes the free flow of 1-particle given in (4.63).
and
\[ \left\| \int_0^t S_{\lambda}^{t-\tau} Q_3(g, g, g)(\tau) \, d\tau \right\|_{\beta, \mu} \leq C(d, \beta_0, \mu_0, T, \lambda) \|g\|_{\beta, \mu}^3, \]

where \(C(d, \beta_0, \mu_0, T, \lambda)\) is given by (6.8).

Choosing \(\lambda = \beta_0/2T\), this estimate implies local well-posedness of the ternary Boltzmann equation up to short times.

Let us write \(B_{X_{\beta, \mu}}\) for the unit ball of \(X_{\beta, \mu}\).

**Theorem 6.11.1** (LWP for the ternary Boltzmann equation). Let \(\beta_0 > 0\) and \(\mu_0 \in \mathbb{R}\). Then there is \(T = T(d, \beta_0, \mu_0) > 0\) such that for any initial data \(f_0 \in X_{\beta_0, \mu_0}\), with \(\|f_0\|_{\beta_0, \mu_0} \leq 1/2\), there is a unique mild solution \(f \in B_{X_{\beta, \mu}}\) to the ternary Boltzmann equation in \([0, T]\) with initial data \(f_0\), where \(\beta, \mu : [0, T] \to \mathbb{R}\) are the functions given by (6.15). The solution \(f\) satisfies the bound
\[ \|f\|_{\beta, \mu} \leq 2\|f_0\|_{\beta_0, \mu_0}. \] (6.25)

Moreover, for any \(h, g \in X_{\beta, \mu}\), the following estimates hold:
\[ \left\| \int_0^t S_{\lambda}^{t-\tau} Q_3(g - h, g - h, g - h)(\tau) \, d\tau \right\|_{\beta, \mu} \leq \frac{1}{8} \left( \|g\|_{\beta, \mu} + \|h\|_{\beta, \mu} \right) \|g - h\|_{\beta, \mu}, \] (6.26)
\[ \left\| \int_0^t S_{\lambda}^{t-\tau} Q_3(g, g, g)(\tau) \, d\tau \right\|_{\beta, \mu} \leq \frac{1}{8} \|g\|_{\beta, \mu}^3. \] (6.27)

The time \(T\) is explicitly given by (6.18).

**Proof.** Choosing \(T\) as in (6.18), (6.8) implies that
\[ C(d, \beta_0, \mu_0, T, \beta_0/2T) = \frac{1}{8}. \]
Thus, Lemma 6.11 implies estimates (6.26)-(6.27). Therefore, for any \( g \in B_{X_{\beta,\mu}} \), we obtain
\[
\left\| \int_0^t S_1^{t-\tau} Q_3(g, g, g)(\tau) \, d\tau \right\|_{\beta,\mu} \leq \frac{1}{8} \|g\|_{\beta,\mu}^3 \leq \frac{1}{8} \|g\|_{\beta,\mu}.
\] (6.28)

Let us define the nonlinear operator \( \mathcal{L} : X_{\beta,\mu} \to X_{\beta,\mu} \) by
\[
\mathcal{L} g(t) = S_1^t f_0 + \int_0^t S_1^{t-\tau} Q_3(g, g, g)(\tau) \, d\tau.
\]

By triangle inequality, expression (6.23), Remark 6.5, (6.28) and the bound \( |f_0|_{\beta,\mu} \leq 1/2 \), for any \( g \in B_{X_{\beta,\mu}} \) and \( t \in [0, T] \), we have
\[
|\mathcal{L} g|_{\beta(t),\mu(t)} \leq |S_1^t f_0|_{\beta(t),\mu(t)} + \frac{1}{8} \|g\|_{\beta,\mu}
\]
\[
= |f_0|_{\beta(t),\mu(t)} + \frac{1}{8} \|g\|_{\beta,\mu}
\]
\[
\leq |f_0|_{\beta,\mu} + \frac{1}{8} \|g\|_{\beta,\mu}
\]
\[
< 1.
\]

Therefore, \( \|\mathcal{L} g\|_{\beta(t),\mu(t)} < 1 \), thus \( \mathcal{L} : B_{X_{\beta,\mu}} \to B_{X_{\beta,\mu}} \). Moreover, for any \( g, h \in B_{X_{\beta,\mu}} \), we have
\[
\|\mathcal{L} g - \mathcal{L} h\|_{\beta,\mu} \leq \frac{1}{8} (\|g\|_{\beta,\mu} + \|h\|_{\beta,\mu})^2 \|g - h\|_{\beta,\mu}
\]
\[
\leq \frac{1}{2} \|g - h\|_{\beta,\mu}.
\] (6.29)

Therefore, the operator \( \mathcal{L} : B_{X_{\beta,\mu}} \to B_{X_{\beta,\mu}} \) is a contraction, so it has a unique fixed point \( f \in B_{X_{\beta,\mu}} \) which is clearly the unique mild solution of the ternary Boltzmann equation in \([0, T]\) with initial data \( f_0 \).

To prove (6.25), we use the fact that \( f = \mathcal{L} f \). Then for any \( t \in [0, T] \), triangle inequality, definition of \( \mathcal{L} \), estimate (6.29) (for \( h = f \) and \( g = 0 \)), expression (6.23),
and Remark 6.5 yield

\[
\|f\|_{\beta(t), \mu(t)} = |L f|_{\beta(t), \mu(t)} \\
\leq |L 0|_{\beta(t), \mu(t)} + |L f - L 0|_{\beta(t), \mu(t)} \\
\leq |S^t f_0|_{\beta(t), \mu(t)} + \frac{1}{2} \|f\|_{\beta, \mu} \\
= |f_0|_{\beta(t), \mu(t)} + \frac{1}{2} \|f\|_{\beta, \mu} \\
\leq |f_0|_{\beta_0, \mu_0} + \frac{1}{2} \|f\|_{\beta, \mu},
\]

thus \( \|f\|_{\beta, \mu} \leq |f_0|_{\beta_0, \mu_0} + \frac{1}{2} \|f\|_{\beta, \mu} \), and (6.25) follows. \( \square \)

We can now prove that chaos is propagated by the Boltzmann hierarchy.

**Theorem 6.11.2 (Propagation of chaos).** Let \( \beta_0 > 0 \), \( \mu_0 \in \mathbb{R} \), \( T > 0 \) the time obtained by Theorem 6.11.1 and \( \beta, \mu : [0, T] \to \mathbb{R} \) the functions defined by (6.15).

Consider \( f_0 \in X_{\beta_0, \mu_0} \) with \( |f_0|_{\beta_0, \mu_0} \leq \frac{1}{2} \). Assume \( f \in B X_{\beta, \mu} \) is the corresponding mild solution of the ternary Boltzmann equation in \([0, T]\), with initial data \( f_0 \) given by Theorem 6.11.1. Then the following hold:

(i) \( F_0 = (f_0^\otimes s)_{s \in \mathbb{N}} \in X_{\infty, \beta_0, \mu_0} \).

(ii) \( F = (f^\otimes s)_{s \in \mathbb{N}} \in X_{\infty, \beta, \mu} \).

(iii) \( F \) is the unique mild solution of the Boltzmann hierarchy in \([0, T]\), with initial data \( F_0 \).

**Proof.** (i) is trivially verified by the bound on the initial data and the definition of the norms. By the the same bound again, we may apply Theorem 6.11.1 to...
obtain the unique mild solution $f \in B_{X_{\beta,\mu}}$ of the corresponding ternary Boltzmann equation. Since $\|f\|_{\beta,\mu} \leq 1$, the definition of the norms directly imply (ii). It is also straightforward to verify that $F$ is a mild solution of the Boltzmann hierarchy in $[0, T]$, with initial data $F_0$. Uniqueness of the mild solution to the Boltzmann hierarchy, obtained by Theorem [6.8.1] implies that $F$ is the unique mild solution.

The goal of the rest of this part of the dissertation is to show that if the BBGKY hierarchy initial data converge in observables to the Boltzmann hierarchy initial data, then the corresponding mild solution, under the scaling imposed, to the BBGKY hierarchy will converge to the mild solution of the Boltzmann hierarchy.
Chapter 7

Convergence Statement

In this chapter we define an appropriate notion of convergence, namely convergence in observables, and we state the main convergence result.

7.1 Approximation of Boltzmann initial data

In this section, we approximate Boltzmann hierarchy initial data by BBGKY hierarchy initial data. Let us first introduce some notation we are using from now on.

Given $\sigma > 0$, we introduce the set of well-separated spatial configurations as follows:

For $m \geq 2$, we define

$$ \Delta^X_m(\sigma) := \{ \tilde{X}_m \in \mathbb{R}^{dm} : |\tilde{x}_i - \tilde{x}_j| > \sigma, \ \forall 1 \leq i < j \leq m \} . $$

(7.1)

For $m = 1$, we trivially define

$$ \Delta^X_1(\sigma) := \mathbb{R}^d . $$

(7.2)

For $m \in \mathbb{N}$, we also define the set of well-separated configurations as:

$$ \Delta_m(\sigma) := \Delta^X_m(\sigma) \times \mathbb{R}^{dm} = \{ (\tilde{X}_m, \tilde{V}_m) \in \mathbb{R}^{2dm} : |\tilde{x}_i - \tilde{x}_j| > \sigma, \ \forall 1 \leq i < j \leq m \} . $$

(7.3)
Recall we consider \((N, \epsilon)\) in the scaling
\[
N \epsilon^{d-\frac{1}{2}} \simeq 1. \tag{7.4}
\]
Let us write \(\epsilon_N\) for the \(\epsilon\) associated to \(N\) under \((7.4)\) i.e.
\[
\epsilon_N \simeq N^{\frac{2}{2d-1}}. \tag{7.5}
\]
Clearly the sequence \((\epsilon_N)_{N \in \mathbb{N}}\) is decreasing and \(\epsilon_N \to 0^+\) as \(N \to \infty\).

We define the following approximating sequence:

**Definition 7.1.** Let \(s \in \mathbb{N}, \beta > 0, \mu \in \mathbb{R}\) and \(G = (g_s)_{s \in \mathbb{N}} \in X_{\infty, \beta, \mu}\). We define
\[
G_N = (g_{N,s})_{s \in \mathbb{N}}, \quad \text{where} \quad g_{N,s} = \mathbb{1}_{\Delta_s(\epsilon_N)} g_s. \tag{7.6}
\]
The sequence \((G_N)_{N \in \mathbb{N}}\) is called approximating BBGKY hierarchy sequence of \(G\).

The associated BBGKY hierarchy sequence has the following approximation property:

**Proposition 7.2.** Let \(s \in \mathbb{N}, \beta > 0, \mu \in \mathbb{R}\), \(G = (g_s)_{s \in \mathbb{N}} \in X_{\infty, \beta, \mu}\) and \((G_N)_{N \in \mathbb{N}}\) the associated BBGKY hierarchy sequence of \(G\). Then the following hold:

(i) \(G_N \in X_{N, \beta, \mu}\) for all \(N \in \mathbb{N}\). In particular,
\[
\sup_{N \in \mathbb{N}} \|G_N\|_{N, \beta, \mu} \leq \|G\|_{\infty, \beta, \mu} \tag{7.7}
\]
(ii) For any $s \in \mathbb{N}$ and $\sigma > 0$, we have
\[
\lim_{N \to \infty} \|g_{N,s} - g_s\|_{L^\infty(\Delta_s(\sigma))} = 0.
\] (7.8)

Proof. Part (i) is immediate from the definition of the norms and $G_N$.

To prove (ii), let us fix $s \in \mathbb{N}$ and $\sigma > 0$. Consider $N_0$ large enough such that $\epsilon_{N_0} < \sigma$, where the sequence $\epsilon_N$ is given by (7.5). Then
\[
\Delta_s(\sigma) \subseteq \Delta_s(\epsilon_N), \quad \forall N \geq N_0.
\]

Therefore, for any $N \geq N_0$, we get
\[
\|g_{N,s} - g_s\|_{L^\infty(\Delta_s(\sigma))} = \|\mathbb{1}_{\Delta_s(\sigma)}(g_{N,s} - g_s)\|_{L^\infty} \leq \|g_s\|_{\infty,\beta,s} \|\mathbb{1}_{\Delta_s(\sigma)}\|_{L^\infty} = 0,
\]

so
\[
\lim_{N \to \infty} \|g_{N,s} - g_s\|_{L^\infty(\Delta_s(\sigma))} = 0.
\]

\[\Box\]

7.3 Convergence in observables

In this section we define the convergence in observables. Let us first introduce some notation. Given $s \in \mathbb{N}$, we define the space of test functions
\[
C_c(\mathbb{R}^{ds}) = \{ \phi_s : \mathbb{R}^{ds} \to \mathbb{R} : \phi_s \text{ is continuous and compactly supported} \}.
\] (7.9)

Definition 7.2. Consider $T > 0$, $s \in \mathbb{N}$ and $g_s \in L^\infty([0,T], L^\infty(\mathbb{R}^{2ds}))$. Given a test function $\phi_s \in C_c(\mathbb{R}^{ds})$, we define the $s$-observable functional as
\[
I_{\phi_s, g_s}(t)(X_s) = \int_{\mathbb{R}^{ds}} \phi_s(V_s)g_s(t, X_s, V_s) dV_s.
\]
Recalling the set of initially good spatial configurations \( \Delta^X_s(\sigma) \) from (7.1)-(7.2), we give the definition of the convergence in observables:

**Definition 7.3.** Let \( T > 0 \). For each \( N \in \mathbb{N} \), consider

\[
G_N = (g_{N,s})_{s \in \mathbb{N}} \in \prod_{s=1}^{\infty} L^\infty([0,T], L^\infty(\mathbb{R}^{2d_s}))
\]

and \( G = (g_s)_{s \in \mathbb{N}} \in \prod_{s=1}^{\infty} L^\infty([0,T], L^\infty(\mathbb{R}^{2d_s})) \). We say that the sequence \( (G_N)_{N \in \mathbb{N}} \) converges in observables to \( G \), and write

\[
G_N \xrightarrow{\sim} G
\]

if for any \( s \in \mathbb{N}, \sigma > 0 \) and \( \phi_s \in C_c(\mathbb{R}^{d_s}) \), we have

\[
\lim_{N \to \infty} \|I_{\phi_s}g_{N,s}(t) - I_{\phi_s}g_s(t)\|_{L^\infty(\Delta^X_s(\sigma))} = 0, \quad \text{uniformly in } [0,T].
\]

### 7.4 Statement of the main convergence result

We are now in the position to state our main convergence result. The rest of this part of the dissertation will be devoted to its proof.

**Theorem 7.4.1** (Convergence). Let \( \beta_0 > 0, \mu_0 \in \mathbb{R} \) and \( T = T(d, \beta_0, \mu_0) > 0 \) given by (6.18). Consider an initial Boltzmann hierarchy datum \( F_0 = (f_{0,s})_{s \in \mathbb{N}} \in X^{\infty, \beta_0, \mu_0} \) with approximating BBGKY hierarchy sequence \( (F_N,0)_{N \in \mathbb{N}} \). Assume that

- for each \( N \), \( F_N \in X_{N, \beta, \mu} \) is the mild solution (given by Theorem 6.4.1) of the BBGKY hierarchy in \([0,T]\) with initial data \( F_{N,0} \).

- \( F \in X^{\infty, \beta, \mu} \) is the mild solution (given by Theorem 6.8.1) of the Boltzmann hierarchy in \([0,T]\) with initial data \( F_0 \).
• $F_0$ satisfies the following uniform continuity growth condition: There is a constant $C > 0$ such that, for any $\zeta > 0$, there is $q = q(\zeta) > 0$ such that for all $s \in \mathbb{N}$, and for all $Z_s, Z'_s \in \mathbb{R}^{2d_s}$ with $|Z_s - Z'_s| < q$, we have

$$|f_0^{(s)}(Z_s) - f_0^{(s)}(Z'_s)| < C^{s-1}\zeta. \quad (7.10)$$

Then, the following convergence in observables holds:

$$F_N \xrightarrow{\sim} F.$$ 

Remark 7.1. Using the definition of convergence, proving Theorem 7.4.1 is equivalent to proving that for any $s \in \mathbb{N}$, $\phi_s \in C_c(\mathbb{R}^{d_s})$ and $\sigma > 0$ we have

$$\lim_{N \to \infty} \|I_N^s(t) - I_\infty^s(t)\|_{L^\infty(\Delta_s^s(\sigma))} = 0, \quad \text{uniformly in } [0, T],$$

where

$$I_N^s(t)(X_s) := I_{\phi_s}f_N^{(s)}(t)(X_s) = \int_{\mathbb{R}^{d_s}} \phi_s(V_s)f_N^{(s)}(t, X_s, V_s) dV_s, \quad (7.11)$$

and

$$I_\infty^s(t)(X_s) := I_{\phi_s}f^{(s)}(t)(X_s) = \int_{\mathbb{R}^{d_s}} \phi_s(V_s)f^{(s)}(t, X_s, V_s) dV_s. \quad (7.12)$$

The following Corollary of Theorem 7.4.1 justifies the derivation of our ternary Boltzmann equation from finitely many particle systems.

**Corollary 7.5.** Let $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ and $f_0 \in X_{\beta_0, \mu_0}$, with $|f_0|_{\beta_0, \mu_0} \leq 1/2$. Assume as well that $f_0$ is uniformly continuous. Then for any $s \in \mathbb{N}$, $\phi_s \in C_c(\mathbb{R}^{d_s})$ and $\sigma > 0$, the following convergence holds:

$$\lim_{N \to \infty} \|I_{\phi_s}f^{\otimes^s}_N \mathbb{1}_{\Delta_s(\epsilon_N)} - I_{\phi_s}f^{\otimes^s}_0\|_{L^\infty(\Delta_s(\sigma))} = 0, \quad (7.13)$$

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where $f$ is the mild solution to the ternary Boltzmann equation in $[0,T]$, with initial data $f_0$, given by Theorem 6.11.1 and $T$ is given by (6.18).

Proof. It is enough to show that $F_0 = (f_0^{\otimes s})_{s \in \mathbb{N}}$ satisfies (7.10), and claim (7.13) follows by Theorem 6.11.2 and Theorem 7.4.1.

Fix $\zeta > 0$. Since $f_0$ is uniformly continuous, there is $q = q(\zeta) > 0$ such that for any $z, z' \in \mathbb{R}^{2d}$ with $|z - z'| < q$, we have

$$|f_0^{\otimes 1}(z) - f_0^{\otimes 1}(z')| = |f_0(z) - f_0(z')| < \zeta. \quad (7.14)$$

Consider $C > 2\|f_0\|_{L^\infty}$. It suffices to prove the following claim:

Claim: For any $s \in \mathbb{N}$, $\ell \in \{1, \ldots, s\}$ and $|Z_\ell - Z'_\ell| < q$ there holds:

$$|f_0^{\otimes \ell}(Z_\ell) - f_0^{\otimes \ell}(Z'_\ell)| < C^{\ell-1} \zeta. \quad (7.15)$$

Proof of the claim: Fix $s \in \mathbb{N}$. We prove that claim (7.15) holds for $\ell \in \{1, \ldots, s\}$.

We will use induction on $\ell \in \{1, \ldots, s\}$.

- $\ell = 1$: Claim (7.15) comes directly from (7.14), since $f_0^{\otimes 1} = f_0$.

- Assume claim (7.15) holds for $\ell \in \{1, \ldots, s - 1\}$ i.e. for each $Z_\ell, Z'_\ell \in \mathbb{R}^{2d\ell}$, with $|Z_\ell - Z'_\ell| < q$, there holds:

$$|f_0^{\otimes \ell}(Z_\ell) - f_0^{\otimes \ell}(Z'_\ell)| < C^{\ell-1} \zeta. \quad (7.16)$$

We will show (7.15) holds for $\ell + 1 \in \{2, \ldots, s\}$. Consider $Z_{\ell+1}, Z'_{\ell+1} \in \mathbb{R}^{2d(\ell+1)}$, with

$$|Z_{\ell+1} - Z'_{\ell+1}| < q \quad (7.17)$$

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Let us write $Z_{\ell+1} = (X_\ell, x_{\ell+1}, V_\ell, v_{\ell+1})$, $Z'_\ell = (X'_\ell, V'_\ell) \in \mathbb{R}^{2d\ell}$. By (7.17), we have $|Z_\ell - Z'_\ell| < q$ and $|z_{\ell+1} - z'_{\ell+1}| < q$, where $z_{\ell+1} = (x_{\ell+1}, v_{\ell+1})$, $z'_{\ell+1} = (x'_{\ell+1}, v'_{\ell+1})$. Therefore (7.14), (7.16) and the fact that $C > 2\|f_0\|_{L^\infty}$ imply

$$
|f_0^{\otimes(\ell+1)}(Z_{\ell+1}) - f_0^{\otimes(\ell+1)}(Z'_{\ell+1})| = |f_0^{\otimes\ell}(Z_\ell)f_0(z_{\ell+1}) - f_0^{\otimes\ell}(Z'_\ell)f_0(z'_{\ell+1})|
\leq |f_0(z_{\ell+1})||f_0^{\otimes\ell}(Z_\ell) - f_0^{\otimes\ell}(Z'_\ell)| + |f_0^{\otimes\ell}(Z'_\ell)||f_0(z_{\ell+1}) - f_0(z'_{\ell+1})|
\leq \|f_0\|_{L^\infty}C^{\ell-1}\zeta + \|f_0^{\otimes\ell}\|_{L^\infty}\zeta
\leq \|f_0\|_{L^\infty}C^{\ell-1}\zeta + \|f_0\|^{\ell}_{L^\infty}\zeta
\leq \frac{1}{2}C^{\ell}\zeta + \frac{1}{2}\zeta C^{\ell}
\leq C^{\ell}\zeta.
$$

Claim (7.15) is proved, and the result follows. \qed

In order to prove Theorem 7.4.1, we will first use the local estimates developed in Chapter 6 to reduce the proof to finitely many observables of bounded energy, which are also well separated in time. Then, we will develop some geometric estimates which will enable us to eliminate recollisions of the backwards interaction flow.
Chapter 8

Reduction to term by term convergence

In this chapter, we reduce the proof of Theorem 7.4.1 to term by term convergence after truncating the observables. For this purpose we will strongly rely on the local estimates developed in Chapter 6.

Throughout this chapter, we consider \( \beta_0 > 0, \mu_0 \in \mathbb{R}, T = T(d, \beta_0, \mu_0) > 0 \)
given by (6.18), the functions \( \beta, \mu : [0, T] \rightarrow \mathbb{R} \) defined by (6.15), \((N, \epsilon)\) in the scaling (5.43) and initial data \( F_{N,0} \in X_{N,\beta_0,\mu_0}, F_0 \in X_{\infty,\beta_0,\mu_0} \). Let \( F_N \in X_{N,\beta,\mu}, F \in X_{\infty,\beta,\mu} \) be the mild solutions of the corresponding BBGKY hierarchy and Boltzmann hierarchy, respectively, in \([0, T]\), given by Theorem 6.4.1 and Theorem 6.8.1. Let us note that by (6.15), we obtain

\[
\beta(T) = \frac{\beta_0}{2}, \quad \mu(T) = \mu_0 - \frac{\beta_0}{2},
\]

thus \( \beta(T), \mu(T) \) do not depend on \( T \).

For convenience, we introduce the following notation. Given \( k \in \mathbb{N} \) and \( t \geq 0 \), we denote

\[
\mathcal{I}_k(t) := \{(t_1, ..., t_k) \in \mathbb{R}^k : 0 \leq t_k < ... < t_1 \leq t\}.
\]
8.1 Series expansion

Let us fix $s \in \mathbb{N}$. Using iteratively the Duhamel’s formula for the mild solution of the BBGKY hierarchy, given by (5.42), we get the following formal series expansion for the mild solution:

$$f_N^{(s)}(t, Z_s) = T_s f_{N,0}^{(s)}(Z_s) + \int_0^t T_s^{t-t_1} C_{s,s+2}^N f_N^{(s+2)}(t_1, Z_s) \, dt_1$$

$$= T_s f_{N,0}^{(s)}(Z_s) + \int_0^t T_s^{t-t_1} C_{s,s+2}^N T_{s+2} f_{N,0}^{(s+2)}(Z_s) \, dt_1 +$$

$$+ \int_0^t \int_0^{t_1} T_s^{t-t_1} T_{s+2} C_{s,s+4}^N f_N^{(s+4)}(t_2, Z_s) \, dt_2 \, dt_1$$

$$= T_s f_{N,0}^{(s)}(Z_s) + \int_0^t T_s^{t-t_1} C_{s,s+2}^N T_{s+2} f_{N,0}^{(s+2)}(Z_s) \, dt_1 +$$

$$+ \int_0^t \int_0^{t_1} T_s^{t-t_1} T_{s+2} C_{s,s+4}^N T_{s+4} f_{N,0}^{(s+4)}(Z_s) \, dt_2 \, dt_1$$

$$+ \int_0^t \int_0^{t_1} \int_0^{t_2} T_s^{t-t_1} C_{s,s+2}^N T_{s+2} C_{s,s+4}^N T_{s+4} f_{N,0}^{(s+6)}(t_3, Z_s) \, dt_3 \, dt_2 \, dt_1$$

$$= \ldots$$

$$= \sum_{k=0}^n f_N^{(s,k)}(Z_s) + R_N^{(n+1)}(t, Z_s),$$

(8.2)

where, recalling (8.1), we denote

$$f_N^{(s,0)}(t, Z_s) := T_s f_{N,0}^{(s)}(Z_s),$$

(8.3)

for $1 \leq k \leq n$:

$$f_N^{(s,k)}(t, Z_s) := \int_{J_k(t)} T_s^{t-t_1} C_{s,s+2}^N T_{s+2}^{t_1-t_2} \ldots T_{s+2k-2}^{t_{k-1}-t_k} C_{s+2k-2,s+2k}^N T_{s+2k} f_N^{(s+2k)}(Z_s) \, dt_k \ldots dt_1,$$

(8.4)

and

$$R_N^{(s,n+1)}(t, Z_s) := \int_{J_{n+1}(t)} T_s^{t-t_1} C_{s,s+2}^N T_{s+2}^{t_1-t_2} \ldots$$

$$\ldots T_{s+2n-2}^{t_{n+1}-t_{n+1}} C_{s+2n-2,s+2n}^N \, dt_{n+1} \, dt_n \ldots dt_1,$$

(8.5)
Similarly, using iteratively the Duhamel's formula for the mild solution of the Boltzmann hierarchy, one gets

\[ f^{(s,0)}(t, Z_s) := S^t_s f^{(s)}(Z_s), \quad (8.6) \]

for \( 1 \leq k \leq n \):

\[ f^{(s,k)}(t, Z_s) := \int_{\mathcal{I}_k(t)} S^{t-t_1}_s C^\infty_{s,s+2} S^{t_1-t_2}_s ... S^{t_{k-1}-t_k}_s C^\infty_{s+2k-2,s+2k} S^{t_k}_s f^{(s+2k)}(Z_s) dt_k ... dt_1, \quad (8.7) \]

and

\[ R^{(s,n+1)}(t, Z_s) := \int_{\mathcal{I}_{n+1}(t)} S^{t-t_1}_s C^\infty_{s,s+2} S^{t_1-t_2}_s ... \]

\[ ... C^{t_{n+1}}_{s+2n-2} C^{t_{n+1}}_{s+2n} ... C^{t_{n+1}}_{s+2n} f^{(s+2n+2)}(t_{n+1}, Z_s) dt_{n+1} dt_n ... dt_1. \quad (8.8) \]

Given \( \phi_s \in C_c(\mathbb{R}^{ds}) \) and \( k \in \mathbb{N} \), let us denote

\[ I^N_{s,k}(X_s) := I_{\phi_s} f^{(s,k)}(t)(X_s) = \int_{\mathbb{R}^{ds}} \phi_s(V_s) f^{(s,k)}(t, X_s, V_s) dV_s, \quad (8.9) \]

\[ I^\infty_{s,k}(X_s) := I_{\phi_s} f^{(s,k)}(t)(X_s) = \int_{\mathbb{R}^{ds}} \phi_s(V_s) f^{(s,k)}(t, X_s, V_s) dV_s. \quad (8.10) \]

Recalling the observables \( I^N_s, I^\infty_s \) defined in (7.11)-(7.12), we obtain the following estimates:

**Lemma 8.2.** For any \( s, n \in \mathbb{N} \) and \( t \in [0, T] \), the following estimates hold:

\[ \| I^N_s(t) - \sum_{k=0}^n I^N_{s,k}(t) \|_{L^\infty_{X_s}} \leq C_{s,\beta_0,\mu_0} S^{-n} \| \phi_s \|_{L^\infty_{V_s}} \| F_{N,0} \|_{L^\infty_{X_s}}, \]

and

\[ \| I^\infty_s(t) - \sum_{k=0}^n I^\infty_{s,k}(t) \|_{L^\infty_{X_s}} \leq C_{s,\beta_0,\mu_0} S^{-n} \| \phi_s \|_{L^\infty_{V_s}} \| F_0 \|_{L^\infty_{X_s}}. \]
Proof. Let us first prove the estimate for the BBGKY hierarchy. For any $Z_s = (X_s, V_s) \in \mathbb{R}^{2ds}$ and $t \in [0, T]$, definition of the norms and repeated use of estimate (6.17) imply

$$e^{s\mu(t)+\beta(t)E_s(Z_s)}|R_N^{(s,n+1)}(t, X_s, V_s)| \leq 8^{-(n+1)}|||F_N|||_{N,\beta,\mu},$$

so by estimate (6.16) and the definition of the norms, we obtain

$$|\phi_s(V_s)R_N^{(s,n+1)}(t, X_s, V_s)| \lesssim 8^{-(n+1)}e^{-s\mu(t)}\|\phi_s\|_{L\infty_{V_s}}|||F_N|||_{N,\beta,\mu}e^{-\beta(t)E_s(Z_s)} \leq 8^{-n}e^{-s\mu(T)}\|\phi_s\|_{L\infty_{V_s}}||F_{N,0}||_{N,\beta,\mu}e^{-\beta(T)E_s(Z_s)}.$$

Thus, integrating with respect to velocities and recalling (8.2), (8.9), we obtain

$$\|I_N^s(t) - \sum_{k=0}^n I_N^{s,k}(t)\|_{L\infty_{X_s}} \leq C_s,\beta_0,\mu_0 \|\phi_s\|_{L\infty_{V_s}}8^{-n}||F_{N,0}||_{N,\beta,\mu} \int_{\mathbb{R}^{2ds}} e^{-\beta(T)E_s(Z_s)} dV_s \leq C_s,\beta_0,\mu_0 \|\phi_s\|_{L\infty_{V_s}}8^{-n}||F_{N,0}||_{N,\beta,\mu}.$$

In the case of the Boltzmann hierarchy, we follow a similar argument using estimates (6.21)-(6.22) instead. \hfill \Box

### 8.3 High energy truncation

We will now truncate energies, so that we can focus on bounded energy domains.

Let us fix $s, n \in \mathbb{N}$ and $R > 1$. As usual we denote $B^{2d}_R$ to be the $2d$-ball of radius $R$ centered at the origin.

We first define the truncated BBGKY hierarchy and Boltzmann hierarchy collisional operators. Recalling (4.45) and notation from (1.16), given $\ell \in \mathbb{N}$, we
define
\[
\mathcal{E}_{\ell, \ell + 2}^{N_R} g_{N, \ell + 2} = \mathcal{E}_{\ell, \ell + 2}^{N} (g_{N, \ell + 2} \mathbf{1}_{[E_{\ell + 2} \leq R^2]}),
\]
(8.11)

For the BBGKY hierarchy, we define
\[
f^{(s,0)}_{N,R}(t, Z_s) := T_t^s (f_{N,0} \mathbf{1}_{[E_s \leq R^2]})(Z_s),
\]
(8.12)

and for \(1 \leq k \leq n\):
\[
f^{(s,k)}_{N,R}(t, Z_s) := \int_{J_k(t)} T_{t_1}^{t-t_1} \mathcal{E}_{s,s+2}^{N_R} T_{t_2}^{t_1-t_2} \ldots T_{t_s}^{t_{s-1}-t_s} \mathcal{E}_{s+2k-2,s+2k}^{N,R} T_{t_k}^{t_{k-1}-t_k} f^{(s+2k)}_{N,0}(Z_s) dt_k \ldots dt_1.
\]
(8.13)

For the Boltzmann hierarchy, we define
\[
f^{(s,0)}_R(t, Z_s) := S_t^s (f_{0} \mathbf{1}_{[E_s \leq R^2]})(Z_s),
\]
(8.14)

and for \(1 \leq k \leq n\):
\[
f^{(s,k)}_R(t, Z_s) := \int_{J_k(t)} S_{t_1}^{t-t_1} \mathcal{E}_{s,s+2}^{\infty,R} S_{t_2}^{t_1-t_2} \ldots S_{t_s}^{t_{s-1}-t_s} \mathcal{E}_{s+2k-2,s+2k}^{\infty,R} S_{t_k}^{t_{k-1}-t_k} f^{(s+2k)}_0(Z_s) dt_k \ldots dt_1.
\]
(8.15)

Given \(\phi_s \in C_c(\mathbb{R}^{ds})\) and \(k \in \mathbb{N}\), let us denote
\[
I^{N}_{s,k,R}(t)(X_s) := I_{\phi_s} f^{(s,k)}_{N,R}(t)(X_s) = \int_{\mathbb{R}^{ds}} \phi_s(V_s) f^{(s,k)}_{N,R}(t, X_s, V_s) dV_s
\]
\[
\quad = \int_{B^d_R} \phi_s(V_s) f^{(s,k)}_{N,R}(t, X_s, V_s) dV_s.
\]
(8.16)

\[
I^{\infty}_{s,k,R}(t)(X_s) := I_{\phi_s} f^{(s,k)}_{R}(t)(X_s) = \int_{\mathbb{R}^{ds}} \phi_s(V_s) f^{(s,k)}_{R}(t, X_s, V_s) dV_s
\]
\[
\quad = \int_{B^d_R} \phi_s(V_s) f^{(s,k)}_{R}(t, X_s, V_s) dV_s.
\]
(8.17)

Recalling the observables \(I^{N}_{s,k}, I^{\infty}_{s,k}\), defined in (8.9)-(8.10), we obtain the following estimates:
Lemma 8.4. For any \( s, n \in \mathbb{N}, R > 1 \) and \( t \in [0, T] \), the following estimates hold:

\[
\sum_{k=0}^{n} \| I_{N,s,k,R}^N(t) - I_{N,s,k}^N(t) \|_{L^n_{X_s}} \leq C_{s,\beta_0,\mu_0,T} \| \phi_s \|_{L^n_{V_s}} e^{-\frac{\beta_0}{3}R^2} \| F_{N,0} \|_{N,\beta_0,\mu_0},
\]

and

\[
\sum_{k=0}^{n} \| I_{N,s,k,R}^\infty(t) - I_{N,s,k}^\infty(t) \|_{L^n_{X_s}} \leq C_{s,\beta_0,\mu_0,T} \| \phi_s \|_{L^n_{V_s}} e^{-\frac{\beta_0}{3}R^2} \| F_0 \|_{\infty,\beta_0,\mu_0}.
\]

Proof. We first prove it for the BBGKY hierarchy case. Let \( \beta'_0 = 2\beta_0/3 \) and \( \lambda' = \beta'_0/4T \). Let us define the functions

\[
\beta'_\lambda(t) = \beta'_0 - \lambda t, \quad \mu'_\lambda(t) = \mu_0 - \lambda t.
\]

It is clear that \( \beta'_\lambda(T) = \beta_0/2 \). Then, a straightforward calculation and (6.18) imply that

\[
C(d, \beta'_0, \mu'_0, T, \lambda') \leq \frac{3}{4}, \quad (8.18)
\]

where \( C(d, \beta'_0, \mu'_0, T, \lambda') \) is given by (6.8).

We define

\[
G_{N,0} = (g_{N,0,m})_{m \in \mathbb{N}}, \quad \text{where} \quad g_{N,0,m} = \int_{V_m \notin B^m_R}^{(m)} N,0.111
\]

Notice that

\[
\| G_{N,0} \|_{N,\beta'_0,\mu_0} \leq e^{-\frac{\beta_0}{3}R^2} \| F_{N,0} \|_{N,\beta_0,\mu_0}. \quad (8.19)
\]

We first assume \( 1 \leq k \leq n \). Applying \( k-1 \) times Lemma 6.4 for \( \beta'_0, \mu_0, \lambda', \beta'_\lambda, \mu'_\lambda, \)
part (ii) of Proposition 6.2 and (8.19), we get

$$
|f(s,k)(t, Z_s) - f(s,k)(t, Z_s)| \leq e^{-s\mu'_\lambda(T) - \beta'_\lambda(T) E_s(Z_s)} \left( \frac{3}{4} \right)^{k-1} \left| \int_0^t \mathcal{T}^r G_{N,0} \, d\tau \right|_{N, \beta', \mu'} \\
\leq T e^{-s\mu'_\lambda(T) - \beta'_\lambda(T) E_s(Z_s)} \left( \frac{3}{4} \right)^{k-1} \left\| G_{N,0} \right\|_{N, \beta'_0, \mu_0} \\
\leq C_{s,\beta_0, \mu_0, T} \left( \frac{3}{4} \right)^{k-1} e^{-\frac{\beta_0}{3} R^2} \left\| F_{N,0} \right\|_{N, \beta_0, \mu_0} e^{-\beta'_\lambda(T) E_s(Z_s)}. 
$$

(8.20)

For $k = 0$, part (i) of Proposition 6.2 and Remark 6.2 yield

$$
|f(s,0)(t, Z_s) - f(s,0)(t, Z_s)| \leq e^{-s\mu'_\lambda(T) - \beta'_\lambda(T) E_s(Z_s)} \left| \int_0^t \mathcal{T}^r G_{N,0} \, d\tau \right|_{N, \beta'_0, \mu_0} \\
= e^{-s\mu'_\lambda(T) - \beta'_\lambda(T) E_s(Z_s)} \left| G_{N,0} \right|_{N, \beta'_0, \mu_0} \\
\leq C_{s,\beta_0, \mu_0} e^{-\frac{\beta_0}{3} R^2} \left\| F_{N,0} \right\|_{N, \beta_0, \mu_0} e^{-\beta'_\lambda(T) E_s(Z_s)}. 
$$

(8.21)

Combining (8.20)-(8.21), and adding for $k = 0, ..., n$, we obtain

$$
\sum_{k=0}^n \left| I_{s,k,R}(t) - I_{s,k}(t) \right|_{L_{N,s}^\infty} \leq C_{s,\beta_0, \mu_0, T} \left\| \phi_s \right\|_{L_{N,s}^\infty} e^{-\frac{\beta_0}{3} R^2} \left\| F_{N,0} \right\|_{N, \beta_0, \mu_0} \int_{\mathbb{R}^d} e^{-\beta'_\lambda(T) E_s(Z_s)} \, dV_s \\
\leq C_{s,\beta_0, \mu_0, T} \left\| \phi_s \right\|_{L_{N,s}^\infty} e^{-\frac{\beta_0}{3} R^2} \left\| F_{N,0} \right\|_{N, \beta_0, \mu_0}.
$$

The proof for the Boltzmann hierarchy case is similar, using Lemma 6.8 and Proposition 6.6 instead.

8.5 Separation of collision times

We will now separate the time intervals we are integrating at, so that collisions occurring are separated in time. For this purpose consider a small time parameter $\delta > 0$. 

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For convenience, given \( t \geq 0 \) and \( k \in \mathbb{N} \), we define
\[
\mathcal{T}_{k,\delta}(t) := \{(t_1, ..., t_k) \in \mathcal{T}_k(t) : 0 \leq t_{i+1} \leq t_i - \delta, \quad \forall i \in [0, k]\},
\] (8.22)
where we denote \( t_{k+1} = 0, t_0 = t \).

For the BBGKY hierarchy, we define
\[
f^{(s,0)}_{N,R,\delta}(t, Z_s) := T^t_s \left(f_{N,0}[E_s \leq R^2]\right)(Z_s),
\] (8.23)
and for \( 1 \leq k \leq n \):
\[
f^{(s,k)}_{N,R,\delta}(t, Z_s) := \int_{\mathcal{T}_{k,\delta}(t)} T^{t-t_1}_{s} \Theta_{s,s+2}^{N,R} T^{t_1-t_2}_{s+2} \Theta_{s+2k-2,s+2k}^{N,R} T^{t_{k-1}-t_k}_{s+2k-2} \Theta_{s+2k-2,s+2k}^{N,R} f^{(s+2k)}_{N,0}(Z_s) \, dt_k \ldots \, dt_1.
\] (8.24)

In the same spirit, for the Boltzmann hierarchy we define
\[
f^{(s,0)}_{R,\delta}(t, Z_s) := S^t_s \left(f_{0}[E_s \leq R^2]\right)(Z_s),
\] (8.25)
and for \( 1 \leq k \leq n \):
\[
f^{(s,k)}_{R,\delta}(t, Z_s) := \int_{\mathcal{T}_{k,\delta}(t)} S^{t-t_1}_{s} \Theta_{s,s+2}^{\infty,R} S^{t_1-t_2}_{s+2} \Theta_{s+2k-2,s+2k}^{\infty,R} S^{t_{k-1}-t_k}_{s+2k-2} \Theta_{s+2k-2,s+2k}^{\infty,R} S^{t_m}_{s+2k} f^{(s+2k)}_{0}(Z_s) \, dt_k \ldots \, dt_1.
\] (8.26)

Given \( \phi_s \in C_c(\mathbb{R}^{ds}) \) and \( k \in \mathbb{N} \), let us denote
\[
I^{N}_{s,k,R,\delta}(t)(X_s) := \int_{\mathbb{R}^{ds}} \phi_s(V_s) f^{(s,k)}_{N,R,\delta}(t)(X_s, V_s) \, dV_s
\] (8.27)
and
\[
I^{\infty}_{s,k,R,\delta}(t)(X_s) := \int_{\mathbb{R}^{ds}} \phi_s(V_s) f^{(s,k)}_{R,\delta}(t)(X_s, V_s) \, dV_s.
\] (8.28)
Remark 8.1. For $0 \leq t \leq \delta$, we trivially obtain $T_{k,\delta}(t) = \emptyset$. In this case the functionals $I_{s,k,R,\delta}^N(t), I_{s,k,R,\delta}^\infty(t)$ are identically zero.

Using Lemmas 6.3, 6.7 that express continuity estimates on the collisional operators, we obtain the following continuity estimates:

Lemma 8.6. Let $s, k \in \mathbb{N}$, $1 \leq j \leq k$, $t > 0$ and $\mathcal{F}(t) \subseteq [0,t]$ measurable. Then the following estimates hold:

(i) Assume $g_{N,s+2j}(\tau, \cdot) \in X_{N, \frac{\beta_0}{2} + \frac{j\beta_0}{2k}, s+2j}$, $\forall \tau \in \mathcal{F}(t)$. Then there holds the estimate:

$$\left| \int_{\mathcal{F}(t)} T_{s+2j-2}^{t-\tau} e^{N,R}_{s+2j-2,s+2j} g_{N,s+2j}(\tau, Z_{s+2j-2}) \, d\tau \right|_{N, \frac{\beta_0}{2} + \frac{(j-1)\beta_0}{2k}, s+2j-2} \leq C_{d,\beta_0} (s + 2k) \int_{\mathcal{F}(t)} |g_{N,s+2j}(\tau)|_{N, \frac{\beta_0}{2} + \frac{j\beta_0}{2k}, s+2j} \, d\tau,$$

(ii) Assume $g_{s+2j}(\tau, \cdot) \in X_{\infty, \frac{\beta_0}{2} + \frac{j\beta_0}{2k}, s+2j}$, $\forall \tau \in \mathcal{F}(t)$. Then there holds the estimate:

$$\left| \int_{\mathcal{F}(t)} S_{s+2j-2}^{t-\tau} e^{\infty,R}_{s+2j-2,s+2j} g_{s+2j}(\tau, Z_{s+2j-2}) \, d\tau \right|_{\infty, \frac{\beta_0}{2} + \frac{(j-1)\beta_0}{2k}, s+2j-2} \leq C_{d,\beta_0} (s + 2k) \int_{\mathcal{F}(t)} |g_{s+2j}(\tau)|_{\infty, \frac{\beta_0}{2} + \frac{j\beta_0}{2k}, s+2j} \, d\tau.$$

Proof. We prove it first for the BBGKY hierarchy. Consider $t > 0$, $k \in \mathbb{N}$, $1 \leq j \leq k$ and $Z_{s+2j-2} \in \mathbb{R}^{2d(s+2j-2)}$. Recalling notation from (6.9), we write

$$Z_{s+2j-2}^{t-\tau} = (X_{s+2j-2}^{t-\tau}, V_{s+2j-2}^{t-\tau}) = \Psi_{s+2j-2}^{t-\tau} Z_{s+2j-2}.$$

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Conservation of energy (4.48) yields

\[ E_{s+2j-2}(Z_{s+2j-2}) = E_{s+2j-2}(Z_{s+2j-2}), \quad \forall \tau \in \mathcal{F}(t). \] (8.29)

Using Lemma 6.3 with \( \beta = \frac{\beta_0}{2} + \frac{j\beta_0}{2k}, \) \( m = s + 2j - 2, \) we obtain

\[
\left| T_{s+2j-2}^{t-\tau} \mathcal{C}^{N,R}_{s+2j-2,s+2j}(\tau, Z_{s+2j-2}) \right| \\
\leq C_d \left( \frac{\beta_0}{2} + \frac{j\beta_0}{2k} \right)^{-d} \left( s + 2j - 2 \right) \left( \frac{\beta_0}{2} + \frac{j\beta_0}{2k} \right)^{-\frac{1}{2} s + 2j^{-2}} \sum_{i=1}^{s+2j^{-2}} |v_i^{\tau-\tau}| \\
\times e^{-\left( \frac{\beta_0}{2} + \frac{j\beta_0}{2k} \right) E_{s+2j-2}(Z_{s+2j-2})} \left| g_{N,s+2j} \mathbb{1}_{[E_{s+2j} \leq R^2]}(\tau) \right|_{N, \frac{\beta_0}{2} + \frac{j\beta_0}{2k}, s+2j} \\
\leq C_d \left( \frac{\beta_0}{2} \right)^{-d} \left( s + 2j - 2 \right) \left( \frac{\beta_0}{2} \right)^{-\frac{1}{2} s + 2j^{-2}} \sum_{i=1}^{s+2j^{-2}} |v_i^{\tau-\tau}| \\
\times e^{-\left( \frac{\beta_0}{2} + \frac{j\beta_0}{2k} \right) E_{s+2j-2}(Z_{s+2j-2})} \left| g_{N,s+2j} \mathbb{1}_{[E_{s+2j} \leq R^2]}(\tau) \right|_{N, \frac{\beta_0}{2} + \frac{j\beta_0}{2k}, s+2j} \\
= C_d e^{-\frac{\beta_0}{2} E_{s+2j-2}(Z_{s+2j-2})} \left( \frac{\beta_0}{2} \right)^{-d} \left( s + 2k \right) \left( \frac{\beta_0}{2} \right)^{-\frac{1}{2} s + 2j^{-2}} \sum_{i=1}^{s+2j^{-2}} |v_i^{\tau-\tau}| \\
\times e^{-\frac{\beta_0}{2} E_{s+2j-2}(Z_{s+2j-2})} \left| g_{N,s+2j} \mathbb{1}_{[E_{s+2j} \leq R^2]}(\tau) \right|_{N, \frac{\beta_0}{2} + \frac{j\beta_0}{2k}, s+2j}, \] (8.31)

where to obtain (8.31) we use (8.29), and to obtain (8.32) we use the fact that \( j \leq k. \)
But Cauchy-Schwartz inequality implies
\[
e^{-\frac{\beta_0}{2k} E_{s+2j-2}(Z_{s+2j-2})} \sum_{i=1}^{s+2j-2} |v_i^l - \tau| = e^{-\frac{\beta_0}{2k} E_{s+2j-2}(Z_{s+2j-2})} \left( \frac{4k}{\beta_0} \right)^{1/2} \left( \frac{\beta_0}{4k} \right)^{1/2} \sum_{i=1}^{s+2j-2} |v_i^l - \tau| \\
\leq e^{-\frac{\beta_0}{2k} E_{s+2j-2}(Z_{s+2j-2})} \left( \frac{4k(s + 2j - 2)}{\beta_0} \right)^{1/2} \left( \frac{\beta_0}{4k} \right)^{1/2} \left( \sum_{i=1}^{s+2j-2} |v_i^l - \tau|^2 \right)^{1/2} \\
= \left( \frac{4k(s + 2j - 2)}{\beta_0} \right)^{1/2} \left( \frac{\beta_0}{2k} E_{s+2j-2}(Z_{s+2j-2}) \right)^{1/2} e^{-\frac{\beta_0}{2k} E_{s+2j-2}(Z_{s+2j-2})} \\
= \left( \frac{4k(s + 2j - 2)}{\beta_0} \right)^{1/2} \left( \frac{\beta_0}{2k} E_{s+2j-2}(Z_{s+2j-2}) \right)^{1/2} e^{-\frac{\beta_0}{2k} E_{s+2j-2}(Z_{s+2j-2})} \quad (8.33)
\]
\[
\leq 2\beta_0^{-1/2}(s + 2k) \sup_{x \geq 0} |\sqrt{x}e^{-x^2}| \\
\leq C\beta_0(s + 2k), \quad (8.34)
\]

where to obtain (8.33) we use (8.29), and to obtain (8.34) we use the elementary bound:
\[
\sup_{x \geq 0} |\sqrt{x}e^{-x^2}| \leq C < \infty.
\]

Therefore, (8.32), (8.34) yield
\[
e^{\left( \frac{\beta_0}{2} + \frac{(j-1)\beta_0}{2k} \right) E_{s+2j-2}(Z_{s+2j-2})} \int_{j(t)} T_{s+2j-2}^{l-\tau} e^{N,R}_{s+2j-2,s+2j} g_{N,s+2j}(\tau, Z_{s+2j-2}) d\tau \\
\leq \int_{j(t)} e^{\left( \frac{\beta_0}{2} + \frac{(j-1)\beta_0}{2k} \right) E_{s+2j-2}(Z_{s+2j-2})} T_{s+2j-2}^{l-\tau} e^{N,R}_{s+2j-2,s+2j} g_{N,s+2j}(\tau, Z_{s+2j-2}) d\tau \\
\leq C_{d,\beta_0}(s + 2k) \int_{j(t)} |g_{N,s+2j}(\tau)|_{N, \frac{\beta_0}{2} + \frac{(j-1)\beta_0}{2k}, s+2j} d\tau.
\]

Hence,
\[
\int_{j(t)} T_{s+2j-2}^{l-\tau} e^{N,R}_{s+2j-2,s+2j} g_{N,s+2j}(\tau, Z_{s+2j-2}) d\tau \bigg|_{N, \frac{\beta_0}{2} + \frac{(j-1)\beta_0}{2k}, s+2j} \leq C_{d,\beta_0}(s + 2k) \int_{j(t)} |g_{N,s+2j}(\tau)|_{N, \frac{\beta_0}{2} + \frac{(j-1)\beta_0}{2k}, s+2j} d\tau.
\]

For the Boltzmann hierarchy, the proof is identical using Lemma 6.7 instead. ☐
Recalling the observables $I_{s,k,R}^N$, $I_{s,k,R}^\infty$ defined in (8.16)-(8.17), and using iteratively Lemma 8.6, we obtain the following estimates:

**Lemma 8.7.** For any $s,n \in \mathbb{N}$, $R > 0$, $\delta > 0$ and $t \in [0, T]$, the following estimates hold:

$$
\sum_{k=0}^{n} \| I_{s,k,R,\delta}^N(t) - I_{s,k,R}^N(t) \|_{L^\infty_{X_s}} \leq \delta \| \phi_s \|_{L^\infty_{Y_s}} C^N_{d,s,\beta_0,\mu_0,T} \| F_{N,0} \|_{N,\beta_0,\mu_0},
$$

and

$$
\sum_{k=0}^{n} \| I_{s,k,R,\delta}^\infty(t) - I_{s,k,R}^\infty(t) \|_{L^\infty_{X_s}} \leq \delta \| \phi_s \|_{L^\infty_{Y_s}} C^\infty_{d,s,\beta_0,\mu_0,T} \| F_0 \|_{\infty,\beta_0,\mu_0}.
$$

**Proof.** We first prove it for the BBGKY hierarchy case. For $k = 0$, the corresponding difference trivially vanishes, so we may assume $1 \leq k \leq n$. Recalling (8.1), (8.22), notice that

$$
T_k(t) \setminus T_k,\delta(t) = \bigcup_{i=0}^{k-1} F_i(t),
$$

where

$$
F_i(t) = \{(t_1, ..., t_k) \in T_k(t) : t_i - \delta < t_{i+1} \leq t_i\}, \quad t_0 = t, \quad t_{k+1} = 0.
$$
We obtain

\[
|\mathcal{F}_i(t)| \leq \int_0^t \cdots \int_0^{t_{i-1}} \int_0^{t_i} \int_0^{\hat{t}_{i+1}} \cdots \int_0^{\hat{t}_{k-1}} \ dt_{k-1} \cdots dt_1
\]

\[
\leq \int_0^t \cdots \int_0^{t_{i-1}} \int_0^{t_i} \frac{\hat{t}_{i+1}^{k-i-1}}{(k-i-1)!} dt_{i+1} \cdots dt_1
\]

\[
= \int_0^t \cdots \int_0^{t_{i-1}} \frac{1}{(k-i)!} \left( \hat{t}_i^{k-i} - (t_i - \delta)^{k-i} \right) dt_i \cdots dt_1
\]

\[
\leq \int_0^t \cdots \int_0^{t_{i-1}} \frac{\delta(k-i)\hat{t}_i^{k-i-1}}{(k-i)!} dt_i \cdots dt_1
\]

\[
= \delta \int_0^t \cdots \int_0^{t_{i-1}} \frac{\hat{t}_i^{k-i-1}}{(k-i-1)!} dt_i \cdots dt_1
\]

\[
= \frac{\delta t^{k-1}}{(k-1)!} \leq \frac{\delta T^{k-1}}{(k-1)!}.
\]  

We also have

\[
|I_{s,k,R,\delta}^N(t)(X_s) - I_{s,k,R}^N(t)(X_s)| \leq
\]

\[
\leq \|\phi_s\|_{L_{\infty}^\infty} \left| f_{N,R,\delta}^{(s,k)}(t) - f_{N,R}^{(s,k)}(t) \right|_{N, \frac{\beta_0}{2}, s} \int_{dV_s} e^{-\frac{\beta_0}{2} E_s(Z_s)} dV_s
\]

\[
\leq C_{s,\beta_0} \|\phi_s\|_{L_{\infty}^\infty} \left| f_{N,R,\delta}^{(s,k)}(t) - f_{N,R}^{(s,k)}(t) \right|_{N, \frac{\beta_0}{2}, s}.
\]  

But by (8.35)-(8.37) and an inductive application of the first estimate of Lemma 8.6
for $j = 1, \ldots, k$, we obtain
\[
|f_{N,R,\delta}^{(s,k)}(t) - f_{N,R}^{(s,k)}(t)|_{N,\beta_0,\mu_0} \leq \sum_{i=0}^{k-1} \int_{J_i(t)} T_{s+2k}^{d} f_{N,0}^{(s+2k)}(Z_s) \left| \frac{k! \delta T^{k-1}}{(k-1)!} \right| dt_k \ldots dt_1
\]
\[
\leq C_{d,\beta_0}^k (s + 2k) \left| f_{N,0}^{(s+2k)} \right|_{N,\beta_0,\mu_0} \frac{k^2 \delta T^{k-1}}{k!}
\]
\[
\leq \delta C_{d,\beta_0,\mu_0}^k \left| f_{N,0} \right|_{N,\beta_0,\mu_0}
\]
\[
\leq \delta C_{d,\beta_0,\mu_0,0}^k \left| f_{N,0} \right|_{N,\beta_0,\mu_0}
\]
(8.39)

where to obtain (8.39) we use Remark 6.1, to obtain (8.40) we use (8.37), and to obtain (8.41) we use the elementary inequality:
\[
\frac{(s + 2k)^k}{k!} \leq 2^k \frac{(s + k)^k}{k!} \leq 2^k \sum_{\ell=0}^{\infty} \frac{(s + k)^\ell}{\ell!} = 2^k e^{s+k} \leq C_s^k.
\]
(8.42)

Using (8.38), (8.41), and adding for $k = 1, \ldots, n$, we obtain
\[
\sum_{k=0}^{n} \left| f_{s,k,R,\delta}^{N}(t) - f_{s,k,R}^{N}(t) \right|_{s,k,\beta_0,\mu_0} \leq \delta \left| \phi_s \right|_{s,k,\beta_0,\mu_0} C_{d,s,\beta_0,\mu_0,0}^n \left| f_{N,0} \right|_{N,\beta_0,\mu_0}.
\]
The proof for the Boltzmann hierarchy is similar using the second estimate of Lemma 8.6.

Now we put together the results obtained throughout this chapter by approximating:
• $I_s^N$ with $\sum_{k=0}^N I_{s,k}^N$, by estimate (i) of Lemma 8.2.

• $\sum_{k=0}^N I_{s,k}^N$ with $\sum_{k=0}^N I_{s,k,R}^N$, by estimate (i) of Lemma 8.4.

• $\sum_{k=0}^N I_{s,k,R}^N$ with $\sum_{k=0}^N I_{s,k,R,\delta}^N$, by estimate (i) Lemma 8.7.

for the BBGKY hierarchy, and similarly, using estimate (ii), for the Boltzmann hierarchy, to obtain the following Proposition:

**Proposition 8.8.** For any $s, n \in \mathbb{N}$, $R > 1$, $\delta > 0$ and $t \in [0,T]$, the following estimates hold:

\[
\|I_s^N(t) - \sum_{k=0}^n I_{s,k,R,\delta}^N(t)\|_{L^\infty_X} \leq C_{s,\beta_0,\mu_0,T} \|\phi_s\|_{L^\infty_{V_s}} \left(2^{-n} + e^{-\frac{\beta_0}{3} R^2} + \delta C_{d,s,\beta_0,\mu_0,T}^n\right) \times \|F_{N,0}\|_{N,\beta_0,\mu_0},
\]

and

\[
\|I_s^\infty(t) - \sum_{k=0}^n I_{s,k,R,\delta}^\infty(t)\|_{L^\infty_X} \leq C_{s,\beta_0,\mu_0,T} \|\phi_s\|_{L^\infty_{V_s}} \left(2^{-n} + e^{-\frac{\beta_0}{3} R^2} + \delta C_{d,s,\beta_0,\mu_0,T}^n\right) \times \|F_0\|_{\infty,\beta_0,\mu_0}.
\]

Proposition 8.8 and triangle inequality imply that the convergence proof reduces to controlling the differences

\[
I_{s,k,R,\delta}^N(t) - I_{s,k,R,\delta}^\infty(t),
\]

for given $0 \leq k \leq n$, $R > 1$, $\delta > 0$, where the observables $I_{s,k,R,\delta}$, $I_{s,k,R,\delta}^\infty$ are given by (8.27)-(8.28). However obtaining such a control requires some delicate analysis because of possible recollisions of the interaction backwards flow i.e. the backwards flow does not coincide in general with the backwards free flow.
Chapter 9

Geometric estimates

In this chapter we provide the crucial geometric estimates required to eliminate recollisions of the backwards interaction flow. This elimination will enable us to compare the difference of the BBGKY hierarchy truncated observable, given in (8.27) and the Boltzmann hierarchy truncated observable, given in (8.28), in the scaled limit.

Let us introduce some notation which we will be using from now on. Given \( w \in \mathbb{R}^d \), \( y \in \mathbb{R}^d \setminus \{0\} \) and \( \rho > 0 \), we write \( K_\rho^d(w, y) \) for the closed \( d \)-dimensional cylinder of center \( w \), direction \( y \) and radius \( \rho \). More precisely,

\[
K_\rho^d(w, y) = \{ x \in \mathbb{R}^d : \text{dist}(x, L_{w, y}) \leq \rho \},
\]

where

\[
L_{w, y} = \{ w + \lambda y : \lambda \in \mathbb{R} \}.
\]

In case we do not need to specify the center and direction we will just be writing \( K_\rho^d \) for convenience. Moreover, given a cylinder \( K_\rho^d \) we will write \( \tilde{K}_\rho^d \) to denote a cylinder relative to \( K_\rho^d \) with proportional radius, meaning \( \tilde{K}_\rho^d \) is of the form \( K_{c\rho}^d \) where \( c \) does not depend on \( \rho \).
9.1 Spherical estimates

In this section, we derive the spherical estimates which will enable us to control pre-collisional configurations. We will strongly rely on the following estimate, see [24]:

**Lemma 9.2.** Given $\rho, r > 0$ the following estimate holds for the $d$-spherical measure of radius $r > 0$:

$$|\mathbb{S}_{r}^{d-1} \cap K_{\rho}^{d}|_{\mathbb{S}_{r}^{d-1}} \lesssim r^{d-1} \min \left\{ 1, \left( \frac{\rho}{r} \right)^{\frac{d-1}{2}} \right\} .$$

**Proof.** After re-scaling we may clearly assume that $r = 1$. We refer to the work of R. Denlinger [24], p.30, for the proof. \qed

Iterating this estimate we obtain the following spherical estimates, which will be crucially used in Chapter 10:

**Proposition 9.3.** Given $0 < \rho \leq 1 \leq R$, the following estimates hold:

(i) $|B_{R}^{d} \cap K_{\rho}^{d}|_{d} \lesssim R^{d} \rho^{\frac{d-1}{2}} .$

(ii) $|B_{R}^{2d} \cap (K_{\rho}^{d} \times \mathbb{R}^{d})|_{2d} \lesssim R^{2d} \rho^{\frac{d-1}{2}} .$

(iii) $|B_{R}^{2d} \cap (B_{\rho}^{d} \times \mathbb{R}^{d})|_{2d} \lesssim R^{d} \rho^{d} .$

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Proof. Let us first prove (i). Lemma 9.2 implies

\[ |B_R^d \cap K^d_{\rho}|_d \simeq \int_0^R |S_{\rho r}^{d-1} \cap K^d_{\rho}|_{S_{\rho r}^{d-1}} \, dr \]

\[ \lesssim \int_0^R r^{d-1} \min \left\{ 1, \left( \frac{\rho}{r} \right)^{\frac{d-1}{2}} \right\} \, dr \]

\[ \leq \int_0^{\rho} r^{d-1} \, dr + \rho^{d-1} \int_0^R r^{\frac{d-1}{2}} \, dr \]

\[ \simeq \rho^d + \rho^{\frac{d-1}{2}} R^{\frac{d+1}{2}}, \text{ since } d \geq 2 \]

\[ \leq R^d \rho^{\frac{d-1}{2}}, \text{ since } 0 < \rho \leq 1 \leq R, \]

and (i) is proved.

Part (ii) comes immediately since \( B_R^{2d} \subseteq B_R^d \times B_R^d \), thus

\[ |B_R^{2d} \cap (K^d_{\rho} \times \mathbb{R}^d)|_{2d} \leq |(B_R^d \cap K^d_{\rho}) \times B_R^d|_d \simeq R^d |B_R^d \cap K^d_{\rho}|_d \lesssim R^d \rho^{\frac{d-1}{2}}. \] (9.2)

Estimate (iii) is straightforward since

\[ |B_R^{2d} \cap (B_{\rho}^d \times \mathbb{R}^d)|_{2d} \leq |B_{\rho}^d \times B_R^d|_{2d} = R^d \rho^d. \]

\[ \square \]

Iterating Lemma 9.2, we obtain new geometric estimates which will be essential to derive the ellipsoidal estimates, which will enable us to control post-collisional configurations. In particular, we estimate the intersection of \( S_1^{2d-1} \) with appropriate 2d-solids. To achieve those estimates we strongly rely on the following symmetric representations of \( S_1^{2d-1} \):

\[ S_1^{2d-1} = \left\{ (\omega_1, \omega_2) \in \mathbb{R}^d \times B_1^d : \omega_1 \in \mathbb{S}_1^{d-1} \right\}, \] (9.3)
and
\[ S_1^{2d-1} = \left\{ (\omega_1, \omega_2) \in B_1^d \times \mathbb{R}^d : \omega_2 \in S^{d-1}_{\sqrt{1-|\omega_1|^2}} \right\}, \quad (9.4) \]

**Lemma 9.4.** For any \( r, \rho > 0 \) the following estimates hold for the \((2d-1)\)-spherical measure of radius \( r > 0 \):

\[ \left| S_r^{2d-1} \cap (K_{\rho}^d \times \mathbb{R}^d) \right|_{S_{r}^{2d-1}} \lesssim r^{2d-1} \min \left\{ 1, \left( \frac{\rho}{r} \right)^{\frac{d-1}{2}} \right\}, \]

and
\[ \left| S_r^{2d-1} \cap (\mathbb{R}^d \times K_{\rho}^d) \right|_{S_{r}^{2d-1}} \lesssim r^{2d-1} \min \left\{ 1, \left( \frac{\rho}{r} \right)^{\frac{d-1}{2}} \right\}, \]

**Proof.** By symmetry it suffices to prove the first estimate. Also, after rescaling we may assume \( r = 1 \). The idea is to iterate Lemma 9.2 using the representation (9.3). Using (9.3), we have

\[ \left| S_1^{2d-1} \cap (K_{\rho}^d \times \mathbb{R}^d) \right|_{S_{1}^{2d-1}} = \int_{B_1^d} \left| \frac{S^{d-1}}{\sqrt{1-|\omega_2|^2}} \cap K_{\rho}^d \right|_{S^{d-1}_{\sqrt{1-|\omega_2|^2}}} d\omega_2 \]

\[ \lesssim \int_{B_1^d} (1-|\omega_2|^2)^{\frac{d-1}{2}} \min \left\{ 1, \left( \frac{\rho}{\sqrt{1-|\omega_2|^2}} \right)^{\frac{d-1}{2}} \right\} d\omega_2 \]

\[ \lesssim \int_0^1 s^{d-1}(1-s^2)^{\frac{d-1}{2}} \min \left\{ 1, \left( \frac{\rho}{\sqrt{1-s^2}} \right)^{\frac{d-1}{2}} \right\} ds, \quad \text{by Lemma 9.2} \]

Let us write

\[ I(\rho) := \int_0^1 s^{d-1}(1-s^2)^{\frac{d-1}{2}} \min \left\{ 1, \left( \frac{\rho}{\sqrt{1-s^2}} \right)^{\frac{d-1}{2}} \right\} ds, \quad (9.6) \]
In the case where $\rho \geq 1$, we have

$$I(\rho) \lesssim \int_0^1 s^{d-1}(1 - s^2)^{\frac{d+1}{2}} ds \simeq 1. \quad (9.7)$$

Assume now $0 < \rho < 1$. Then, we may decompose $I(\rho)$ as follows:

$$I(\rho) = \int_0^{\sqrt{1-\rho^2}} s^{d-1}(1 - s^2)^{\frac{d-1}{2}} \left( \frac{\rho}{\sqrt{1-s^2}} \right)^{\frac{d}{2}} ds + \int_{\sqrt{1-\rho^2}}^1 s^{d-1}(1 - s^2)^{\frac{d-1}{2}} ds. \quad (9.8)$$

Performing the change of variables $u = 1 - s^2$, equation (9.8) can be written as:

$$I(\rho) = \frac{1}{2} \rho^{\frac{d}{2} - 1} \int_{\rho^2}^1 (1 - u)^{\frac{d-1}{2}} u^{-\frac{1}{4}} du + \frac{1}{2} \int_0^{\rho^2} (1 - u)^{\frac{d-1}{2}} u^{-\frac{1}{4}} du$$

$$\lesssim \rho^{\frac{d}{2} - 1} \int_{\rho^2}^1 u^{-\frac{1}{4}} du + \int_0^{\rho^2} u^{-\frac{1}{4}} du$$

$$\simeq \rho^{\frac{d}{2} - 1} \left( 1 - \rho^{\frac{d+1}{2}} \right) + \rho^{d+1}$$

$$\lesssim \rho^{\frac{d}{2} - 1}, \quad \text{since } \rho < 1. \quad (9.9)$$

Thus, (9.7) and (9.9) yield

$$I(\rho) \lesssim \min\{1, \rho^{\frac{d+1}{2}}\}, \quad (9.10)$$

and by (9.5) we obtain the result. \hfill \square

In the same spirit as in Lemma 9.4, we obtain the following estimate for the intersection of $S^{2d-1}$ with the strip:

$$W_{\rho, \mu, \lambda}^{2d} := \{ (\omega_1, \omega_2) \in \mathbb{R}^{2d} : |\mu \omega_1 - \lambda \omega_2| \leq \rho \}, \quad (9.11)$$

where $\mu, \lambda \neq 0$. 

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Lemma 9.5. For any \( r, \rho > 0 \) the following estimate holds for the \((2d-1)\)-spherical measure of radius \( r > 0 \):

\[
|S^{2d-1} \cap W_{\rho,\mu,\lambda}^{2d}|_{S^{2d-1}} \lesssim r^{2d-1} \min \left\{ 1, \left( \frac{\rho}{|\mu|r} \right)^{\frac{d-1}{2}}, \left( \frac{\rho}{|\lambda|r} \right)^{\frac{d-1}{2}} \right\}.
\]

Proof. Clearly after rescaling we may assume \( r = 1 \). Notice that

\[
W_{\rho,\mu,\lambda}^{2d} = \{ (\omega_1, \omega_2) \in \mathbb{R}^{2d} : \omega_1 \in B_{\rho/|\mu|}^d (\lambda \mu^{-1} \omega_2) \} \quad (9.12)
\]

\[
\subseteq \{ (\omega_1, \omega_2) \in \mathbb{R}^{2d} : \omega_1 \in K_{\rho/|\mu|}^d (\lambda \mu^{-1} \omega_2) \}, \quad (9.13)
\]

where given \( \omega_2 \in \mathbb{R}^d \), \( K_{\rho/|\mu|}(\lambda \mu^{-1} \omega_1) \) is any cylinder of radius \( \rho/|\mu| \) centered at \( \lambda \mu^{-1} \omega_1 \). Let us write \( \rho_\mu = \rho/|\mu| \). Then, we have

\[
|S^{2d-1} \cap W_{\rho,\mu,\lambda}^{2d}|_{S^{2d-1}} = \int_{B^d_1} \left| S^{d-1}_{\sqrt{1-|\omega_2|^2}} \cap B_{\rho_\mu}^d (\lambda \mu^{-1} \omega_1) \right|_{S^{d-1}_{\sqrt{1-|\omega_2|^2}}} \, d\omega_2 \quad (9.14)
\]

\[
\leq \int_{B^d_1} \left| S^{d-1}_{\sqrt{1-|\omega_2|^2}} \cap K_{\rho_\mu}^d (\lambda \mu^{-1} \omega_1) \right|_{S^{d-1}_{\sqrt{1-|\omega_2|^2}}} \, d\omega_2 \quad (9.15)
\]

\[
\lesssim \int_{B^d_1} (1 - |\omega_2|^2)^{\frac{d-1}{2}} \min \left\{ 1, \left( \frac{\rho_\mu}{\sqrt{1 - |\omega_2|^2}} \right)^{\frac{d-1}{2}} \right\} \, d\omega_2 \quad (9.16)
\]

\[
\approx \int_0^1 s^{d-1} (1 - s^2)^{\frac{d-1}{2}} \min \left\{ 1, \left( \frac{\rho_\mu}{\sqrt{1 - s^2}} \right)^{\frac{d-1}{2}} \right\} \, ds = I(\rho_\mu) \quad (9.17)
\]

\[
\leq \min \left\{ 1, \rho_\mu^{\frac{d-1}{2}} \right\} \quad (9.18)
\]

\[
= \min \left\{ 1, \left( \frac{\rho}{|\mu|} \right)^{\frac{d-1}{2}} \right\}. \quad (9.19)
\]
where to obtain (9.14) we use representation (9.3) and (9.12), to obtain (9.15) we use (9.13), to obtain (9.16) we use Lemma 9.2 to obtain (9.17) we recall notation from (9.6), and to obtain (9.18) we use estimate (9.10).

Exchanging the roles of $\omega_1, \omega_2$ and $\mu, \lambda$, an entirely symmetric argument shows that

$$|S_1^{2d-1} \cap W_{\rho,\mu,\lambda}|_{S_2^{2d-1}} \lesssim \min \left\{ 1, \left( \frac{\rho}{|\lambda|} \right)^{\frac{d-1}{2}} \right\}.$$  (9.20)

Combining (9.19)-(9.20), we conclude the proof. □

9.6 The transition map

In this section, we construct a transition map which will allow us to control post-collisional configurations using some appropriate ellipsoidal estimates developed in Section 9.8. Since, up to our knowledge, ternary interactions have not been rigorously addressed in the past, it is the first time that such a transition map appears in literature.

We first state the following elementary result from Linear Algebra which will be useful for the calculation of Jacobians:

We are now ready to introduce the transition map. We will rely on Lemma A.2 (see Appendix). Before that, let us introduce some notation. Recall from (3.5) the cross-section

$$b(\omega_1, \omega_2, \nu_1, \nu_2) = \langle \omega_1, \nu_1 \rangle + \langle \omega_2, \nu_2 \rangle, \quad (\omega_1, \omega_2) \in \mathbb{R}^{2d}, \quad (\nu_1, \nu_2) \in \mathbb{R}^{2d}.$$
Given \( v_1, v_2, v_3 \in \mathbb{R}^d \), we define the domain

\[
\Omega = \{ \omega = (\omega_1, \omega_2) \in \mathbb{R}^d : |\omega_1|^2 + |\omega_2|^2 < \frac{3}{2} \text{ and } b(\omega_1, \omega_2, v_2 - v_1, v_3 - v_1) > 0 \},
\]
and the set

\[
S_{v_1, v_2, v_3}^+ := \{ (\omega_1, \omega_2) \in S_1^{2d-1} : b(\omega_1, \omega_2, v_2 - v_1, v_3 - v_1) > 0 \} \subseteq \Omega. \tag{9.21}
\]

We also define the smooth map \( \Psi : \mathbb{R}^{2d} \to \mathbb{R} \) by:

\[
\Psi(\nu_1, \nu_2) = |\nu_1|^2 + |\nu_2|^2 + |\nu_1 - \nu_2|^2, \quad (\nu_1, \nu_2) \in \mathbb{R}^{2d}, \tag{9.22}
\]
and the \((2d - 1)\)-ellipsoid

\[
E_{1}^{2d-1} := [\Psi = 1] = \{ (\nu_1, \nu_2) \in \mathbb{R}^{2d} : |\nu_1|^2 + |\nu_2|^2 + |\nu_1 - \nu_2|^2 = 1 \}. \tag{9.23}
\]

**Proposition 9.7.** Consider \( v_1, v_2, v_3 \in \mathbb{R}^d \) and \( r > 0 \) such that

\[
|v_1 - v_2|^2 + |v_1 - v_3|^2 + |v_2 - v_3|^2 = r^2. \tag{9.24}
\]

We define the transition map \( \mathcal{J}_{v_1, v_2, v_3} : \Omega \to \mathbb{R}^{2d} \setminus \left\{ \frac{1}{r} \left( v_1 - v_2 \right) \right\} \) by

\[
\nu = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \mathcal{J}_{v_1, v_2, v_3}(\omega) := \frac{1}{r} \begin{pmatrix} v_1^* - v_2^* \\ v_1^* - v_3^* \end{pmatrix}, \quad \omega = (\omega_1, \omega_2) \in \Omega. \tag{9.25}
\]

(i) \( \mathcal{J}_{v_1, v_2, v_3} \) is well-defined and smooth in \( \Omega \) with bounded derivative uniformly in \( r \) i.e.

\[
\|D\mathcal{J}_{v_1, v_2, v_3}(\omega)\|_{\infty} \leq C_d, \quad \forall \omega \in \Omega, \tag{9.26}
\]

\[\text{i.e.}^1\] by a small abuse of notation we extend the collisional operator \( T_{\omega_1, \omega_2} \) for \( (\omega_1, \omega_2) \in \Omega \), see Section 3.
(ii) The Jacobian of $J_{v_1, v_2, v_3}$ is given by:

$$\text{Jac}(J_{v_1, v_2, v_3})(\omega) \simeq r^{-2d} \frac{b_2 d(\omega_1, \omega_2, v_2 - v_1, v_3 - v_1)}{(1 + (\omega_1, \omega_2))^{2d+1}} > 0, \quad \forall \omega = (\omega_1, \omega_2) \in \Omega.$$  \hspace{1cm} (9.27)

Moreover, for any $\omega = (\omega_1, \omega_2) \in \Omega$, there holds the estimate:

$$\text{Jac}(J_{v_1, v_2, v_3})(\omega) \approx r^{-2d} b_2 d(\omega_1, \omega_2, v_2 - v_1, v_3 - v_1).$$  \hspace{1cm} (9.28)

(iii) The map $J_{v_1, v_2, v_3} : S_{v_1, v_2, v_3}^+ \to \mathbb{E}_{1}^{2d-1} \setminus \left\{ r^{-1} \left( v_1 - v_2 \atop v_1 - v_3 \right) \right\}$ is bijective. Moreover, there holds

$$S_{v_1, v_2, v_3}^+ = [\Psi \circ J_{v_1, v_2, v_3} = 1].$$  \hspace{1cm} (9.29)

(iv) For any measurable $g : \mathbb{R}^{2d} \to [0, +\infty]$, there holds the estimate

$$\int_{S_{v_1, v_2, v_3}^+} |g \circ J_{v_1, v_2, v_3}(\omega)| \text{Jac} J_{v_1, v_2, v_3}(\omega) d\omega \lesssim \int_{\mathbb{E}_{1}^{2d-1}} g(\nu) d\nu.$$  \hspace{1cm} (9.30)

Proof. For convenience, let us use the following notation:

$$\nu = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}, \quad v = \begin{pmatrix} v_1 - v_2 \\ v_1 - v_3 \end{pmatrix}, \quad \omega = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, \quad \pi(\omega) = \langle \omega_1, \omega_2 \rangle.$$  \hspace{1cm} (9.31)

Under this notation, and recalling (3.1) we also have\footnote{by a small abuse of notation we write $\langle \cdot, \cdot \rangle$ for the inner product in $\mathbb{R}^{2d}$ as well.}

$$c = c_{\omega_1, \omega_2, v_1, v_2, v_3} = - \frac{\langle \omega, v \rangle}{1 + \pi(\omega)}.$$  \hspace{1cm} (9.32)

We prove each claim separately:

(i): By (9.25) and (3.4), we have

$$J_{v_1, v_2, v_3}(\omega) = r^{-1} (v + cA\omega).$$  \hspace{1cm} (9.33)
where
\[ A = \begin{pmatrix} 2I_d & I_d \\ I_d & 2I_d \end{pmatrix}, \] (9.34)
and \( I_d \) is the \( d \times d \) identity matrix. Let us now calculate the derivative of \( J_{v_1,v_2,v_3} \).

Using (9.33), we obtain
\[ DJ_{v_1,v_2,v_3}(\omega) = r^{-1} A (cI_{2d} + \omega \nabla^T c), \] (9.35)

Using notation from (9.31)-(9.32), we obtain
\[ \nabla c = -\frac{v}{1 + \pi(\omega)} + \frac{\langle \omega, v \rangle \bar{\omega}}{(1 + \pi(\omega))^2}, \] (9.36)

where \( \bar{\omega} = \nabla \pi(\omega) = \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} \). Combining (9.35)-(9.36), we obtain
\[ DJ_{v_1,v_2,v_3}(\omega) = r^{-1} \left( -\frac{\langle \omega, v \rangle A}{1 + \pi(\omega)} - A \omega v^T + \frac{\langle \omega, v \rangle A \omega \bar{\omega}^T}{(1 + \pi(\omega))^2} \right). \] (9.37)

Recall we have assumed \( \omega \in \Omega \Rightarrow |\omega_1|^2 + |\omega_2|^2 < \frac{3}{2}, \) so Cauchy-Schwartz inequality implies
\[ \frac{1}{4} < 1 + \pi(\omega) < \frac{7}{4}. \] (9.38)

therefore \( J_{v_1,v_2,v_3} \) is differentiable in \( \Omega \). It is clear from (9.37)-(9.38) that \( J_{v_1,v_2,v_3} \) is in fact smooth.

For \( \omega \in \Omega \), Cauchy-Schwartz inequality and (9.37)-(9.38) imply that
\[ \|DJ_{v_1,v_2,v_3}(\omega)\|_\infty \leq r^{-1} C_d (|v_1 - v_2| + |v_1 - v_3|) \leq C_d, \]

since \( |v_1 - v_2|^2 + |v_1 - v_3|^2 + |v_2 - v_3|^2 = r^2 \). Claim (9.26) is proved.
(ii): To calculate the Jacobian, we use (9.35) and apply Lemma A.1 for \( n = 2d, w = \omega, u = \nabla \omega c \), to obtain

\[
\text{Jac}(\mathcal{J}_{v_1,v_2,v_3})(\omega) = \det(r^{-1}A) \det(c I_{2d} + \omega \nabla^T \omega c) = (4^d - 1)r^{-2d}c^{2d} \left( 1 + c^{-1}\langle \omega, \nabla \omega c \rangle \right)
\]

(9.39)

By (9.32)-(9.36), we have

\[
\langle \omega, \nabla \omega c \rangle = \langle 1 + \pi(\omega) \rangle^{-1} \langle \omega, v \rangle + (1 + \pi(\omega))^{-2} \langle \omega, v \rangle \langle \omega, \tilde{\omega} \rangle
\]

\[
= c - c \left( (1 + \pi(\omega))^{-1} \langle \omega, \tilde{\omega} \rangle \right)
\]

\[
= c \left( 1 - \frac{2\pi(\omega)}{1 + \pi(\omega)} \right),
\]

(9.40)

since

\[
\langle \omega, \tilde{\omega} \rangle = \langle \omega_1, \omega_2 \rangle + \langle \omega_2, \omega_1 \rangle = 2\pi(\omega).
\]

Hence (9.39)-(9.40) and (3.6) imply (9.27). To obtain (9.28), we combine (9.27) and estimate (9.38). (iii): Let us first show that \( \mathcal{J}_{v_1,v_2,v_3} \) maps in \( \mathbb{E}^2 d^{-1} \). Fix \( \omega = (\omega_1, \omega_2) \in S_{v_1,v_2,v_3}^+ \). Using conservation of relative velocities (3.12) and (9.24), we get

\[
|\nu_1|^2 + |\nu_2|^2 + |\nu_1 - \nu_2|^2 = \frac{|v_1^* - v_2^*|^2 + |v_1^* - v_3^*|^2 + |v_2^* - v_3^*|^2}{r^2} = \frac{|v_1 - v_2|^2 + |v_1 - v_3|^2 + |v_2 - v_3|^2}{r^2} = 1,
\]

thus \( \mathcal{J}_{v_1,v_2,v_3} \) maps in \( \mathbb{E}^2 d^{-1} \). Let us now show that \( \mathcal{J}_{v_1,v_2,v_3} : \Omega \to \mathbb{R}^{2d} \setminus \{r^{-1}v\} \). Indeed consider \( \omega \in S_{v_1,v_2,v_3}^+ \) with

\[
\mathcal{J}_{v_1,v_2,v_3}(\omega) = r^{-1}v.
\]

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Then (9.33) implies
\[ c A \omega = 0 \implies c = 0, \]
since \( A \) (given in (9.34)) is invertible and \( \omega \neq 0 \). But (9.32) implies \( \langle \omega, v \rangle = 0 \) which is a contradiction, since \( \omega \in S_{v_1,v_2,v_3}^+ \).

**Injectivity:** Consider \( \omega, \omega' \in S_{v_1,v_2,v_3}^+ \) such that
\[ J_{v_1,v_2,v_3}(\omega) = J_{v_1,v_2,v_3}(\omega'). \]
We have
\[ J_{v_1,v_2,v_3}(\omega) = r^{-1}(v + c A \omega), \]
\[ J_{v_1,v_2,v_3}(\omega') = r^{-1}(v + c' A \omega'), \]
Since \( A \) is invertible, we have
\[ J_{v_1,v_2,v_3}(\omega) = J_{v_1,v_2,v_3}(\omega') \iff -c \omega = -c' \omega' \iff \frac{\langle \omega, v \rangle}{1 + \langle \omega_1, \omega_2 \rangle} \omega = \frac{\langle \omega', v \rangle}{1 + \langle \omega'_1, \omega'_2 \rangle} \omega'. \]
\[ (9.41) \]
Since \( \omega, \omega' \in S_{v_1,v_2,v_3}^+ \), (9.41) implies that \( \omega, \omega' \) are linearly dependent i.e. there is \( \lambda \neq 0 \) such that
\[ \omega' = \lambda \omega. \]
Since \( \omega, \omega' \in S_{v_1,v_2,v_3}^+ \), we also obtain \( \lambda = 1 \), thus \( \omega = \omega' \). Therefore, \( J_{v_1,v_2,v_3} : S_{v_1,v_2,v_3}^+ \to \mathbb{E}_1^{2d-1} \setminus \{r^{-1}v\} \) is injective.

**Surjectivity:** Consider \( \nu \in \mathbb{E}_1^{2d-1} \setminus \{r^{-1}v\} \). Let us investigate the possible solutions of the equation:
\[ \nu = J_{v_1,v_2,v_3}(\omega). \]
\[ (9.42) \]
Using notation from (9.31)-(9.34), equation (9.42) is equivalent to the equation

\[
\frac{\langle \omega, v \rangle}{1 + \langle \omega_1, \omega_2 \rangle} \omega = A^{-1}(v - r\nu).
\] (9.43)

Since \( \nu \neq r^{-1}v \) and \( A \) is invertible, (9.43) implies there is \( \lambda \neq 0 \) such that

\[
\omega = \lambda B(v - r\nu),
\] (9.44)

where we write

\[
B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} := A^{-1}.
\]

Replacing expression (9.44) into (9.43), we obtain the following equation for \( \lambda \):

\[
\lambda^2(\langle B(v - r\nu), v \rangle - \langle B_1(v - r\nu), B_2(v - r\nu) \rangle) = 1.
\] (9.45)

To prove surjectivity, we consider \( \nu \in E_2^{d-1} \setminus \{r^{-1}v\} \). Motivated by (9.45), let us define

\[
\omega := \frac{-\text{sgn}(\langle B(v - r\nu), v \rangle)}{\sqrt{\langle B(v - r\nu), v \rangle - \langle B_1(v - r\nu), B_2(v - r\nu) \rangle}} B(v - r\nu),
\]

Assumption (9.24), the fact that \( \nu \in E_1^{2d-1} \setminus \{r^{-1}v\} \), and a straightforward algebraic calculation imply that

\[
\omega \in S^{+}_{v_1,v_2,v_3}, \quad \nu = J_{v_1,v_2,v_3}(\omega).
\] (9.46)

Therefore \( J_{v_1,v_2,v_3} : S_{v_1,v_2,v_3}^{+} \rightarrow E_1^{2d-1} \setminus \{r^{-1}v\} \) is surjective.

**Proof of (9.29):** We have already seen that

\[
J_{v_1,v_2,v_3}(S_{v_1,v_2,v_3}^{+}) \subseteq E_1^{2d-1} = [\Psi = 1],
\]

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so

\[ S_{v_1,v_2,v_3}^+ \subseteq [\Psi \circ J_{v_1,v_2,v_3} = 1]. \] \hspace{1cm} (9.47)

Let us prove that \([\Psi \circ J_{v_1,v_2,v_3} = 1] \subseteq S_{v_1,v_2,v_3}^+\) as well. Consider \(\omega \in [\Psi \circ J_{v_1,v_2,v_3} = 1]\).

This means that \(\nu := J_{v_1,v_2,v_3}(\omega) \in E^{d-1}_1\). Since \(\omega \in \Omega\), we also have \(\nu \neq r^{-1}\nu\), thus the calculation made to prove surjectivity yields that

\[ \omega' := \frac{-\text{sgn}(\langle B(v - r\nu), v \rangle)}{\sqrt{\langle B(v - r\nu), v \rangle - \langle B_1(v - r\nu), B_2(v - r\nu) \rangle}}B(v - r\nu) \in S_{v_1,v_2,v_3}^+ \subseteq \Omega, \]

satisfies \(J_{v_1,v_2,v_3}(\omega') = \nu\) as well. Since \(J_{v_1,v_2,v_3}\) is injective on \(S_{v_1,v_2,v_3}^+\), we obtain \(\omega = \omega'\), thus

\[ S_{v_1,v_2,v_3}^+ \supseteq [\Psi \circ J_{v_1,v_2,v_3} = 1]. \] \hspace{1cm} (9.48)

By (9.47)-(9.48), we obtain (9.29).

(iv): Since \(J_{v_1,v_2,v_3} : S_{v_1,v_2,v_3}^+ \rightarrow E^{d-1} \setminus \{r^{-1}\nu\}\) is bijective, we have

\[ S_{v_1,v_2,v_3}^+ = [\Psi \circ J_{v_1,v_2,v_3} = 1], \]

thus using notation from (A.3) (see Appendix), we have

\[ N_{J_{v_1,v_2,v_3}}(\nu, [\Psi \circ J_{v_1,v_2,v_3} = 1]) = 1, \quad \forall \nu \in E^{d-1}_1 \setminus \{r^{-1}\nu\}. \] \hspace{1cm} (9.49)

We easily calculate \(4\Psi(\nu) \leq |\nabla \Psi(\nu)|^2 \leq 16\Psi(\nu)\), for all \(\nu \in \mathbb{R}^d\). Hence

\[ \nabla \Psi(\nu) \neq 0, \quad \forall \nu \in \left(\frac{1}{2} < \Psi < \frac{3}{2}\right). \] \hspace{1cm} (9.50)

By (9.50), (9.27) we may use Lemma A.2 for the function \(g\), and \(\gamma = 1, \delta = 1/2\).
\[ F = J_{v_1,v_2,v_3}, \Psi \text{ given by (9.22). We have} \]

\[
\int_{S^+_{v_1,v_2,v_3}} (g \circ J_{v_1,v_2,v_3})(\omega) |\text{Jac} J_{v_1,v_2,v_3}(\omega)| \frac{|\nabla \Psi(J_{v_1,v_2,v_3}(\omega))|}{|\nabla (\Psi \circ J_{v_1,v_2,v_3})(\omega)|} d\omega \]

\[= \int_{\Psi_0 J_{v_1,v_2,v_3} = 1} (g \circ J_{v_1,v_2,v_3})(\omega) |\text{Jac} J_{v_1,v_2,v_3}(\omega)| \frac{|\nabla \Psi(J_{v_1,v_2,v_3}(\omega))|}{|\nabla (\Psi \circ J_{v_1,v_2,v_3})(\omega)|} d\omega \quad (9.51) \]

\[= \int_{[\Psi = 1]} g(\nu) N_{J_{v_1,v_2,v_3}}(\nu, [\Psi \circ J_{v_1,v_2,v_3} = 1]) d\nu \quad (9.52) \]

\[= \int_{\mathbb{R}^{2d-1}} g(\nu) N_{J_{v_1,v_2,v_3}}(\nu, S^+_{v_1,v_2,v_3}) d\nu, \quad (9.53) \]

where to obtain (9.51) we use (9.29), to obtain (9.52) we use Lemma A.2 to obtain (9.53) we use (9.23) and (9.49). Moreover, by the chain rule and (9.26), we obtain

\[
|\nabla (\Psi \circ J_{v_1,v_2,v_3})(\omega)| = \frac{|D^T J_{v_1,v_2,v_3}(\omega) \nabla \Psi(J_{v_1,v_2,v_3}(\omega))|}{|\nabla \Psi(J_{v_1,v_2,v_3}(\omega))|} = C_d \|D J_{v_1,v_2,v_3}(\omega)\|_\infty \leq C_d, \]

and (9.30) follows, since \( g \geq 0. \) \[\square\]

### 9.8 Ellipsoidal estimates

We will now derive the ellipsoidal estimates which will enable us to control post-collisional configurations.

**Lemma 9.9.** Let \( v_1, v_2, v_3 \in \mathbb{R}^d \) and \( r > 0 \) satisfying

\[ |v_1 - v_2|^2 + |v_1 - v_3|^2 + |v_2 - v_3|^2 = r^2. \]

Denoting \((v_1, v_2) = J_{v_1,v_2,v_3}(\omega_1, \omega_2)\) and considering \( \rho > 0 \), the following hold:

\[
\begin{aligned}
\begin{pmatrix} v_1^* \\ v_2^* \\ v_3^* \end{pmatrix} \in K_\rho \iff \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in S_{12}^{-1} \tilde{K}_{\rho/r} \\
\begin{pmatrix} v_1^* \\ v_2^* \end{pmatrix} \in K_\rho \iff \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in S_{13}^{-1} \tilde{K}_{\rho/r},
\end{aligned}
\]

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where

\[ S_{12} = \begin{pmatrix} I_d & I_d \\ -2I_d & I_d \end{pmatrix}, \quad (9.54) \]

\[ S_{13} = \begin{pmatrix} I_d & I_d \\ I_d & -2I_d \end{pmatrix}, \quad (9.55) \]

and \( K_\rho \) is either of the form \( K_\rho^d \times \mathbb{R}^d \) or \( \mathbb{R}^d \times K_\rho^d \) while \( \bar{K}_{\rho/r} \) is either of the form \( \bar{K}_{\rho/r}^d \times \mathbb{R}^d \) or \( \mathbb{R}^d \times \bar{K}_{\rho/r}^d \) respectively, and \( K_\rho^d, \bar{K}_{\rho/r}^d \) are \( d \)-cylinders or radius \( \rho \) and \( \rho/r \) respectively.

**Proof.** Using (9.25) to eliminate \( c\omega_1, c\omega_2 \) from (3.4), we obtain

\[
\begin{align*}
    v^*_1 &= \frac{v_1 + v_2 + v_3}{3} + \frac{r}{3} (\nu_1 + \nu_2), \\
    v^*_2 &= \frac{v_1 + v_2 + v_3}{3} + \frac{r}{3} (-2\nu_1 + \nu_2), \\
    v^*_3 &= \frac{v_1 + v_2 + v_3}{3} + \frac{r}{3} (\nu_1 - 2\nu_2).
\end{align*}
\]

The conclusion is immediate after a translation and a dilation.

Recalling from (9.23) the \((2d-1)\)-ellipsoid \( E^{2d-1}_1 \), one can see that

\[ S_{12}(E^{2d-1}_1) = S_{13}(E^{2d-1}_1) = \left\{ (y_1, y_2) \in \mathbb{R}^{2d} : |y_1|^2 + |y_2|^2 + \langle y_1, y_2 \rangle = \frac{3}{2} \right\}. \quad (9.56) \]

For convenience we write

\[ S := S_{12}(E^{2d-1}_1) = S_{13}(E^{2d-1}_1) = \left\{ (y_1, y_2) \in \mathbb{R}^{2d} : |y_1|^2 + |y_2|^2 + \langle y_1, y_2 \rangle = \frac{3}{2} \right\}. \quad (9.57) \]

We state the following useful Lemma:

**Lemma 9.10.** There exist linear bijections \( T_1, T_2, P_1, P_2 : \mathbb{R}^{2d} \to \mathbb{R}^{2d} \), with the following properties:
(i) \( T_1(S) = S_1^{2d-1} \). Furthermore for any \( \rho > 0 \), we have: \( T_1(K_\rho^d \times \mathbb{R}^d) \subseteq \tilde{K}_\rho^d \times \mathbb{R}^d \).

(ii) \( T_2(S) = S_1^{2d-1} \) and for any \( \rho > 0 \), there holds: \( T_2(\mathbb{R}^d \times K_\rho^d) \subseteq \tilde{K}_\rho^d \times \mathbb{R}^d \).

(iii) \( P_1(E_2^{d-1}) = S_1^{2d-1} \) and for any \( \rho > 0 \), there holds: \( P_1(K_\rho^d \times \mathbb{R}^d) \subseteq \tilde{K}_\rho^d \times \mathbb{R}^d \).

(iv) \( P_2(E_2^{d-1}) = S_1^{2d-1} \) and for any \( \rho > 0 \), there holds: \( P_2(\mathbb{R}^d \times K_\rho^d) \subseteq \tilde{K}_\rho^d \times \mathbb{R}^d \),

where \( K_\rho^d \) is a \( d \)-cylinder of radius \( \rho \) and \( \tilde{K}_\rho^d \) is a \( d \)-cylinder relative to \( K_\rho^d \) (see the notation defined prior in Section 9.1).

Proof. A direct algebraic calculation shows that the maps given by:

\[
T_1 = \begin{pmatrix}
-\sqrt{2} I_d & 0 \\
\sqrt{\frac{6}{3}} I_d & \sqrt{\frac{6}{3}} I_d
\end{pmatrix}, \quad T_2 = \begin{pmatrix}
0 & -\sqrt{2} I_d \\
\sqrt{\frac{6}{3}} I_d & \sqrt{\frac{6}{3}} I_d
\end{pmatrix},
\]

\[
P_1 = \begin{pmatrix}
\sqrt{\frac{6}{3}} I_d & 0 \\
-\sqrt{\frac{6}{3}} I_d & \sqrt{\frac{6}{3}} I_d
\end{pmatrix}, \quad P_2 = \begin{pmatrix}
0 & \sqrt{\frac{6}{3}} I_d \\
\sqrt{2} I_d & -\sqrt{2} I_d
\end{pmatrix},
\]

where \( I_d \) is the \( d \times d \) identity matrix, satisfy the properties listed above. \( \square \)

Now we are ready to apply the spherical estimates from Section 9.1 to obtain ellipsoidal estimates. We are thankful to Ryan Denlinger for suggesting us that it is sufficient to use spherical estimates and the topological observation of the type that we obtained in Lemma 9.10.

Given \( \rho > 0 \), recall from (9.11) the notation

\[
W_{\rho,1,1}^{2d} = \left\{(w_1, w_2) \in \mathbb{R}^{2d} : |w_1 - w_2| \leq \rho\right\}.
\]

(9.59)
We obtain the following ellipsoidal estimates:

**Proposition 9.11.** For any \( r, \rho > 0 \), the following estimates hold:

(i) \[ |S \cap (K_{\rho/r}^d \times \mathbb{R}^d)|_{S} \lesssim \min \left\{ 1, \left( \frac{\rho}{r} \right)^{\frac{d-1}{2}} \right\}. \]

(ii) \[ |S \cap (\mathbb{R}^d \times K_{\rho/r}^d)|_{S} \lesssim \min \left\{ 1, \left( \frac{\rho}{r} \right)^{\frac{d-1}{2}} \right\}. \]

(iii) \[ |E_1^{2d-1} \cap (B_{\rho/r}^d \times \mathbb{R}^d)|_{E_1^{2d-1}} \lesssim \min \left\{ 1, \left( \frac{\rho}{r} \right)^{\frac{d-1}{2}} \right\}. \]

(iv) \[ |E_1^{2d-1} \cap (\mathbb{R}^d \times B_{\rho/r}^d)|_{E_1^{2d-1}} \lesssim \min \left\{ 1, \left( \frac{\rho}{r} \right)^{\frac{d-1}{2}} \right\}. \]

(v) \[ |E_1^{2d-1} \cap W_{\rho/r,1,1}^{2d}|_{E_1^{2d-1}} \lesssim \min \left\{ 1, \left( \frac{6\rho}{(3\sqrt{2} + \sqrt{6})r} \right)^{\frac{d-1}{2}} \right\}. \]

**Proof.** Let us first provide the proof of (i). Lemma 9.10 asserts \( T_1 : S \rightarrow \mathbb{S}_1^{2d-1} \) is a linear bijection which preserves sets of the form \( K_{\rho/r}^d \times \mathbb{R}^d \) up to inclusion, thus we have

\[
|S \cap (K_{\rho/r}^d \times \mathbb{R}^d)|_{S} = \int_S 1_{K_{\rho/r}^d \times \mathbb{R}^d}(\omega) \, d\omega
\]

\[
= \int_S 1_{T_1(K_{\rho/r}^d \times \mathbb{R}^d)}(T_1\omega) \, d\omega
\]

\[
\lesssim \int_{\mathbb{S}_1^{2d-1}} 1_{T_1(K_{\rho/r}^d \times \mathbb{R}^d)}(\theta) \, d\theta
\]

\[
\lesssim \int_{\mathbb{S}_1^{2d-1}} 1_{K_{\rho/r}^d \times \mathbb{R}^d}(\theta) \, d\theta
\]

\[
= |S_1^{2d-1} \cap (K_{\rho/r}^d \times \mathbb{R}^d)|_{S_1^{2d-1}}\]

\[
\lesssim \min \left\{ 1, \left( \frac{\rho}{r} \right)^{\frac{d-1}{2}} \right\},
\]

(9.60)
where to obtain (9.60) we use Lemma A.2, and to obtain (9.61) we use Lemma 9.4. The proof for (ii) is identical using the bijection $T_2$ defined in Lemma 9.10.

For estimates (iii) and (iv) we use in a similar way the bijections $P_1, P_2$ defined in Lemma 9.10, obtain the estimates

$$|E_{1}^{2d-1} \cap (K_{\rho/r} \times \mathbb{R}^{d})|_{g_{i}^{2d-1}} \lesssim \min \left\{ 1, \left( \frac{\rho}{r} \right)^{\frac{d-1}{2}} \right\} \quad (9.62)$$

$$|E_{1}^{2d-1} \cap (\mathbb{R}^{d} \times K_{\rho/r}^{d})|_{g_{i}^{2d-1}} \lesssim \min \left\{ 1, \left( \frac{\rho}{r} \right)^{\frac{d-1}{2}} \right\} , \quad (9.63)$$

which together with the observation that the ball $B_{\rho/r}^{d}$ embeds in a cylinder of the form $K_{\rho/r}^{d}$ imply (iii) and (iv). For estimate (v), recalling notation from (9.11), notice that

$$P_{1}(W_{\eta/r,1,1}^{2d}) = W_{\eta/r,\mu,\lambda}^{2d},$$

for $\mu = (3\sqrt{2} + \sqrt{6})/6$ and $\lambda = -\sqrt{6}/3$ (we could use any of the other maps as well). Then the claim comes with a similar argument using Lemma 9.5 instead of Lemma 9.4.

\[\square\]
Chapter 10

Good configurations and stability

In this chapter we define good configurations and study their stability properties under the adjunction of a collisional pair of particles. Heuristically speaking, given $m \in \mathbb{N}$, a configuration $Z_m \in \mathbb{R}^{2dm}$ is called good configuration if the backwards interaction flow coincides with the backwards free flow. The aim of this chapter is to investigate conditions under which a given good configuration $Z_m$ remains a good configurations after adding a pair of particles. This is possible on the complement of a small measure set of particles which is constructed in Proposition 10.4. Proposition 10.7 uses the geometric tools developed in Chapter 9 to derive a measure estimate for this pathological set.

This chapter is the heart of our contribution, since we will strongly rely on Proposition 10.4 and Proposition 10.7 when we use them inductively to control the differences (8.43) of the BBGKY hierarchy truncated observable, given in (8.27), and the Boltzmann hierarchy truncated observable, given in (8.28).

Let us recall the notation introduced in Chapter 9. Given $w \in \mathbb{R}^n$, $y \in \mathbb{R}^n \backslash \{0\}$ and $\rho > 0$, we write $K_\rho^d(w,y)$ for the closed $d$-dimensional cylinder of center $w$, direction $y$ and radius $\rho$. When we refer to an arbitrary cylinder of radius $\rho$, we will be writing $K_\rho^d$. Finally, given a cylinder $K_\rho^d$, we write $\tilde{K}_\rho^d$ for a cylinder relative to
\( K^d_\rho \), with radius \( c\rho \), where \( c > 0 \) is independent of \( \rho \).

### 10.1 Adjunction of new particles

We start with some definitions on the configurations we are using. Consider \( m \in \mathbb{N} \) and \( \sigma > 0 \), and recall from (7.1)-(7.3) the set of well-separated configurations

\[
\Delta_m(\sigma) = \{ \tilde{Z}_m = (\tilde{X}_m, \tilde{V}_m) \in \mathbb{R}^{2dm} : |\tilde{x}_i - \tilde{x}_j| > \sigma, \quad \forall 1 \leq i < j \leq m \}, \quad m \geq 2,
\]

\[
\Delta_1(\sigma) = \mathbb{R}^{2d}.
\]

(10.1)

Roughly speaking, a good configuration is a configuration which remains well-separated under backwards time evolution. More precisely, given \( \sigma > 0, \ t_0 > 0 \), we define the set of good configurations as:

\[
G_m(\sigma, t_0) = \{ Z_m = (X_m, V_m) \in \mathbb{R}^{2dm} : Z_m(t) \in \Delta_m(\sigma), \quad \forall t \geq t_0 \},
\]

(10.2)

where \( Z_m(t) \) denotes the backwards in time free flow of \( Z_m = (X_m, V_m) \), given by:

\[
Z_m(t) = (X_m(t), V_m(t)) := (X_m - tV_m, V_m), \quad t \geq 0.
\]

(10.3)

Notice that \( Z_m \) is the initial point of the trajectory i.e. \( Z_m(0) = Z_m \). In other words for \( m \geq 2 \), we have

\[
G_m(\sigma, t_0) = \{ Z_m \in \Delta_m(\sigma) : |x_i(t) - x_j(t)| > \sigma, \quad \forall t \geq t_0, \quad \forall i < j \in \{1, \ldots, m\} \},
\]

(10.4)

while for \( m = 1 \), we have

\[
G_1(\sigma, t_0) = \mathbb{R}^{2d}.
\]

(10.5)
From now on, we consider parameters $R >> 1$ and $0 < \delta, \eta, \epsilon_0, \alpha << 1$ satisfying:
\[
\alpha << \epsilon_0 << \eta, \quad R\alpha << \eta \epsilon_0.
\] (10.6)

For convenience we choose the parameters in (10.6) in the very end of Part I of the dissertation, see (??)-(??).

The following Lemma, due to [33], is useful for the adjunction of particles to a given configuration. We provide the proof for convenience of a reader.

**Lemma 10.2.** Consider parameters $\alpha, \epsilon_0, R, \eta, \delta$ as in (10.6) and $\epsilon << \alpha$. Let $\bar{y}_1, \bar{y}_2 \in \mathbb{R}^d$, with $|\bar{y}_1 - \bar{y}_2| > \epsilon_0$ and $v_1 \in B^d_R$. Then there is a $d$-cylinder $K^d_\eta \subseteq \mathbb{R}^d$, such that for any $y_1 \in B^d_\alpha(\bar{y}_1)$, $y_2 \in B^d_\alpha(\bar{y}_2)$ and $v_2 \in B^d_R \setminus K^d_\eta$, we have

(i) $Z_2 \in G_2(\sqrt{2}\epsilon, 0)$,

(ii) $Z_2 \in G_2(\epsilon_0, \delta),

where $Z_2 = (y_1, y_2, v_1, v_2)$.

**Proof.** Let $y_1 \in B^d_\alpha(\bar{y}_1), y_2 \in B^d_\alpha(\bar{y}_2), v_2 \in B^d_R$ and let us write $Z_2 = (y_1, y_2, v_1, v_2)$.

To prove (i) assume first that $Z_2 \notin G_2(\sqrt{2}\epsilon, 0)$ i.e. there is $\tau \geq 0$ such that

\[
|y_1 - y_2 - \tau(v_1 - v_2)| \leq \sqrt{2}\epsilon.
\]

Triangle inequality and the facts $y_1 \in B^d_\alpha(\bar{y}_1), y_2 \in B^d_\alpha(\bar{y}_2)$ yield

\[
|\bar{y}_1 - \bar{y}_2 - \tau(v_1 - v_2)| \leq 2\alpha + |y_1 - y_2 - \tau(v_1 - v_2)| \leq 2\alpha + \sqrt{2}\epsilon < 3\alpha, \quad (10.7)
\]

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since $\epsilon << \alpha$. This, together with the fact $|\bar{y}_1 - \bar{y}_2| > \epsilon_0 > 3\alpha$ (thanks to $\alpha << \epsilon_0$), implies

$$v_1 - v_2 \in C_{3\alpha}^d(0, \bar{y}_1 - \bar{y}_2),$$

where $C_{3\alpha}^d(0, \bar{y}_1 - \bar{y}_2)$ is the $d$-dimensional cone of vertex 0, supported on the ball of center $\bar{y}_1 - \bar{y}_2$ and radius $3\alpha$. Since $v_1, v_2 \in B^d_{R}$, we obtain that

$$v_1 - v_2 \in B^d_{2R} \cap C_{3\alpha}^d(0, \bar{y}_1 - \bar{y}_2).$$ (10.8)

By a similar triangles argument, to the triangles $(OBC), (ODE)$ below,

\begin{align*}
\overline{OB} &= \bar{y}_2 - \bar{y}_1, & (BC) &= 3\alpha, & (OD) &= 2R.
\end{align*}

Figure 10.1

we obtain

$$\rho := (OE) = \frac{6R\alpha}{|\bar{y}_1 - \bar{y}_2|}.$$
Therefore, (10.8) and Figure 10.1 yield

\[ v_1 - v_2 \in B_{2R}^d \cap C_{3\alpha}^d (0, \bar{y}_1 - \bar{y}_2) \subseteq K_{\rho}^d (0, \bar{y}_1 - \bar{y}_2). \]  

(10.9)

But

\[ \rho = \frac{6R\alpha}{|\bar{y}_1 - \bar{y}_2|} < \eta, \]  

(10.10)

since \( |\bar{y}_1 - \bar{y}_2| > \epsilon_0 \) and \( R\alpha << \eta \epsilon_0 \). Inclusion (10.9) and inequality (10.10) imply that \( v_2 \in K_{\eta}^d (v_1, \bar{y}_1 - \bar{y}_2) := K_{\eta}^d \). Therefore, for any \( v_2 \in B_R^d \setminus K_{\eta}^d \), we obtain \( Z_2 \in G_2(\sqrt{2}\epsilon_0, 0) \), and claim (i) is proved.

To prove (ii) assume that \( Z_2 \notin G_2(\epsilon_0, \delta) \) i.e. there is \( \tau \geq \delta \) such that

\[ |y_1 - y_2 - \tau (v_2 - v_2)| \leq \epsilon_0. \]

Again triangle inequality implies

\[ |\bar{y}_1 - \bar{y}_2 - \tau (v_1 - v_2)| \leq 2\alpha + |y_1 - y_2 - \tau (v_1 - v_2)| \leq 2\alpha + \epsilon_0 < 2\epsilon_0, \]  

(10.11)

since \( \alpha << \epsilon_0 \). Since \( \tau \geq \delta \), (10.11) implies

\[ |v_1 - v_2 - \tau^{-1}(\bar{y}_1 - \bar{y}_2)| < \frac{2\epsilon_0}{\tau} \leq \frac{2\epsilon_0}{\delta} \Leftrightarrow v_2 \in K_{2\epsilon_0/\delta}^d (v_1, \bar{y}_1 - \bar{y}_2). \]  

(10.12)

Since \( \epsilon_0 << \eta \delta \), for any \( v_2 \in B_R^d \setminus K_{\eta}^d (v_1, \bar{y}_1 - \bar{y}_2) \), we have \( Z_2 \in G_2(\epsilon_0, \delta) \), and assertion (ii) follows. The result is proved.

10.3 Stability of good configurations under adjunction of collisional pair

We prove a statement and a measure estimate regarding the stability of good configurations under the adjunction of a collisional pair of particles to any of the ini-
tial configurations. This statement will allow us to pass to term by term convergence in the series expansion of the Duhamel formula.

For convenience, given \( v \in \mathbb{R}^d \), let us denote

\[
(S_{1}^{2d-1} \times B_{R}^{2d})^+ (v) = \{ (\omega_1, \omega_2, v_1, v_2) \in S_{1}^{2d-1} \times B_{R}^{2d} : b(\omega_1, \omega_2, v_1 - v, v_2 - v) > 0 \},
\]

(10.13)

where

\[
b(\omega_1, \omega_2, v_1 - v, v_2 - v) = \langle \omega_1, v_1 - v \rangle + \langle \omega_2, v_2 - v \rangle,
\]

is the cross-section given in (3.5).

We prove the following Proposition, which will be the inductive step of the convergence proof. We then provide the corresponding measure estimate.

Recall that given \( m \in \mathbb{N} \) and \( Z_m = (X_m, V_m) \in \mathbb{R}^{2dm} \), we denote

\[
Z_m(t) = (X_m(t), V_m(t)) = (X_m - tV_m, V_m), \quad t \geq 0,
\]

the backwards evolution in time of \( Z_m \). In particular, \( Z_m(0) = Z_m \).

Recall also the notation from (4.5)

\[
\mathcal{D}_{m+2, \epsilon} = \{ Z_{m+2} \in \mathbb{R}^{2d(m+2)} : d^2(x_i, x_j, x_k) > 2\epsilon^2, \quad \forall i < j < k \in \{1, \ldots, m+2\} \}
\]

**Proposition 10.4.** Consider parameters \( \alpha, \epsilon_0, R, \eta, \delta \) as in (10.6) and \( \epsilon \ll \alpha \). Let \( m \in \mathbb{N} \), \( \tilde{Z}_m = (\tilde{X}_m, \tilde{V}_m) \in G_m(\epsilon_0, 0), \ell \in \{1, \ldots, m\} \) and \( X_m \in B_{\alpha/2}(\tilde{X}_m) \). Then there is a subset \( \mathcal{B}_\ell(\hat{Z}_m) \subseteq (S_{1}^{2d-1} \times B_{R}^{2d})^+ (\hat{v}_\ell) \) such that:
(i) For any \((\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S^{2d-1}_1 \times B_R^{2d})^+(\bar{v}_t) \setminus \mathcal{B}_t(Z_m)\), one has:

\[
Z_{m+2}(t) \in \mathcal{D}_{m+2,\epsilon}, \quad \forall t \geq 0, \\
Z_{m+2} \in G_{m+2}(\epsilon_0/2, \delta), \\
\bar{Z}_{m+2} \in G_{m+2}(\epsilon_0, \delta),
\]

where

\[
Z_{m+2} = (x_1, \ldots, x_{\ell}, \ldots, x_m, x_{m+1}, x_{m+2}, \bar{v}_1, \ldots, \bar{v}_\ell, \ldots, \bar{v}_m, v_{m+1}, v_{m+2}),
\]

\[
x_{m+1} = x_{\ell} - \sqrt{2}\epsilon\omega_1,
\]

\[
x_{m+2} = x_{\ell} - \sqrt{2}\epsilon\omega_2,
\]

\[
\bar{Z}_{m+2} = (\bar{x}_1, \ldots, \bar{x}_\ell, \ldots, \bar{x}_m, \bar{x}_\ell, \bar{x}_\ell, \bar{v}_1, \ldots, \bar{v}_\ell, \ldots, \bar{v}_m, v_{m+1}, v_{m+2}).
\]

(ii) For any \((\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S^{2d-1}_1 \times B_R^{2d})^+(\bar{v}_t) \setminus \mathcal{B}_t(Z_m)\), one has:

\[
Z^*_{m+2}(t) \in \mathcal{D}_{m+2,\epsilon}, \quad \forall t \geq 0, \\
Z^*_{m+2} \in G_{m+2}(\epsilon_0/2, \delta), \\
\bar{Z}^*_{m+2} \in G_{m+2}(\epsilon_0, \delta),
\]

where

\[
Z^*_{m+2} = (x_1, \ldots, x_{\ell}, \ldots, x_m, x_{m+1}, x_{m+2}, \bar{v}_1, \ldots, \bar{v}_\ell, \ldots, \bar{v}_m, v^*_{m+1}, v^*_{m+2}),
\]

\[
x_{m+1} = x_{\ell} + \sqrt{2}\epsilon\omega_1,
\]

\[
x_{m+2} = x_{\ell} + \sqrt{2}\epsilon\omega_2,
\]

\[
\bar{Z}^*_{m+2} = (\bar{x}_1, \ldots, \bar{x}_\ell, \ldots, \bar{x}_m, \bar{x}_\ell, \bar{x}_\ell, \bar{v}_1, \ldots, \bar{v}_\ell, \ldots, \bar{v}_m, v^*_{m+1}, v^*_{m+2}),
\]

\[
(\bar{v}^*_t, v^*_{m+1}, v^*_{m+2}) = T_{\omega_1,\omega_2}(\bar{v}_t, v_{m+1}, v_{m+2}).
\]

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Proof. By symmetry, we may assume without loss of generality that \( \ell = m \).

For convenience, let us define the set of indices:

\[
F_{m+2} = \{(i, j) \in \{1, \ldots, m + 2\} \times \{1, \ldots, m + 2\} : i < \min\{j, m\}\}.
\] (10.22)

Proof of (i) Here we use the notation from (10.17).

We start by formulating the following claim, which will imply (10.14).

Lemma 10.5. Under the hypothesis of Proposition 10.4, there is a set \( B^0_m - (\bar{Z}_m) \subseteq S^{2d-1} \times B^d_R \) such that for any \( (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S^{2d-1} \times B^d_R)^+ (\bar{v}_m) \setminus B^0_m - (Z_m) \), there holds:

\[
|x_i(t) - x_j(t)| > \sqrt{2} \epsilon, \quad \forall t \geq 0, \quad \forall (i, j) \in F_{m+2}.
\] (10.23)

\[
d^2(x_m(t); x_{m+1}(t), x_{m+2}(t)) > 2 \epsilon^2, \quad \forall t \geq 0.
\] (10.24)

We observe that (10.23)-(10.24) imply

\[
d^2(x_i(t); x_j(t), x_k(t)) > 2 \epsilon^2, \quad \forall t \geq 0, \quad \forall 1 \leq i < j < k \leq m + 2,
\] (10.25)

which is equivalent to (10.14). In particular, (10.23) implies (10.25) for all \( i < j < k \in \{1, \ldots, m + 2\} \) and \( i < \min\{j, m\} \), while (10.24) implies (10.25) in the remaining case when \( (i, j, k) = (m, m + 1, m + 2) \). Therefore (10.23)-(10.24) imply (10.14).

Proof of Lemma 10.5:

Step 1 - the proof of (10.23).

Fix \((i, j) \in F_{m+2}\). We distinguish the following cases:
• $j \leq m$: Since $\bar{Z}_m \in G_m(\epsilon_0, 0)$ and $i < j \leq m$, we have

$$|\bar{x}_i - \bar{x}_j - t(\bar{v}_i - \bar{v}_j)| > \epsilon_0, \quad \forall t \geq 0.$$ 

Hence, triangle inequality implies

$$|x_i(t) - x_j(t)| = |x_i - x_j - t(\bar{v}_i - \bar{v}_j)|$$

$$\geq |\bar{x}_i - \bar{x}_j - t(\bar{v}_i - \bar{v}_j)| - \alpha \quad (10.26)$$

$$\geq \epsilon_0 - \alpha > \frac{\epsilon_0}{2} > \sqrt{2}\epsilon,$$

since $\epsilon << \alpha << \epsilon_0$. Therefore, (10.23) holds for any $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in S^{2d-1}_1 \times B^{2d}_R$.

• $j = m + 1$: Since $(i, j) \in F_{m+2}$ we have $i \leq m - 1$. Then for $\bar{Z}_m \in G_m(\epsilon_0, 0)$ and $X_m \in B^{dn}_{\alpha/2}(\bar{X}_m)$, we conclude

$$|\bar{x}_i - \bar{x}_m| > \epsilon_0,$$

$$|x_{m+1} - \bar{x}_m| \leq |x_m - \bar{x}_m| + |x_{m+1} - x_m| \leq \frac{\alpha}{2} + \sqrt{2}\epsilon|\omega_1| \leq \frac{\alpha}{2} + \sqrt{2}\epsilon < \alpha.$$

Applying part (i) of Lemma 10.2 with $\bar{y}_1 = \bar{x}_i, \bar{y}_2 = \bar{x}_m, y_1 = x_i, y_2 = x_{m+1}$, we can find a cylinder $K^{d,i}_\eta$ such that for any $v_{m+1} \in B^{d}_R \setminus K^{d,i}_\eta$ we have:

$$|x_i(t) - x_{m+1}(t)| > \sqrt{2}\epsilon, \quad \forall t \geq 0.$$ 

Hence (10.23) holds for any $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S^{2d-1}_1 \times B^{2d}_R) \setminus U^i_{m+1}$, where

$$U^i_{m+1} = S^{2d-1}_1 \times K^{d,i}_\eta \times \mathbb{R}^d. \quad (10.27)$$

• $j = m + 2$: Since $(i, j) \in F_{m+2}$, we obtain $i < m$. Hence, we may repeat the same argument as in the previous case using part (i) of Lemma 10.2.
with \( \bar{y}_1 = \bar{x}_i, \bar{y}_2 = \bar{x}_m, \ y_1 = x_i, \ y_2 = x_{m+2} \). Thus, (10.23) holds for any

\[(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S_1^{2d-1} \times B_R^{2d}) \setminus U_{m+2}^i, \]

where

\[U_{m+2}^i = S_1^{2d-1} \times \mathbb{R}^d \times K_{\eta}^{d,i}. \quad (10.28)\]

Therefore, we conclude that (10.23) holds for any

\[(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S_1^{2d-1} \times B_R^{2d}) \setminus (\bigcup_{i=1}^{m-1} (U_{m+1}^i \cup U_{m+2}^i)). \]

Step 2 - the proof of (10.24). Let us recall notation from (10.13).

Considering \( t \geq 0 \) and \((\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S_1^{2d-1} \times B_R^{2d})^+(\bar{v}_m)\), we have

\[
d^2 (x_m(t) ; x_{m+1}(t) ; x_{m+2}(t)) = |\sqrt{2} \epsilon \omega_1 + t(v_{m+1} - \bar{v}_m)|^2 + |\sqrt{2} \epsilon \omega_2 + t(v_{m+2} - \bar{v}_m)|^2 \\
\geq 2 \epsilon^2 (|\omega_1|^2 + |\omega_2|^2) + 2 \sqrt{2} \epsilon t b(\omega_1, \omega_2, v_{m+1} - \bar{v}_m, v_{m+2} - \bar{v}_m) \\
> 2 \epsilon^2, \quad (10.29)\]

where to obtain (10.29) we use the fact that \((\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S_1^{2d-1} \times B_R^{2d})^+(\bar{v}_m)\).

Defining

\[
\mathcal{B}^0_{m} (\bar{Z}_m) = \bigcup_{i=1}^{m-1} (U_{m+1}^i \cup U_{m+2}^i), \quad (10.30)\]

Lemma 10.5 is proved, and (10.14) follows.

Let us now find a set \( \mathcal{B}^\delta_{m} (\bar{Z}_m) \subseteq S_1^{2d-1} \times B_R^{2d} \) such that (10.15) holds for any

\[(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S_1^{2d-1} \times B_R^{2d}) \setminus \mathcal{B}^\delta_{m} (\bar{Z}_m).\]

We distinguish the following cases:
• \((i,j) \in F_{m+2}, j \leq m\): We use the same argument as in (10.26) to obtain the lower bound \(\epsilon_0/2\).

• \((i,j) \in F_{m+2}, j \in \{m+1, m+2\}\): (10.15) holds for \((\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S^{2d-1}_1 \times B^d_R) \setminus B^0_m(\bar{Z}_m)\), using part (ii) of Lemma 10.2 and similar arguments to the corresponding cases in the proof of Lemma 10.5. Let us note that the lower bound is in fact \(\epsilon_0\).

• \((i,j) = (m, m+1)\): Triangle inequality implies that for \(t \geq \delta\) and for any \((\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in S^{2d-1}_1 \times B^d_R\), such that \(|v_{m+1} - \bar{v}_m| > \eta\), we have

\[
|x_m(t) - x_{m+1}(t)| = |\sqrt{2}\epsilon \omega_1 - t(\bar{v}_m - v_{m+1})| \geq |\bar{v}_m - v_{m+1}| |t - \sqrt{2}\epsilon \omega_1| \\
\geq |\bar{v}_m - v_{m+1}| \delta - \sqrt{2}\epsilon \\
> \eta \delta - \sqrt{2}\epsilon > \epsilon_0, \tag{10.31}
\]

where to obtain (10.31) we use the fact that \(\epsilon < \epsilon_0 < \eta \delta\). Therefore, (10.15) holds for \((\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S^{2d-1}_1 \times B^d_R) \setminus V_{m,m+1}\), where

\[V_{m,m+1} = S^{2d-1}_1 \times B^d_\eta(\bar{v}_m) \times \mathbb{R}^d. \tag{10.32}\]

Let us note that the lower bound is in fact \(\epsilon_0\).

• \((i,j) = (m, m+2)\): Same arguments as in the case \((i,j) = (m, m+1)\) yield that (10.15) holds for \((\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S^{2d-1}_1 \times B^d_R) \setminus V_{m,m+2}\), where

\[V_{m,m+2} = S^{2d-1}_1 \times \mathbb{R}^d \times B^d_\eta(\bar{v}_m). \tag{10.33}\]

The lower bound is in fact \(\epsilon_0\).
• \((i,j) = (m + 1, m + 2)\). Triangle inequality implies that for \(t \geq \delta\) and \((\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in S_1^{2d-1} \times B_R^d\), such that \(|v_{m+1} - v_{m+2}| > \eta\), we have

\[
|x_{m+1}(t) - x_{m+2}(t)| = |\sqrt{2}\epsilon(\omega_2 - \omega_1) - t(v_{m+1} - v_{m+2})| \\
\geq |v_{m+1} - v_{m+2}|t - \sqrt{2}\epsilon|\omega_1 - \omega_2| \\
\geq |v_{m+1} - v_{m+2}|\delta - \sqrt{2}\epsilon(|\omega_1| + |\omega_2|) \\
\geq |v_{m+1} - v_{m+2}|\delta - 2\sqrt{2}\epsilon \\
> \eta\delta - 2\sqrt{2}\epsilon > \epsilon_0,
\]

where to obtain (10.34) we use the fact that \(\epsilon << \epsilon_0 << \eta\delta\).

Recalling from (9.59) the 2d-strip

\[
W_{\eta,1,1}^{2d} = \{(w_1, w_2) \in \mathbb{R}^{2d} : |w_1 - w_2| \leq \eta\},
\]

we obtain that (10.15) holds for \((\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S_1^{2d-1} \times B_R^d) \setminus U_{m+1,m+2}\), where

\[
U_{m+1,m+2} = S_1^{2d-1} \times W_{\eta,1,1}^{2d}.
\]

Notice that the lower bound is in fact \(\epsilon_0\) again.

Defining

\[
\mathcal{B}^{\delta,-}_m(\mathcal{Z}_m) = \mathcal{B}^{0,-}_m(\mathcal{Z}_m) \cup V_{m,m+1} \cup V_{m,m+2} \cup U_{m+1,m+2},
\]

we conclude that (10.15) holds for

\[(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S_1^{2d-1} \times B_R^d) \setminus \mathcal{B}^{\delta,-}_m(\mathcal{Z}_m).
\]
Let us note that the only case which prevents us from having $Z_{m+2} \in G_{m+2}(\epsilon_0, \delta)$ is the case $1 \leq i < j \leq m$, where we obtain a lower bound of $\epsilon_0/2$. In all other cases we can obtain lower bound $\epsilon_0$.

A similar argument shows that, for $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S_t^{2d-1} \times B_{R}^{2d}) \setminus B^\delta_m(\bar{Z}_m)$, (10.16) holds for all $1 \leq i < j \leq m+2$ except the case $1 \leq i < j \leq m$. However in this case, for any $1 \leq i < j \leq m$, we have

$$|\bar{x}(t) - \bar{x}(t)| = |\bar{x}_i - \bar{x}_j - t(\bar{v}_i - \bar{v}_j)| > \epsilon_0,$$

since $\bar{Z}_m \in G_m(\epsilon_0, 0)$. This observation shows that (10.16) holds for

$$(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S_t^{2d-1} \times B_{R}^{2d}) \setminus B^\delta_m(\bar{Z}_m),$$

as well.

We conclude that the set

$$B_m^-(\bar{Z}_m) = (S_t^{2d-1} \times B_{R}^{2d})^+(\bar{v}) \cap B^0_m(\bar{Z}_m) \cup B^\delta_m(\bar{Z}_m),$$

is the set we need for the pre-collisional case.

Proof of (ii) Here we use the notation from (10.21).

The proof follows the steps of the pre-collisional case, but we replace the velocities $(\bar{v}_m, v_{m+1}, v_{m+2})$ by the transformed velocities $(\bar{v}^*_m, v^*_m, v^*_m)$ and then pull-back. It is worth mentioning that the $m$-particle needs special treatment since its velocity is transformed to $\bar{v}^*_m$.

We first prove the following claim:
Lemma 10.6. Under the hypothesis of Proposition 10.4, there is a set $B_m^0(Z_m) \subseteq \mathbb{S}_1^{2d-1} \times B_R^{2d}$ such that for any $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in \left(\mathbb{S}_1^{2d-1} \times B_R^{2d}\right)^+ \setminus B_m^0(Z_m)$, there holds:

$$|x_i(t) - x_j(t)| > \sqrt{2}\epsilon, \quad \forall t \geq 0, \quad \forall (i, j) \in \mathcal{F}_{m+2}. \quad (10.40)$$

$$d^2(x_m(t); x_{m+1}(t), x_{m+2}(t)) > 2\epsilon^2, \quad \forall t \geq 0. \quad (10.41)$$

We observe that (10.40) and (10.41) clearly imply (10.18).

Proof of Lemma 10.6:

Step 1 - the proof of (10.40).

Fix $(i, j) \in \mathcal{F}_{m+2}$. We distinguish the following cases:

- $j \leq m - 1$: Since $Z_m \in G_m(\epsilon_0, 0), i < j \leq m - 1$, we have

$$|\bar{x}_i - \bar{x}_j - t(\bar{v}_i - \bar{v}_j)| > \epsilon_0, \quad \forall t \geq 0.$$

Hence, triangle inequality implies

$$|x_i(t) - x_j(t)| = |x_i - x_j - t(\bar{v}_i - \bar{v}_j)|$$

$$\geq |\bar{x}_i - \bar{x}_j - t(\bar{v}_i - \bar{v}_j)| - \alpha \quad (10.42)$$

$$\geq \epsilon_0 - \alpha \geq \frac{\epsilon_0}{2} > \sqrt{2}\epsilon,$$

since $\epsilon \ll \alpha \ll \epsilon_0$. Therefore, (10.40) holds for any $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in \mathbb{S}_1^{2d-1} \times B_R^{2d}$. 

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\textbullet{} \( j = m \): Since \((i, j) \in \mathcal{F}_{m+2}\) we obtain \( i \leq m - 1 \). Since \( \bar{Z}_m \in G_m(\epsilon_0, 0) \) and \( X_m \in B_{\alpha/2}^d(\bar{X}_m) \), we have

\[
|\bar{x}_i - \bar{x}_m| > \epsilon_0, \\
|x_m - \bar{x}_m| \leq \frac{\alpha}{2} < \alpha.
\]

Applying part (i) of Lemma \ref{lemma:10.2} with \( \bar{y}_1 = \bar{x}_i, \bar{y}_2 = \bar{x}_m, y_1 = x_i, y_2 = x_m \), we can find a cylinder \( K^d_{\eta, i} \) such that for any \( v_m^* \in B_R^d \setminus K^d_{\eta, i} \) we have:

\[
|x_i(t) - x_m(t)| = |x_i - x_m - t(\bar{v}_i - \bar{v}_m^*)| > \sqrt{2}\epsilon, \quad \forall t \geq 0.
\]

Hence (10.40) holds for any \((\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (\mathcal{S}_1^{2d-1} \times B_R^d) \setminus V_{m, i}^* \), where

\[
V_{m, i}^* = \left\{ (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in \mathcal{S}_1^{2d-1} \times \mathbb{R}^{2d} : \bar{v}_m^* \in K^d_{\eta, i} \right\}. \quad (10.43)
\]

\textbullet{} \( j = m + 1 \): Since \((i, j) \in \mathcal{F}_{m+2}\) we obtain \( i \leq m - 1 \). Since \( Z_m \in G_m(\epsilon_0, 0) \) and \( X_m \in B_{\alpha/2}^d(\bar{X}_m) \), we have

\[
|\bar{x}_i - \bar{x}_m| > \epsilon_0, \\
|x_{m+1} - \bar{x}_m| \leq |x_m - \bar{x}_m| + |x_{m+1} - x_m| \leq \frac{\alpha}{2} + \sqrt{2}\epsilon|\omega_1| \leq \frac{\alpha}{2} + \sqrt{2}\epsilon < \alpha.
\]

Applying part (i) of Lemma \ref{lemma:10.2} with \( \bar{y}_1 = \bar{x}_i, \bar{y}_2 = \bar{x}_m, y_1 = x_i, y_2 = x_{m+1} \), we can find a cylinder \( K^d_{\eta, i} \) such that for any \( v_{m+1}^* \in B_R^d \setminus K^d_{\eta, i} \) we have:

\[
|x_i(t) - x_{m+1}(t)| = |x_i - x_{m+1} - t(\bar{v}_i - \bar{v}_{m+1}^*)| > \sqrt{2}\epsilon, \quad \forall t \geq 0.
\]

Hence (10.40) holds for any \((\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (\mathcal{S}_1^{2d-1} \times B_R^d) \setminus U_{m+1, i}^*, \) where

\[
U_{m+1, i}^* = \left\{ (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in \mathcal{S}_1^{2d-1} \times \mathbb{R}^{2d} : v_{m+1}^* \in K^d_{\eta, i} \right\}. \quad (10.44)
\]
\( j = m + 2 \): Since \((i,j) \in \mathcal{F}_{m+2}\), we obtain \(i \leq m - 1\). Hence, we may repeat the same argument as in the previous case using part \((i)\) of Lemma 10.2 with \(\bar{y}_1 = \bar{x}_i, \bar{y}_2 = \bar{x}_m, y_1 = x_i, y_2 = x_{m+2}\) instead. Thus, (10.40) holds for any \((\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (\mathbb{S}^{2d-1}_1 \times B^{2d}_R) \setminus U_{m+2}^{i,i}\), where

\[
U_{m+2}^{i,i} = \left\{ (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in \mathbb{S}^{2d-1}_1 \times \mathbb{R}^d : v^*_m \in K_{\eta}^{d,i} \right\}. \tag{10.45}
\]

Thus, (10.40) holds for any

\[
(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (\mathbb{S}^{2d-1}_1 \times B^{2d}_R) \setminus \bigcup_{i=1}^{m-1} (V^*_i \cup U^{i*}_{m+1} \cup U^{i*}_{m+2}).
\]

**Step 2 - the proof of** (10.41). Let us recall notation from (10.13).

Considering \(t \geq 0\) and \((\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (\mathbb{S}^{2d-1}_1 \times B^{2d}_R)^+ (\bar{v}_m)\), we have

\[
d^2 (x_m (t) ; x_{m+1} (t), x_{m+2} (t))
\]

\[
= |\sqrt{2}\epsilon \omega_1 + t(\bar{v}^*_m - v^*_m)|^2 + |\sqrt{2}\epsilon \omega_2 + t(\bar{v}^*_m - v^*_m + 2)|^2
\]

\[
\geq 2\epsilon^2 (|\omega_1|^2 + |\omega_2|^2) + 2\sqrt{2}\epsilon t b(\omega_1, \omega_2, \bar{v}^*_m - v^*_m + v^*_m - v^*_m + 2)
\]

\[
= 2\epsilon^2 + 2\sqrt{2}\epsilon t b(\omega_1, \omega_2, v_{m+1} - \bar{v}_m, v_{m+2} - \bar{v}_m) \tag{10.46}
\]

\[
> 2\epsilon^2, \tag{10.47}
\]

where to obtain (10.46)-(10.47) we use (3.14) and the fact that

\[(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (\mathbb{S}^{2d-1}_1 \times B^{2d}_R)^+ (\bar{v}_m),\]

respectively. Therefore (10.41) is proved.

Defining

\[
\mathcal{B}_m^{0,+} (\bar{Z}_m) = \bigcup_{i=1}^{m-1} (V^*_i \cup U^{i*}_{m+1} \cup U^{i*}_{m+2}), \tag{10.48}
\]

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Lemma 10.6 is proved, and (10.18) follows.

Let us now find a set $B_{m+ \delta}^+ (\bar{Z}_m) \subseteq S_{1}^{2d-1} \times B_{R}^{2d}$ such that (10.19) holds for any $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S_{1}^{2d-1} \times B_{R}^{2d}) \setminus B_{m+ \delta}^+ (\bar{Z}_m)$.

We distinguish the following cases:

- $(i, j) \in F_{m+2}, j \leq m - 1$: We use the same argument as in (10.42) to obtain the lower bound $\epsilon_0 /2$.

- $(i, j) \in F_{m+2}, j \in \{m, m+1, m+2\}$: (10.19) holds for $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S_{1}^{2d-1} \times B_{R}^{2d}) \setminus B_{m}^0 + (\bar{Z}_m)$, using part (ii) of Lemma 10.2 and similar arguments to the corresponding cases in the proof of Lemma 10.6. Let us note that the lower bound is in fact $\epsilon_0$.

- $(i, j) = (m, m+1)$: Triangle inequality implies that for $t \geq \delta$ and for any $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in S_{1}^{2d-1} \times B_{R}^{2d}$, such that $|\bar{v}_m^* - v_{m+1}^*| > \eta$, we have
  \[
  |x_m(t) - x_{m+1}(t)| = | - \sqrt{2} \epsilon \omega_1 - t(\bar{v}_m^* - v_{m+1}^*)| \geq |\bar{v}_m^* - v_{m+1}^*| t - \sqrt{2} \epsilon |\omega_1| \\
  \geq |\bar{v}_m^* - v_{m+1}^*| \delta - \sqrt{2} \epsilon \\
  > \eta \delta - \sqrt{2} \epsilon > \epsilon_0, \tag{10.49}
  \]
where to obtain (10.49) we use the fact that $\epsilon << \epsilon_0 << \eta \delta$. Therefore, (10.19) holds for $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S_{1}^{2d-1} \times B_{R}^{2d}) \setminus V_{m,m+1}^*$, where

\[
V_{m,m+1}^* = \left\{ (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in S_{1}^{2d-1} \times \mathbb{R}^{2d} : v_m^* - v_{m+1}^* \in B_{\eta}^d \right\} = \left\{ (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in S_{1}^{2d-1} \times \mathbb{R}^{2d} : (v_m^*, v_{m+1}^*) \in W_{\eta,1,1}^{2d} \right\}. \tag{10.50}
\]

Let us note that the lower bound is in fact $\epsilon_0$. 156
\( (i,j) = (m,m+2) \): Similar arguments as in the case \((i,j) = (m,m+1)\) yield that \((10.19)\) holds for \((\omega_1,\omega_2,v_{m+1},v_{m+2}) \in (S_1^{2d-1} \times B^{2d}_R) \setminus V_{m,m+2}^*\), where

\[
V_{m,m+2}^* = \left\{ (\omega_1,\omega_2,v_{m+1},v_{m+2}) \in S_1^{2d-1} \times \mathbb{R}^{2d} : v_m^* - v_{m+2}^* \in B^d_\eta \right\}
\]

(10.51)

The lower bound is in fact \(\epsilon_0\).

\( (i,j) = (m+1,m+2) \). Triangle inequality implies that for \(t \geq \delta\) and \((\omega_1,\omega_2,v_{m+1},v_{m+2}) \in S_1^{2d-1} \times B^{2d}_R\), such that \(|v_{m+1}^* - v_{m+2}^*| > \eta\), we have

\[
|x_{m+1}(t) - x_{m+2}(t)| = |\sqrt{2}\epsilon(\omega_1 - \omega_2) - t(v_{m+1}^* - v_{m+2}^*)| \\
\geq |v_{m+1}^* - v_{m+2}^*| t - \sqrt{2}\epsilon|\omega_1 - \omega_2| \\
\geq |v_{m+1}^* - v_{m+2}^*| \delta - \sqrt{2}\epsilon(|\omega_1| + |\omega_2|) \\
\geq |v_{m+1}^* - v_{m+2}^*| \delta - 2\sqrt{2}\epsilon \\
> \eta \delta - 2\sqrt{2}\epsilon > \epsilon_0,
\]

(10.52)

where to obtain \((10.52)\) we use the fact that \(\epsilon << \epsilon_0 << \eta \delta\). Therefore, \((10.19)\) holds for \((\omega_1,\omega_2,v_{m+1},v_{m+2}) \in (S_1^{2d-1} \times B^{2d}_R) \setminus U_{m+1,m+2}^*\), where

\[
U_{m+1,m+2}^* = \left\{ (\omega_1,\omega_2,v_{m+1},v_{m+2}) \in S_1^{2d-1} \times \mathbb{R}^{2d} : v_{m+1}^* - v_{m+2}^* \in B^d_\eta \right\}
\]

(10.54)

Notice that the lower bound is in fact \(\epsilon_0\) again.

Defining

\[
\mathcal{B}^{\delta,+}(\bar{Z}_m) = \mathcal{B}^{0,+}(\bar{Z}_m) \cup V_{m,m+1}^* \cup V_{m,m+2}^* \cup U_{m+1,m+2}^*
\]

(10.54)
we conclude that (10.19) holds for $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S_1^{2d-1} \times B^R_{2d}) \setminus B^\delta_m(\tilde{Z}_m)$.

With a similar argument as in the pre-collisional case, we obtain (10.20) as well.

Therefore, the set

$$B^+_m(\tilde{Z}_m) = (S_1^{2d-1} \times B^R_{2d})^+(\bar{v}_m) \cap (B^0_m(\tilde{Z}_m) \cup B^\delta_m(\tilde{Z}_m)),$$

is the appropriate set for the post-collisional case.

We conclude that the set we need is

$$B_m(Z_m) = B^-_m(Z_m) \cup B^+_m(Z_m).$$

(10.56)

The proof of Proposition 10.4 is complete.

We now use the results of Chapter 9 to estimate the measure of this set, with respect to the parameters chosen.

**Proposition 10.7.** Consider parameters $\alpha, \epsilon_0, R, \eta, \delta$ as in (10.6) and $\epsilon << \alpha$. Let $m \in \mathbb{N}$, $\tilde{Z}_m \in G_m(\epsilon_0, 0)$, $\ell \in \{1, \ldots, m\}$ and $B_\ell(\tilde{Z}_m)$ the set given in the statement of Proposition 10.4. Then the following measure estimate holds:

$$|B_\ell(\tilde{Z}_m)| \lesssim mR^{2d}\eta^{\frac{d-1}{d+2}},$$

where $|\cdot|$ denotes the product measure on $S_1^{2d-1} \times B^R_{2d}$.

**Proof.** Without loss of generality, we may assume that $\ell = m$. By (10.56), it suffices to estimate the measure of $B^-_m(Z_m)$ and $B^+_m(Z_m)$.
Estimate of $B_m(\bar{Z}_m)$. Recalling (10.39), (10.37), (10.30), we obtain

$$B_m(\bar{Z}_m) = (S^{2d-1}_1 \times B_{R}^{2d})^{+}(\bar{v}_m) \cap \left[ V_{m,m+1} \cup V_{m,m+2} \cup U_{m+1,m+2} \cup \bigcup_{i=1}^{m-1} (U_{m+1}^{i} \cup U_{m+2}^{i}) \right], \quad (10.57)$$

where $V_{m,m+1}, V_{m,m+2}, U_{m+1,m+2}, U_{m+1}^{i}, U_{m+2}^{i}$ are given by (10.32), (10.33), (10.36), (10.27), (10.28) respectively.

- Estimate of the terms corresponding to $V_{m,m+1}, V_{m,m+2}, U_{m+1,m+2}$: By recalling (10.32), we have

$$V_{m,m+1} = S^{2d-1}_1 \times B_{\eta}^{d}(\bar{v}_m) \times \mathbb{R}^d,$$

Recalling (10.13), we obtain

$$(S^{2d-1}_1 \times B_{R}^{2d})^{+}(\bar{v}_m) \cap V_{m,m+1} \subseteq S^{2d-1}_1 \times (B_{R}^{d} \cap B_{\eta}^{d}(\bar{v}_m)) \times B_{R}^{d},$$

so

$$|(S^{2d-1}_1 \times B_{R}^{2d})^{+}(\bar{v}_m) \cap V_{m,m+1}| \leq |S^{2d-1}_1| |S^{2d-1}_1| |B_{R}^{d} \cap B_{\eta}^{d}(\bar{v}_m)|_{d} |B_{R}^{d}|_{d} \lesssim R^{d} \eta^{d}. \quad (10.58)$$

In a similar way, by recalling (10.33), we obtain

$$|(S^{2d-1}_1 \times B_{R}^{2d})^{+}(\bar{v}_m) \cap V_{m,m+2}| \lesssim R^{d} \eta^{d}. \quad (10.59)$$

Moreover, by recalling (10.36), we have

$$U_{m+1,m+2} = S^{2d-1}_1 \times W_{\eta,1,1}^{2d},$$

where

$$W_{\eta,1,1}^{2d} = \{ (w_1, w_2) \in \mathbb{R}^2 : |w_1 - w_2| \leq \eta \} = \{ (w_1, w_2) \in \mathbb{R}^2 : w_2 \in B_{\eta}^{d}(w_1) \},$$

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Recalling (10.13), we have
\[
(S^{2d-1}_1 \times B^d_R) + (\bar{v}_m) \cap U_{m+1,m+2} \subseteq S^{2d-1}_1 \times [(B^d_R \times B^d_R) \cap W_{\eta,1,1}^{2d}],
\]
hence
\[
|(S^{2d-1}_1 \times B^d_R) + (\bar{v}_m) \cap U_{m+1,m+2}| \leq |S^{2d-1}_1| |(B^d_R \times B^d_R) \cap W_{\eta,1,1}^{2d}|^{2d} \leq \int_{B^d_R} \int_{B^d_R} 1_{B^d_R(v_{m+1})}(v_{m+2}) dv_{m+2} dv_{m+1} \lesssim R^d \eta^d.
\]
(10.60)

- Estimate of the terms corresponding to $U^i_{m+1}, U^i_{m+2}$, $i \in \{1, ..., m - 1\}$: Fix $i \in \{1, ..., m - 1\}$. By recalling (10.27)

\[
U^i_{m+1} = S^{2d-1}_1 \times K^{d,i}_\eta \times \mathbb{R}^d,
\]
we have
\[
|(S^{2d-1}_1 \times B^d_R) + (\bar{v}_m) \cap U^i_{m+1}| \subseteq S^{2d-1}_1 \times [B^d_R \cap (K^{d,i}_\eta \times \mathbb{R}^d)].
\]
Since $\eta << 1 << R$, Proposition 9.3 part (ii) implies that
\[
|(S^{2d-1}_1 \times B^d_R) + \cap U^i_{m+1}| \leq |S^{2d-1}_1| |(B^d_R \cap (K^{d,i}_\eta \times \mathbb{R}^d))|^{2d} \lesssim R^{2d} \eta^{d-1}.\]
(10.61)

In a similar way, by recalling (10.28), we obtain
\[
|(S^{2d-1}_1 \times B^d_R) + \cap U^i_{m+2}| \lesssim R^{2d} \eta^{d-1}.\]
(10.62)

Therefore, recalling (10.57), using estimates (10.58)-(10.62) and the facts that $s \geq 1$, $\eta << 1 << R$, sub-additivity implies
\[
|B_m(\tilde{Z}_m)| \lesssim mR^{2d} \eta^{d-1} < mR^{2d} \eta^{d-1} \frac{d-1}{d+2},\]
(10.63)
since \( \eta << 1 \). We note that the last inequality in (10.63) is provided so that we can combine it with the estimate on \( B^+_m(\bar{Z}_m) \) which is obtained in (10.99).

**Estimate of \( B^+_m(\bar{Z}_m) \):** Recalling (10.55), (10.54), (10.48), we obtain

\[
B^+_m(\bar{Z}_m) = (S_1^{2d-1} \times B_{R}^{2d})^+ (\bar{v}_m) \cap \\
\left[ V_{m,m+1}^* \cup V_{m,m+2}^* \cup U_{m+1,m+2}^* \cup \bigcup_{i=1}^{m-1} (V_m^{i,*} \cup U_m^{i,*} \cup U_{m+2}^{i,*}) \right], \tag{10.64}
\]

where \( V_{m,m+1}^*, V_{m,m+2}^*, U_{m+1,m+2}^*, V_m^{i,*}, U_m^{i,*}, U_{m+2}^{i,*} \) are given by (10.50)-(10.51), (10.53), (10.43)-(10.45).

To estimate the measure of \( B^+_m(\bar{Z}_m) \), we will strongly rely on the properties of the transition map defined in Proposition 9.7.

Let us define the smooth map \( \Phi_{\bar{v}_m} : \mathbb{R}^{2d} \to \mathbb{R} \) given by:

\[
\Phi_{\bar{v}_m}(v_{m+1}, v_{m+2}) = |v_{m+1} - \bar{v}_m|^2 + |v_{m+2} - \bar{v}_m|^2 + |v_{m+1} - v_{m+2}|^2. \tag{10.65}
\]

We easily calculate

\[
|\nabla \Phi_{\bar{v}_m}(v_{m+1}, v_{m+2})|^2 = 4|v_{m+1} - \bar{v}_m|^2 + 4|v_{m+2} - \bar{v}_m|^2 + 16|v_{m+1} - v_{m+2}|^2.
\]

Therefore, given \( r > 0 \) and \((v_{m+1}, v_{m+2}) \in \Phi_{\bar{v}_m}^{-1}(\{r^2\})\), we have the estimate:

\[
2r \leq |\nabla \Phi_{\bar{v}_m}(v_{m+1}, v_{m+2})| \leq 4r. \tag{10.66}
\]

Let also define the set

\[
G_{R}^{2d}(\bar{v}_m) := [0 \leq \Phi_{\bar{v}_m} \leq 16R^2] \tag{10.67}
\]

\[
= \{(v_{m+1}, v_{m+2}) \in \mathbb{R}^{2d} : |v_{m+1} - \bar{v}_m|^2 + |v_{m+2} - \bar{v}_m|^2 + v_{m+2}^2 \leq 16R^2\}.
\]
By the triangle inequality and the fact that $\bar{v}_m \in B_R^{2d}$ we have

$$B_R^{2d} \subseteq G_R^{2d}(\bar{v}_m).$$  \hspace{1cm} (10.68)$$

Recall from (9.21) the set:

$$S^+_{\bar{v}_m, v_{m+1}, v_{m+2}} := \{ (\omega_1, \omega_2) \in S_{2d-1}^1 : b(\omega_1, \omega_2, v_{m+1} - \bar{v}_m, v_{m+2} - \bar{v}_m) > 0 \}$$

$$= \{ (\omega_1, \omega_2) \in S_{2d-1}^1 : (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S_{2d-1}^1 \times B_R^{2d})^+(\bar{v}_m) \}. \hspace{1cm} (10.69)$$

Using notation from (10.13), (10.65) and (10.67)-(10.69), Fubini’s Theorem and the co-area formula yield

$$|B_m^+(\bar{Z}_m)| \hspace{1cm} (10.70)$$

$$= \int_{(S_{2d-1}^1 \times B_R^{2d})^+(\bar{v}_m)} 1_{B_m^+(\bar{Z}_m)} \ d\omega_1 \ d\omega_2 \ dv_{m+1} \ dv_{m+2}$$

$$= \int_{B_R^{2d}(\bar{v}_m)} \int_{S^+_{\bar{v}_m, v_{m+1}, v_{m+2}}} 1_{B_m^+(\bar{Z}_m)} \ d\omega_1 \ d\omega_2 \ dv_{m+1} \ dv_{m+2}$$

$$\leq \int_{G_R^{2d}(\bar{v}_m)} \int_{S^+_{\bar{v}_m, v_{m+1}, v_{m+2}}} 1_{B_m^+(\bar{Z}_m)} \ d\omega_1 \ d\omega_2 \ dv_{m+1} \ dv_{m+2}$$

$$= \int_0^{16R^2} \int_{\Phi_{\bar{v}_m}^{-1}(\{s\})} |\nabla \Phi_{\bar{v}_m}(v_{m+1}, v_{m+2})|^{-1} \int_{S_{v_{m+1}, v_{m+2}}^+} 1_{B_m^+(\bar{Z}_m)} \ d\omega_1 \ d\omega_2 \ dv_{m+1} \ dv_{m+2} \ ds$$

$$= \int_0^{4R} 2r \int_{\Phi_{\bar{v}_m}^{-1}(\{r^2\})} |\nabla \Phi_{\bar{v}_m}(v_{m+1}, v_{m+2})|^{-1} \int_{S_{v_{m+1}, v_{m+2}}^+} 1_{B_m^+(\bar{Z}_m)} \ d\omega_1 \ d\omega_2 \ dv_{m+1} \ dv_{m+2} \ dr,$$

$$\leq \int_0^{4R} \int_{\Phi_{\bar{v}_m}^{-1}(\{r^2\})} \int_{S_{v_{m+1}, v_{m+2}}^+} 1_{B_m^+(\bar{Z}_m)}(\omega_1, \omega_2) \ d\omega_1 \ d\omega_2 \ dv_{m+1} \ dv_{m+2} \ dr, \hspace{1cm} (10.71)$$

where to obtain (10.71) we use the lower bound of (10.66).

We estimate the integral:

$$\int_{S_{v_{m+1}, v_{m+2}}^+} 1_{B_m^+(\bar{Z}_m)}(\omega_1, \omega_2) \ d\omega_1 \ d\omega_2,$$

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for fixed $0 < r \leq 4R$ and $(v_{m+1}, v_{m+2}) \in \Phi^{-1}_{\delta_m}(\{r^2\})$. Let us introduce a parameter $0 < \beta < 1$, which will be chosen later in terms of $\eta$. Writing

$$\omega = (\omega_1, \omega_2), \quad v = (v_{m+1} - \bar{v}_m, v_{m+2} - \bar{v}_m),$$

we notice that

$$b(\omega, v) = (\omega_1, v_{m+1} - \bar{v}_m) + (\omega_2, v_{m+2} - \bar{v}_m)$$

$$= (\omega, v).$$

Inspired in part by [24] (Proposition 8.3), we may decompose \( S^+_{v_m, v_{m+1}, v_{m+2}} \) as follows:

$$S^+_{v_m, v_{m+1}, v_{m+2}} = S^{1,+}_{v_m, v_{m+1}, v_{m+2}} \cup S^{2,+}_{v_m, v_{m+1}, v_{m+2}},$$

where

$$S^{1,+}_{v_m, v_{m+1}, v_{m+2}} = \{ \omega \in S^+_{v_m, v_{m+1}, v_{m+2}} : b(\omega, v) > \beta|v| \}$$

$$= \{ \omega \in S^+_{v_m, v_{m+1}, v_{m+2}} : \langle \omega, v \rangle > \beta|v| \},$$

and

$$S^{2,+}_{v_m, v_{m+1}, v_{m+2}} = \{ \omega \in S^+_{v_m, v_{m+1}, v_{m+2}} : 0 < b(\omega, v) \leq \beta|v| \}$$

$$= \{ \omega \in S^+_{v_m, v_{m+1}, v_{m+2}} : 0 < \langle \omega, v \rangle \leq \beta|v| \}.$$

Notice that \( S^{2,+}_{v_m, v_{m+1}, v_{m+2}} \) is the union of two unit \((2d - 1)\)-spherical caps of angle \(\pi/2 - \arccos \beta\). Thus integrating in spherical coordinates, we may estimate the measure of \( S^{2,+}_{v_m, v_{m+1}, v_{m+2}} \) as follows:

$$\int_{S^2_{v_m, v_{m+1}, v_{m+2}}} 1 \ d\omega_1 \ d\omega_2 \lesssim 2 \int_{\arccos \beta}^{\pi/2} \sin^{2d-2}(\theta) \ d\theta \lesssim \frac{\pi}{2} - \arccos \beta = \arcsin \beta.$$
Thus
\[
\int_{\mathbb{S}_{\bar{v}_{m},v_{m+1},v_{m+2}}^{\perp}} 1_{\mathbb{B}^+_{\bar{v}_{m}}(Z_m)}(\omega_1, \omega_2) \, d\omega_1 \, d\omega_2 \lesssim \arcsin \beta.
\] (10.78)

Let us estimate the terms corresponding to \( S_{\bar{v}_{m},v_{m+1},v_{m+2}}^{1,+} \). Our purpose is to change variables under the transition map \( J_{\bar{v}_{m},v_{m+1},v_{m+2}} \), and use part (v) of Proposition 9.7.

Given \( \omega \in S_{\bar{v}_{m},v_{m+1},v_{m+2}}^{1,+}, \) (9.28) from Proposition 9.7 implies that the Jacobian matrix of the transition map is estimated by
\[
r^{-2d}b^{-2d}(\omega, v) \lesssim \text{Jac}(J_{\bar{v}_{m},v_{m+1},v_{m+2}})(\omega) \lesssim r^{-2d}b^{-2d}(\omega, v).
\]

Thus for \( \omega \in S_{\bar{v}_{m},v_{m+1},v_{m+2}}^{1,+}, \) (10.76) implies
\[
\text{Jac}^{-1}(J_{\bar{v}_{m},v_{m+1},v_{m+2}})(\omega) \lesssim r^{2d}b^{2d}(\omega, v) \leq r^{2d}b^{-2d}v^{-2d} \lesssim \beta^{-2d},
\] (10.79)
since
\[
|v_{m+1} - v_{m+2}|^2 \leq 2(|\bar{v}_{m} - v_{m+1}|^2 + |\bar{v}_{m} - v_{m+2}|^2),
\]
hence
\[
r^2 = |\bar{v}_{m} - v_{m+1}|^2 + |\bar{v}_{m} - v_{m+2}|^2 + |v_{m+1} - v_{m+2}|^2 \leq 3|v|^2.
\]

By (10.64) and (10.76), we have
\[
S_{\bar{v}_{m},v_{m+1},v_{m+2}}^{1,+} \cap \mathbb{B}^+_{\bar{v}_{m}}(Z_m) = \mathbb{S}_{\bar{v}_{m},v_{m+1},v_{m+2}}^{1,+} \cap \left[ V_{m,m+1}^* \cup V_{m,m+2}^* \cup U_{m+1,m+2}^* \cup \bigcup_{i=1}^{m-1} (V_{m}^i \cup U_{m+1}^i \cup U_{m+2}^i) \right].
\] (10.80)
Estimate of the terms corresponding to \( V_{m,m+1}^*, V_{m,m+2}^*, U_{m+1,m+2}^* \): By recalling (10.50)

\[
V_{m,m+1}^* = \{ (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in S_1^{2d-1} : \bar{v}_m^* - v_{m+1}^* \in B_{\eta}^d \},
\]

and (9.25), given \( \omega = (\omega_1, \omega_2) \in S_{v_m,v_{m+1},v_{m+2}}^1 \), we have

\[
\bar{v}_m^* - v_{m+1}^* \in B_{\eta}^d \iff \nu = (\nu_1, \nu_2) \in B_{\eta/r}^d \times \mathbb{R}^d.
\] (10.81)

Therefore, we obtain

\[
\int_{S_{v_m,v_{m+1},v_{m+2}}^1} 1_{V_{m,m+1}^*}(\omega) \, d\omega = \int_{S_{v_m,v_{m+1},v_{m+2}}^1} (1_{B_{\eta/r}^d \times \mathbb{R}^d} \circ J_{v_m,v_{m+1},v_{m+2}})(\omega) \, d\omega
\]

\[
\lesssim \beta^{-2d} \int_{S_{v_m,v_{m+1},v_{m+2}}^1} (1_{B_{\eta/r}^d \times \mathbb{R}^d} \circ J_{v_m,v_{m+1},v_{m+2}})(\omega) \text{Jac} J_{v_m,v_{m+1},v_{m+2}}(\omega) \, d\omega
\] (10.82)

\[
\lesssim \beta^{-2d} \int_{\mathbb{R}^{2d-1}} 1_{B_{\eta/r}^d \times \mathbb{R}^d}(\nu) \, d\nu
\] (10.83)

\[
\lesssim \beta^{-2d} \min \left\{ 1, \left( \frac{\eta}{r} \right)^{\frac{d-1}{2}} \right\}
\] (10.84)

where to obtain (10.82) we use (10.79), to obtain (10.83) we use part (v) of Proposition 9.7, and to obtain (10.84) we use part (iii) of Lemma 9.11. Thus

\[
\int_{S_{v_m,v_{m+1},v_{m+2}}^1} 1_{V_{m,m+1}^*}(\omega_1, \omega_2) \, d\omega_1 \, d\omega_2 \lesssim \beta^{-2d} \min \left\{ 1, \left( \frac{\eta}{r} \right)^{\frac{d-1}{2}} \right\}.
\] (10.85)

In a similar manner, recalling from (10.51) and (10.53) the sets

\[
V_{m,m+2}^* = \{ (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in S_1^{2d-1} : \bar{v}_m^* - v_{m+2}^* \in B_{\eta}^d \},
\]

\[
U_{m+1,m+2}^* = \{ (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in S_1^{2d-1} : v_{m+1}^* - v_{m+2}^* \in B_{\eta}^d \},
\]

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and \((9.25)\), given \((\omega_1, \omega_2) \in S_{v_m, v_{m+1}, v_{m+2}}^{1+}\), we have

\((\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in V^*_{m+2} \iff (\nu_1, \nu_2) \in \mathbb{R}^d \times B_{\eta/r}^d,\)

\((\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in U^*_{m+1, m+2} \iff (\nu_1, \nu_2) \in W^2_{\eta/r, 1, 1},\)

where

\(W^2_{\eta/r, 1, 1} = \{(\nu_1, \nu_2) \in \mathbb{R}^d : \nu_1 - \nu_2 \in B_{\eta/r}^d\}.\)

Using parts \((iv), (v)\) of Lemma \(9.11\) respectively, we similarly obtain the estimates

\[
\int_{S_{v_m, v_{m+1}, v_{m+2}}^{1+}} 1_{V^*_{m, m+2}}(\omega_1, \omega_2) \, d\omega_1 \omega_2 \lesssim \beta^{-2d} \min \left\{ 1, \left(\frac{\eta}{r}\right)^{\frac{d+1}{2}} \right\}, \quad (10.86)
\]

\[
\int_{S_{v_m, v_{m+1}, v_{m+2}}^{1+}} 1_{U^*_{m+1, m+2}}(\omega_1, \omega_2) \, d\omega_1 \omega_2 \lesssim \beta^{-2d} \min \left\{ 1, \left(\frac{\eta}{r}\right)^{\frac{d+1}{2}} \right\}. \quad (10.87)
\]

- Estimate of the terms corresponding to \(V^i_m, U^i_{m+1}, U^i_{m+2}, i \in \{1, ..., m - 1\}\): Consider \(i \in \{1, ..., m - 1\}\). By recalling \((10.43)\)

\[V^i_m = \{(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in S_1^{2d-1} \times B_R^d : \vec{v}^* \in K^d_{\eta/i}\} = \{(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in S_1^{2d-1} \times B_R^d : (\vec{v}^*, v^*_{m+1}) \in K^d_{\eta/i} \times \mathbb{R}^d\},\]

and the operator \(S_{12}\) defined in \((9.54),\) Lemma \(9.9\) implies

\[(\vec{v}^*, v^*_{m+1}) \in K^d_{\eta/i} \times \mathbb{R}^d \iff (S_{12} \circ J_{\bar{v}_m, v_{m+1}, v_{m+2}})(\omega_1, \omega_2) \in K^d_{\eta/r} \times \mathbb{R}^d, \quad (10.88)\]

where \(K^d_{\eta/r}, \bar{K}^d_{\eta/r}\) are \(d\)-cylinders of radius \(\eta/r\). We also recall from \((9.57)\), the ellipsoid

\[S = S_{12}(\mathbb{R}^{2d-1}) = \left\{(\nu_1, \nu_2) \in \mathbb{R}^{2d} : |\nu_1|^2 + |\nu_2|^2 + \langle \nu_1, \nu_2 \rangle = \frac{3}{2} \right\}.\]

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Using the same reasoning to change variables under $J_{v_m, v_{m+1}, v_{m+2}}$ as in the estimate for $V_{m,m+1}^*$, we obtain

$$
\int_{S_{v_m, v_{m+1}, v_{m+2}}} 1_{V_i, i'}(\omega_1, \omega_2) d\omega_1 d\omega_2 =
$$

$$
= \int_{S_{v_m, v_{m+1}, v_{m+2}}} 1_{(v_m, v_{m+1}) \in K_{\eta}^d} (\omega_1, \omega_2) d\omega_1 d\omega_2
$$

$$
= \int_{S_{v_m, v_{m+1}, v_{m+2}}} (1_{K_{\eta}^d} \circ S_{12} \circ J_{v_m, v_{m+1}, v_{m+2}}) (\omega_1, \omega_2) d\omega_1 d\omega_2
$$

$$
\lesssim \beta^{-2d} \int_{E_1^{d-1}} 1_{K_{\eta/r}^d \times \mathbb{R}^d} (\nu_1, \nu_2) d\nu_1 d\nu_2
$$

$$
\lesssim \beta^{-2d} \int_{E_1^{d-1}} 1_{K_{\eta/r}^d \times \mathbb{R}^d} (\theta_1, \theta_2) d\theta_1 d\theta_2
$$

$$
\lesssim \beta^{-2d} \min \left\{ 1, \left( \frac{\eta}{r} \right)^{d-1} \right\}
$$

(10.89)

(10.90)

(10.91)

(10.92)

where to obtain (10.89) we use (10.88), to obtain (10.90) we use estimate (10.79) and part (v) of Proposition (9.7), to obtain (10.91) we use the substitution $(\theta_1, \theta_2) = S_{12}(\nu_1, \nu_2)$, and to obtain (10.92) we use part (i) of Lemma 9.11.

Therefore,

$$
\int_{S_{v_m, v_{m+1}, v_{m+2}}} 1_{V_i, i'}(\omega_1, \omega_2) d\omega_1 d\omega_2 \lesssim \beta^{-2d} \min \left\{ 1, \left( \frac{\eta}{r} \right)^{d-1} \right\}
$$

(10.93)

Recalling $U_{m+1}^*, U_{m+2}^*$ from (10.44) and (10.45), and using respectively the map $S_{12}$ from Lemma 9.9 and estimate (ii) from Proposition 9.11, the map $S_{13}$ from Lemma 9.9 and estimate (ii) from Proposition 9.11, we obtain in an analogous
way the following estimates:

\[
\int_{S_{vm+1}^{1+}} 1_{U_{m+1}^{i,*}}(\omega_1, \omega_2) \, d\omega_1 \, d\omega_2 \lesssim \beta^{-2d} \min \left\{ 1, \left( \frac{\eta}{r} \right)^{\frac{d-1}{2}} \right\}.
\]  \tag{10.94}

\[
\int_{S_{vm+1}^{1+}} 1_{U_{m+2}^{i,*}}(\omega_1, \omega_2) \, d\omega_1 \, d\omega_2 \lesssim \beta^{-2d} \min \left\{ 1, \left( \frac{\eta}{r} \right)^{\frac{d-1}{2}} \right\}.
\]  \tag{10.95}

Combining (10.85)-(10.87), (10.93)-(10.95) and (10.80) we obtain

\[
\int_{S_{vm+1}^{1+}} 1_{B_{m+1}^{i}(Z_m)}(\omega_1, \omega_2) \, d\omega_1 \, d\omega_2 \lesssim m \beta^{-2d} \min \left\{ 1, \left( \frac{\eta}{r} \right)^{\frac{d-1}{2}} \right\}.
\]  \tag{10.96}

Therefore, recalling (10.75), and using estimates (10.78), (10.96), we obtain the estimate:

\[
\int_{S_{vm+1}^{1+}} 1_{B_{m}^{i}(Z_m)}(\omega_1, \omega_2) \, d\omega_1 \, d\omega_2 \lesssim \arcsin \beta + m \beta^{-2d} \min \left\{ 1, \left( \frac{\eta}{r} \right)^{\frac{d-1}{2}} \right\}.
\]  \tag{10.97}

Hence, (10.71) yields

\[
|B_{m}(Z_m)| \lesssim \int_0^{4R} \int_{ \Phi_{vm}^{-1}(r^2)} \arcsin \beta + m \beta^{-2d} \min \left\{ 1, \left( \frac{\eta}{r} \right)^{\frac{d-1}{2}} \right\} \, dv_{m+1} \, dv_{m+2} \, dr
\]

\[
\lesssim \int_0^{4R} r^{2d-1} \left( \arcsin \beta + m \beta^{-2d} \min \left\{ 1, \left( \frac{\eta}{r} \right)^{\frac{d-1}{2}} \right\} \right) \, dr
\]

\[
\lesssim m R^{2d} \left( \arcsin \beta + \beta^{-2d} \eta^{\frac{d-1}{2}} \right)
\]

\[
\lesssim m R^{2d} \left( \beta + \beta^{-2d} \eta^{\frac{d-1}{2}} \right),
\]  \tag{10.98}

after using an estimate similar to (9.1) and the fact that \(m \geq 1, \beta << 1\). Choosing \(\beta = \eta^{\frac{d-1}{4d+2}} << 1\), since \(d \geq 2\), we obtain

\[
|B_{m}(Z_m)| \lesssim m R^{2d} \eta^{\frac{d-1}{4d+2}}.
\]  \tag{10.99}

Combining (10.56), (10.63), (10.99), we obtain the required estimate. □
Chapter 11

Elimination of recollisions

In this chapter, we reduce the convergence proof to comparing truncated elementary observables. We first restrict to good configurations and provide the corresponding measure estimate. This is presented in Proposition 11.3. We then inductively apply Proposition 10.4 and Proposition 10.7 to reduce the convergence proof to truncated elementary observables. The convergence proof, completed in Chapter 12, will then follow naturally, since the backwards interaction flow and the free flow will be comparable out of a small measure set.

11.1 Restriction to good configurations

Inductively using Lemma 10.2 we are able to reduce the convergence proof to good configurations, up to a small measure set. The measure of the complement will be negligible in the limit. Throughout this section, we consider $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$, $T = T(d, \beta_0, \mu_0) > 0$ given by (6.18), the functions $\beta, \mu : [0, T] \to \mathbb{R}$ defined by (6.15), $(N, \epsilon)$ in the scaling (5.43) and initial data $F_{N,0} \in X_{N,\beta_0,\mu_0}$, $F_0 \in X_{\infty,\beta_0,\mu_0}$.

Let $F_N \in X_{N,\beta,\mu}$, $F \in X_{\infty,\beta,\mu}$ be the mild solutions of the corresponding BBGKY hierarchy and Boltzmann hierarchy, respectively, in $[0, T]$, given by Theorem 6.4.1 and Theorem 6.8.1.
For the convenience of a reader we recall the notation from Chapter 10. Specifically, given \( m \in \mathbb{N}, \sigma > 0 \) and \( t_0 > 0 \), thanks to (10.1)-(10.3) we have

\[
\Delta_1(\sigma) = \mathbb{R}^{2d},
\]
\[
\Delta_m(\sigma) = \{ \tilde{Z}_m = (\tilde{X}_m, \tilde{V}_m) \in \mathbb{R}^{2dm} : |\tilde{x}_i - \tilde{x}_j| > \sigma, \ \forall 1 \leq i < j \leq m \}, \ m \geq 2,
\]
\[
G_m(\sigma, t_0) = \{ Z_m = (X_m, V_m) \in \mathbb{R}^{2dm} : Z_m(t) \in \Delta_m(\sigma), \ \forall t \geq t_0 \},
\]
where given \( Z_m = (X_m, V_m) \in \mathbb{R}^{2dm}, Z_m(t) \) denotes the backwards free flow, given by:

\[
Z_m(t) = (X_m - tV_m, V_m), \ t \geq 0,
\]
where we recall that \( Z_m \) is the initial point of the trajectory i.e. \( Z_m(0) = Z_m \). Let also recall from (7.1)-(7.2) the set \( \Delta^X_m(\sigma) \) of well-separated spatial configurations:

\[
\Delta^X_m(\sigma) = \{ \tilde{X}_m \in \mathbb{R}^{ds} : |\tilde{x}_i - \tilde{x}_j| > \epsilon_0, \ \forall 1 \leq i < j \leq m \}, \ m \geq 2,
\]
\[
\Delta^X_1(\epsilon_0) = \mathbb{R}^d.
\]

Given \( \epsilon, \epsilon_0 > 0 \) with \( \epsilon << \epsilon_0 \) and \( \delta > 0 \), we define the new set

\[
G_m(\epsilon, \epsilon_0, \delta) := G_m(\epsilon, 0) \cap G_m(\epsilon_0, \delta).
\]

**Proposition 11.2.** Let \( s \in \mathbb{N}, \alpha, \epsilon_0, R, \eta, \delta \) be parameters as in (10.6) and \( \epsilon << \alpha \). Then for any \( X_s \in \Delta^X_s(\epsilon_0) \), there is a subset of velocities \( M_s(X_s) \subseteq B^d_{R} \) of measure

\[
|M_s(X_s)|_{ds} \leq C_{d,s} R^{d \epsilon_0 - \frac{1}{2}},
\]

such that

\[
Z_s = (X_s, V_s) \in G_s(\epsilon, \epsilon_0, \delta), \ \forall V_s \in B^d_{R} \setminus M_s(X_s).
\]
Proof. Let us fix $X_s \in \Delta^X_s(\epsilon_0)$. We prove the following claim:

**Claim:** For any $m \in \{1, \ldots, s\}$ there is a set $N_m(X_m) \subseteq \prod_{i=1}^m B_R^d$ of measure

$$|N_m(X_m)|_{dm} \leq C_{d,m} R^{dm} \eta \frac{d-1}{2},$$

such that

$$Z_m = (X_m, V_m) \in G_m(\epsilon, \epsilon_0, \delta), \quad \forall V_m \in \left(\prod_{i=1}^m B_R^d\right) \setminus N_m(X_m).$$

**Proof of the claim:** We will proceed by induction. For $m = 1$, the claim follows by choosing $N_1(X_1) = \emptyset$.

Assume the claim holds for $m \in \{1, \ldots, s-1\}$ i.e. there is a set $N_m(X_m)$ satisfying (11.4)-(11.5). We want to show the claim holds for $m + 1$.

Fix $V_m = (v_1, \ldots, v_m) \in \left(\prod_{i=1}^m B_R^d\right) \setminus N_m(X_m)$. Since $X_s \in \Delta^X_s(\epsilon_0)$ and $m \in \{1, \ldots, s-1\}$, we have that

$$|x_i - x_{m+1}| > \epsilon_0, \quad \forall i = 1, \ldots, m.$$ 

Therefore, for each fixed $i \in \{1, \ldots, m\}$ we may apply Lemma 10.2 with $y_1 = \bar{y}_1 = x_i$ and $y_2 = \bar{y}_2 = x_{m+1}$ to find a cylinder $K^d_\eta(v_i, x_i - x_{m+1})$ such that:

$$(x_i, x_{m+1}, v_i, v_{m+1}) \in G^2_2(\epsilon, \epsilon_0, \delta), \quad \forall v_{m+1} \in B_R^d \setminus K^d_\eta(v_i, x_i - x_{m+1}).$$ (11.6)

Writing

$$K_{m+1}(X_{m+1}, V_m) := \bigcup_{i=1}^m K^d_\eta(v_i, x_i - x_{m+1}),$$ (11.7)
and defining

\[ N_{m+1}(X_{m+1}) = \left\{ V_{m+1} \in \prod_{i=1}^{m+1} B^d_R : V_m \in N_m(X_m) \right\} \cup \]
\[ \cup \left\{ V_{m+1} \in \prod_{i=1}^{m+1} B^d_R : V_m \notin N_m(X_m) \text{ and } v_{m+1} \in K_{m+1}(X_{m+1}, V_m) \right\}. \]

(11.8)

(11.5)-(11.6) yield

\[(X_{m+1}, V_{m+1}) \in G_{m+1}(\epsilon, \epsilon_0, \delta), \quad \forall V_{m+1} \in (\prod_{i=1}^{m+1} B^d_R) \setminus N_{m+1}(X_{m+1}). \quad (11.9)\]

(11.8), Fubini’s Theorem and (11.4), (11.7) yield

\[ |N_{m+1}(X_{m+1})|_{d(m+1)} \leq \]
\[ \leq C_d R^d |N_m(X_m)|_{d m} + \int (\prod_{i=1}^m B^d_R) \setminus N_m(X_m) \int B^d_R \mathbb{1}_{K_{m+1}(X_{m+1}, V_m)}(v_{m+1}) \, dv_{m+1} \, dV_m \]
\[ \leq C_{d,m} R^{d(m+1)} \eta^{-d-1} \sum_{i=1}^m \int (\prod_{i=1}^m B^d_R) \setminus N_m(X_m) \int B^d_R \mathbb{1}_{K^d_{\eta}(v_i, x_i - x_{m+1})}(v_{m+1}) \, dv_{m+1} \, dV_m \]
\[ \leq \left( C_{d,m} + m \prod_{i=1}^m B^d_1 \right) R^{d(m+1)} \eta^{-d-1} \]
\[ := C_{d,m+1} R^{d(m+1)} \eta^{-d-1}, \quad (11.10)\]

where to obtain (11.10) we use part (i) of Proposition 9.3. Clearly (11.9), (11.11) complete the induction and the claim is proved.

The set

\[ M_s(X_s) := B^d_{R_s} \cap N_s(X_s), \quad (11.12)\]

satisfies (11.2)-(11.3) and the proof is complete. \qed
Consider $s, n \in \mathbb{N}$, parameters $\alpha, \epsilon_0, R, \eta, \delta$ as in (10.6), $(N, \epsilon)$ in the scaling (5.43) with $\epsilon \ll \alpha$, $0 \leq k \leq n$ and $t \in [0, T]$. Let us recall the observables $I_{s,k,R,\delta}^N(t)$, $I_{s,k,R,\delta}^\infty(t)$ defined in (8.27)-(8.28). We restrict the domain of integration to velocities giving good configurations. For convenience, given $X_s \in \Delta^X_s(\epsilon_0)$, we write

$$M^c_s(X_s) = B_{ds}^R \setminus M_s(X_s).$$

We define

$$I_{s,k,R,\delta}^N(t)(X_s) = \int_{M^c_s(X_s)} \phi_s(V_s) f^{(s,k)}_{N,R,\delta}(t, X_s, V_s) dV_s,$$  

(11.13)

$$I_{s,k,R,\delta}^\infty(t)(X_s) = \int_{M^c_s(X_s)} \phi_s(V_s) f^{(s,k)}_{R,\delta}(t, X_s, V_s) dV_s.$$  

(11.14)

Let us apply Proposition 11.2 to restrict to initially good configurations.

**Proposition 11.3.** Let $s, n \in \mathbb{N}$, $\alpha, \epsilon_0, R, \eta, \delta$ be parameters as in (10.6), $(N, \epsilon)$ in the scaling (5.43) with $\epsilon \ll \alpha$, and $t \in [0, T]$. Then, the following estimates hold:

$$\sum_{k=0}^n \| I_{s,k,R,\delta}^N(t) - I_{s,k,R,\delta}^N(t) \|_{L^\infty(\Delta^X_s(\epsilon_0))} \leq C_{d,s,\mu_0,T} R^{ds/d} \eta^{d+1/2} \| F_{N,0} \|_{N,\beta_0,\mu_0},$$

$$\sum_{k=0}^n \| I_{s,k,R,\delta}^\infty(t) - I_{s,k,R,\delta}^\infty(t) \|_{L^\infty(\Delta^X_s(\epsilon_0))} \leq C_{d,s,\mu_0,T} R^{ds/d} \eta^{d+1/2} \| F_0 \|_{\infty,\beta_0,\mu_0}.$$  

**Proof.** We present the proof for the BBGKY hierarchy case only. The proof for the Boltzmann hierarchy case is analogous. Let us fix $X_s \in \Delta^X_s(\epsilon_0)$.

We first assume that $k \in \{1, ..., n\}$. Applying $k-1$ times estimate (6.17) from
Theorem 6.4.1 and part (ii) of Proposition 6.2 we obtain

\[ |I_{s,k,R,\delta}(t)(X_s) - \tilde{I}_{s,k,R,\delta}(t)(X_s)| \leq \int_{M_s(X_s)} |\phi_s(V_s)f(s,k)_{N,R,\delta}(t, X_s, V_s)| dV_s \]

\[ \leq T\|\phi_s\|_{L^\infty} e^{-s\mu(T)} \left( \frac{1}{8} \right)^{k-1} \|F_{N,0}\|_{N,\beta_0,\mu_0} \int_{M_s(X_s)} e^{-\beta(T)E_s(Z_s)} dV_s \]

\[ \leq T\|\phi_s\|_{L^\infty} e^{-s\mu(T)} \left( \frac{1}{8} \right)^{k-1} \|M_s(X_s)\| \|F_{N,0}\|_{N,\beta_0,\mu_0}. \quad (11.15) \]

For \( k = 0 \), Remark (6.3), part (i) of Proposition 6.2 and Remark 6.3 yield

\[ |I_{s,0,R,\delta}(t)(X_s) - \tilde{I}_{s,0,R,\delta}(t)(X_s)| \leq \int_{M_s(X_s)} |\phi_s(V_s)\rho(s,0)_{N,R,\delta}(t, X_s, V_s)| dV_s \]

\[ \leq \|\phi_s\|_{L^\infty} e^{-s\mu(T)} \|T^f F_{N,0}\|_{N,\beta_0,\mu} \int_{M_s(X_s)} e^{-\beta(T)E_s(Z_s)} dV_s \]

\[ \leq \|\phi_s\|_{L^\infty} e^{-s\mu(T)} \|T^f F_{N,0}\|_{N,\beta_0,\mu} \|M_s(X_s)\| \|F_{N,0}\|_{N,\beta_0,\mu_0} \]

\[ = \|\phi_s\|_{L^\infty} e^{-s\mu(T)} \|M_s(X_s)\| \|F_{N,0}\|_{N,\beta_0,\mu_0} \quad (11.16) \]

The claim comes after using (11.15)-(11.16), adding over \( k = 0, \ldots, n \), and using the measure estimate of Proposition 11.2.

\[ \square \]

Remark 11.1. Under the assumptions of Proposition 11.3, given \( X_s \in \Delta_s^X(\epsilon_0) \), the definition of \( M_s(X_s) \) implies that:

\[ \tilde{I}_{s,0,R,\delta}(t)(X_s) = \tilde{I}_{s,0,R,\delta}^\infty(t)(X_s), \quad \forall t \in [0, T]. \]

Therefore, Proposition 11.3 allows us to reduce the convergence to controlling the differences

\[ \tilde{I}_{s,k,R,\delta}(t) - \tilde{I}_{s,k,R,\delta}^\infty(t), \]

for \( k = 1, \ldots, n \), in the scaled limit.
11.4 Reduction to elementary observables

In this section, given \( s, n \in \mathbb{N} \), parameters \( \alpha, \epsilon, R, \eta, \delta \) as in (10.6) \( 1 \leq k \leq n \), \((N, \epsilon)\) in the scaling (5.43) with \( \epsilon \ll \alpha \), and \( t \in [0, T] \), inspired by notation used in [49] and [33], we express the observables \( \hat{I}^N_{s, k, R, \delta}(t) \) and \( \hat{I}^\infty_{s, k, R, \delta}(t) \), defined in (11.13)-(11.14), as a superposition of elementary observables.

For this purpose, given \( \ell, N \in \mathbb{N} \) with \( \ell < N \), \( R > 1 \), and recalling (5.40), we decompose the truncated BBGKY hierarchy collisional operator in the following way:

\[
C_{N,R}^\ell,\ell+2 = \sum_{i=1}^{\ell} C_{N,R,+1,i}^\ell,\ell+2 - \sum_{i=1}^{\ell} C_{N,R,-1,i}^\ell,\ell+2,
\]

where

\[
C_{N,R,+1,i}^\ell,\ell+2 g_{\ell+2}(Z_{\ell+2}) := A_{N,\epsilon,\ell}^s \int_{S^d_{2} \times \mathbb{R}^d} b_2(\omega_{\ell+1}, \omega_{\ell+2}, v_{\ell+1} - v_i, v_{\ell+2} - v_i)
\times g_{\ell+2} 1_{|E_{\ell+2} \leq R^2}(Z_{\ell+2,i}) d\omega_{\ell+1} d\omega_{\ell+2} dv_{\ell+1} dv_{\ell+2},
\]

and

\[
C_{N,R,-1,i}^\ell,\ell+2 g_{\ell+2}(Z_{\ell+2}) := A_{N,\epsilon,\ell}^s \int_{S^d_{2} \times \mathbb{R}^d} b_2(\omega_{\ell+1}, \omega_{\ell+2}, v_{\ell+1} - v_i, v_{\ell+2} - v_i)
\times g_{\ell+2} 1_{|E_{\ell+2} \leq R^2}(Z_{\ell+2,i}) d\omega_{\ell+1} d\omega_{\ell+2} dv_{\ell+1} dv_{\ell+2}.
\]

For \( s \in \mathbb{N} \) and \( k \in \mathbb{N} \), let us denote

\[
A_{s,k} := \{ J = (j_1, \ldots, j_k) \in \mathbb{N}^k : j_i \in \{-1, 1\}, \forall i \in \{1, \ldots, k\} \}, \quad (11.17)
\]

\[
B_{s,k} := \{ M = (m_1, \ldots, m_k) \in \mathbb{N}^k : m_i \in \{1, \ldots, s + 2i - 2\}, \forall i \in \{1, \ldots, k\} \}, \quad (11.18)
\]

\[
\mathcal{U}_{s,k} := A_{s,k} \times B_{s,k}. \quad (11.19)
\]

Under this notation, given \( s, n \in \mathbb{N} \), parameters \( \alpha, \epsilon, R, \eta, \delta \) as in (10.6) \( 1 \leq k \leq n \), \((N, \epsilon)\) in the scaling (5.43) with \( \epsilon \ll \alpha \), and \( t \in [0, T] \), the BBGKY hierarchy
Observable functional \( \tilde{I}_{s,k,R,\delta}^N(t) \) can be expressed as a superposition of elementary observables

\[
\tilde{I}_{s,k,R,\delta}^N(t)(X_s) = \sum_{(J,M) \in U_{s,k}} \left( \prod_{i=1}^k j_i \right) \tilde{I}_{s,k,R,\delta}^N(t,J,M)(X_s),
\]

(11.20)

where the elementary observables are defined by

\[
\tilde{I}_{s,k,R,\delta}^N(t,J,M)(X_s) = \hat{M}^{s,c_s}(X_s) \hat{T}_k^{t_1} T_{s,s+2}^{t-t_1} \cdots \sum_{(J,M) \in U_{s,k}} \left( \prod_{i=1}^k j_i \right) \tilde{I}_{s,k,R,\delta}^N(t,J,M)(X_s),
\]

(11.21)

Similarly, given \( \ell, N \in \mathbb{N} \) with \( \ell < N \), \( R > 1 \), and recalling (5.46), we decompose the truncated Boltzmann hierarchy collisional operator as:

\[
C_{\infty,R}^{\ell} = \sum_{i=1}^\ell C_{\infty,R}^{\ell+1,i} - \sum_{i=1}^\ell C_{\infty,R}^{-1,i},
\]

where for \( i = 1, \ldots, \ell \), we denote:

\[
C_{\infty,R}^{\ell+1,i} g_{\ell+2}(Z_{\ell}) := \int_{S_2^{d-1} \times B_R^d} b_2(\omega_{\ell+1}, \omega_{\ell+2}, v_{\ell+1} - v_i, v_{\ell+2} - v_i) \times g_{\ell+2} \mathbb{I}_{[E_{\ell+2} \leq R^2]}(Z_{\ell+2}) \, dw_{\ell+1} \, dw_{\ell+2} \, dv_{\ell+1} \, dv_{\ell+2},
\]

and

\[
C_{\infty,R}^{-1,i} g_{\ell+2}(Z_{\ell}) := \int_{S_2^{d-1} \times B_R^d} b_2(\omega_{\ell+1}, \omega_{\ell+2}, v_{\ell+1} - v_i, v_{\ell+2} - v_i) \times g_{\ell+2} \mathbb{I}_{[E_{\ell+2} \leq R^2]}(Z_{\ell+2}) \, dw_{\ell+1} \, dw_{\ell+2} \, dv_{\ell+1} \, dv_{\ell+2}.
\]

Under this notation, given \( s, n \in \mathbb{N} \), \( t \in [0,T] \), parameters \( \alpha, \epsilon_0, R, \eta, \delta \) as in (10.6), \( 1 \leq k \leq n \), and \( t \in [0,T] \), the Boltzmann hierarchy observable functional \( \tilde{I}_{s,k,R,\delta}^\infty(t) \) can be expressed as a superposition of elementary observables

\[
\tilde{I}_{s,k,R,\delta}^\infty(t)(X_s) = \sum_{(J,M) \in U_{s,k}} \left( \prod_{i=1}^k j_i \right) \tilde{I}_{s,k,R,\delta}^\infty(t,J,M)(X_s),
\]

(11.22)
where the elementary observables are defined by

\[
\tilde{I}_{s,k,R,\delta}^\infty(t,J,M)(X_s) = \int_{M^s_s(X_s)} \phi_s(V_s) \int_{T_{k,\delta}(t)} S^{t-t_1}_{s+1,m_1} S^{t_{k+1}-t_2}_{s+2} \ldots C_{s,R,j,1,m_1}^{\infty}(Z_s) dt_1 \ldots dt_{k+1} dV_s. \tag{11.23}
\]

11.5 Boltzmann hierarchy pseudo-trajectories

In this section, we introduce an explicit discrete backwards in time construction of so-called Boltzmann hierarchy pseudo-trajectory, which lets us keep track of the collisions. Similar constructions, although continuous in time, can be found in [49], [33], [24]. Let \( s \in \mathbb{N} \), \( Z_s = (X_s, V_s) \in \mathbb{R}^{2d} \), \( k \in \mathbb{N} \) and \( t \in [0,T] \). Let us recall from [8.1] the set

\[\mathcal{T}_k(t) = \{(t_1, \ldots, t_k) \in \mathbb{R}^k : 0 = t_{k+1} < t_k < \ldots < t_1 < t_0 = t\},\]

where we use the convention \( t_0 = t \) and \( t_{k+1} = 0 \).

Consider \((t_1, \ldots, t_k) \in \mathcal{T}_k(t)\), \( J = (j_1, \ldots, j_k) \), \( M = (m_1, \ldots, m_k) \), \((J, M) \in U_{s,k}\), and for each \( i = 1, \ldots, k \), we consider \((\omega_{s+2i-1}, \omega_{s+2i}, v_{s+2i-1}, v_{s+2i}) \in S^{2d-1}_{s+1} \times \mathbb{R}^d\).

We inductively define the Boltzmann hierarchy pseudo-trajectory of \( Z_s \). Roughly speaking, the Boltzmann pseudo-trajectory is formulated in the following way:

Assume we are given a configuration \( Z_s = (X_s, V_s) \in \mathbb{R}^{2d} \) at time \( t_0 = t \). \( Z_s \) evolves under backwards free flow until the time \( t_1 \) when a pair of particles \((\omega_{s+1}, \omega_{s+2}, v_{s+1}, v_{s+2})\) is added to the \( m_1 \)-particle, the adjunction being precollisional if \( j_1 = -1 \) and post-collisional if \( j_1 = 1 \). We then form an \((s+2)\)-configuration and continue this process inductively until time \( t_{k+1} = 0 \).
More precisely, we inductively construct the Boltzmann hierarchy pseudo-trajectory of \( Z_s = (X_s, V_s) \in \mathbb{R}^{2ds} \) as follows:

**Time** \( t_0 = t \): We initially define

\[
Z_s^\infty(t_0^+) = (x_s^\infty(t_0^+), \ldots, x_s^\infty(t_0^+), v_s^\infty(t_0^+), \ldots, v_s^\infty(t_0^+)) := Z_s.
\]

**Time** \( t_1 \): We define \( Z_s^\infty(t_1^+) = (x_s^\infty(t_1^+), \ldots, x_s^\infty(t_1^+), v_s^\infty(t_1^+), \ldots, v_s^\infty(t_1^+)) \) as follows:

\[
Z_s^\infty(t_1^+) := (X_s^\infty(t_0^-) - (t - t_1)V_s^\infty(t_0^-), V_s^\infty(t_0^-)) = (X_s - (t - t_1)V_s, V_s).
\]

We also define \( Z_{s+2}^\infty(t_1^-) = (x_{s+2}^\infty(t_1^-), \ldots, x_{s+2}^\infty(t_1^-), v_{s+2}^\infty(t_1^-), \ldots, v_{s+2}^\infty(t_1^-)) \) as follows:

\[
(x_j^\infty(t_1^-), v_j^\infty(t_1^-)) := (x_j^\infty(t_1^+), v_j^\infty(t_1^+)), \quad \forall j \in \{1, \ldots, s\} \setminus \{m_1\},
\]

and if \( j_1 = -1 \):

\[
(x_{m_1}^\infty(t_1^-), v_{m_1}^\infty(t_1^-)) := (x_{m_1}^\infty(t_1^+), v_{m_1}^\infty(t_1^+)) ;
\]

\[
(x_{s+1}^\infty(t_1^-), v_{s+1}^\infty(t_1^-)) := (x_{s+1}^\infty(t_1^+), v_{s+1}^\infty(t_1^+)),
\]

\[
(x_{s+2}^\infty(t_1^-), v_{s+2}^\infty(t_1^-)) := (x_{s+2}^\infty(t_1^+), v_{s+2}^\infty(t_1^+)),
\]

while if \( j_1 = 1 \):

\[
(x_{m_1}^\infty(t_1^-), v_{m_1}^\infty(t_1^-)) := (x_{m_1}^\infty(t_1^+), v_{m_1}^\infty(t_1^+)) ;
\]

\[
(x_{s+1}^\infty(t_1^-), v_{s+1}^\infty(t_1^-)) := (x_{s+1}^\infty(t_1^+), v_{s+1}^\infty(t_1^+)),
\]

\[
(x_{s+2}^\infty(t_1^-), v_{s+2}^\infty(t_1^-)) := (x_{s+2}^\infty(t_1^+), v_{s+2}^\infty(t_1^+)),
\]

where

\[
(v_{m_1}^\infty(t_1^+), v_{s+1}^\infty(t_1^+), v_{s+2}^\infty(t_1^+)) = T_{\omega_{s+1}, \omega_{s+2}} (v_{m_1}^\infty(t_1^+), v_{s+1}, v_{s+2}).
\]

**Time** \( t_i, i \in \{2, \ldots, k\} \): Consider \( i \in \{2, \ldots, k\} \), and assume we know

\[
Z_{s+2i-2}^\infty(t_{i-1}^-) = (x_{s+2i-2}^\infty(t_{i-1}^-), \ldots, x_{s+2i-2}^\infty(t_{i-1}^-), v_1^\infty(t_{i-1}^-), \ldots, v_{s+2i-2}^\infty(t_{i-1}^-)).
\]
We define \( Z_{s+2i-2}^{\infty}(t_i^+) = (x_1^\infty(t_i^+), \ldots, x_{s+2i-2}^\infty(t_i^+), v_1^\infty(t_i^+), \ldots, v_{s+2i-2}^\infty(t_i^+)) \) as:
\[
Z_{s+2i-2}^{\infty}(t_i^+) := (X_{s+2i-2}^\infty(t_{i-1}^-) - (t_{i-1} - t_i) V_{s+2i-2}^\infty(t_{i-1}^-), V_{s+2i-2}^\infty(t_{i-1}^-)) .
\]

We also define \( Z_s^{\infty}(t_i^-) = (x_1^\infty(t_i^-), \ldots, x_s^\infty(t_i^-), v_1^\infty(t_i^-), \ldots, v_s^\infty(t_i^-)) \) as:
\[
(x_j^\infty(t_i^-), v_j^\infty(t_i^-)) := (x_j^\infty(t_i^+), v_j^\infty(t_i^+)) \quad \forall j \in \{1,\ldots,s+2i-2\} \setminus \{m_i\} ,
\]

and if \( j_i = -1 \):
\[
(x_{m_i}^\infty(t_i^-), v_{m_i}^\infty(t_i^-)) := (x_{m_i}^\infty(t_i^+), v_{m_i}^\infty(t_i^+)) ,
\]
\[
(x_{s+2i-1}^\infty(t_i^-), v_{s+2i-1}^\infty(t_i^-)) := (x_{m_i}^\infty(t_i^+), v_{s+2i-1}^\infty(t_i^+)) ,
\]
\[
(x_{s+2i}^\infty(t_i^-), v_{s+2i}^\infty(t_i^-)) := (x_{m_i}^\infty(t_i^+), v_{s+2i}^\infty(t_i^+)) ,
\]

while if \( j_i = 1 \):
\[
(x_{m_i}^\infty(t_i^-), v_{m_i}^\infty(t_i^-)) := (x_{m_i}^\infty(t_i^+), v_{m_i}^\infty(t_i^+)) ,
\]
\[
(x_{s+2i-1}^\infty(t_i^-), v_{s+2i-1}^\infty(t_i^-)) := (x_{m_i}^\infty(t_i^+), v_{s+2i-1}^\infty(t_i^+)) ,
\]
\[
(x_{s+2i}^\infty(t_i^-), v_{s+2i}^\infty(t_i^-)) := (x_{m_i}^\infty(t_i^+), v_{s+2i}^\infty(t_i^+)) ,
\]

where
\[
(v_{m_i}^\infty(t_i^-), v_{s+2i-1}^\infty(t_i^-), v_{s+2i}^\infty(t_i^-)) = T_{\omega_{s+2i-1},\omega_{s+2i}}(v_{m_i}^\infty(t_i^+), v_{s+2i-1}^\infty(t_i^+), v_{s+2i}^\infty(t_i^+)) .
\]

**Time \( t_{k+1} = 0 \):** We finally obtain
\[
Z_s^{\infty}(0^+) = Z_s^{\infty}(t_{k+1}^+) = (X_{s+2k}^\infty(t_{k+1}^-) - t_k V_{s+2k}^\infty(t_k^-), V_{s+2k}^\infty(t_k^-)) .
\]

The process is illustrated in the following diagram (to be read from right to left):
We give the following definition:

**Definition 11.1.** Let \( Z_s = (X_s, V_s) \in \mathbb{R}^{2d s} \), \((t_1, \ldots, t_k) \in \mathcal{T}_k(t)\), \( J = (j_1, \ldots, j_k) \), \( M = (m_1, \ldots, m_k) \), \((J, M) \in \mathcal{U}_{s,k} \) and for each \( i = 1, \ldots, k \), we consider

\[
(\omega_{s+2i-1}, \omega_{s+2i}, v_{s+2i-1}, v_{s+2i}) \in S^{2d-1}_1 \times B^{2d}_{R}.
\]

The sequence \( \{Z_{s+2i-2}(t^+_i)\}_{i=0,\ldots,k+1} \) constructed above is called the Boltzmann hierarchy pseudo-trajectory of \( Z_s \).

### 11.6 Reduction to truncated elementary observables

We will now use the Boltzmann hierarchy pseudo-trajectory to define the truncated observables for the BBGKY hierarchy and Boltzmann hierarchy. The convergence proof will then be reduced to the convergence of the corresponding truncated elementary observables. Given \( \ell \in \mathbb{N} \), parameters \( \alpha, \epsilon_0, R, \eta, \delta \) as in (10.6) and \( \epsilon < \alpha \), recall the notation from (11.1):

\[
G_\ell(\epsilon, \epsilon_0, \delta) = G_\ell(\epsilon, 0) \cap G_\ell(\epsilon_0, \delta).
\]

Let \( s \in \mathbb{N} \), \( X_s \in \Delta_{s}(\epsilon_0) \), \( 1 \leq k \leq n \), \((J, M) \in \mathcal{U}_{s,k} \) and \( t \in [0, T] \). By Proposition 11.2 for any \( V_s \in \mathcal{M}_c(X_s) \), we have

\[
Z_s = (X_s, V_s) \in G_s(\epsilon, \epsilon_0, \delta) \Rightarrow Z_s^\infty(t^+_1) \in G_s(\epsilon_0, 0).
\]

Recalling notation from (10.13) Proposition 10.4 (see (10.16) for the pre-collisional case or (10.20) for the post-collisional case) yields there is a set \( \mathcal{B}_{m_1}(Z_s^\infty(t^+_1)) \subseteq (S^{2d-1}_1 \times B^{2d}_{R}) \cup (v_c^{\infty}(t^+_1)) \) such that

\[
Z_{s+2}(t^+_2) \in G_{s+2}(\epsilon_0, 0), \quad \forall (\omega_{s+1}, \omega_{s+2}, v_{s+1}, v_{s+2}) \in \mathcal{B}_{m_1}(Z_s^\infty(t^+_1)),
\]

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where
\[ \mathcal{B}_{m_1}^c \left( Z_{s+1}^\infty \left( t_1^+ \right) \right) := \left( S_1^{2d-1} \times B_R^d \right)^+ \left( v_{m_1}^\infty \left( t_1^+ \right) \right) \setminus \mathcal{B}_{m_1} \left( Z_{s}^\infty \left( t_1^+ \right) \right). \]

Clearly this process can be iterated. In particular, given \( i \in \{2, ..., k\} \), we have
\[ Z_{s+2i-2}^\infty (t_i^+) \in G_{s+2i-2}(\epsilon_0, 0), \]
so there exists a set \( \mathcal{B}_{m_i} \left( Z_{s+2i-2}^\infty (t_i^+) \right) \subseteq \left( S_1^{2d-1} \times B_R^d \right)^+ \left( v_{m_i}^\infty \left( t_i^+ \right) \right) \) such that:
\[ Z_{s+2i}^\infty (t_{i+1}^+) \in G_{s+2i}(\epsilon_0, 0), \quad \forall (\omega_{s+2i-1}, \omega_{s+2i}, v_{s+2i-1}, v_{s+2i}) \in \mathcal{B}_{m_i} \left( Z_{s+2i-2}^\infty (t_i^+) \right), \]
where
\[ \mathcal{B}_{m_i} \left( Z_{s+2i-2}^\infty (t_i^+) \right) := \left( S_1^{2d-1} \times B_R^d \right)^+ \left( v_{m_i}^\infty \left( t_i^+ \right) \right) \setminus \mathcal{B}_{m_i} \left( Z_s^\infty \left( t_i^+ \right) \right). \]

We finally obtain \( Z_{s+2k}^\infty (0^+) \in G_{s+2k}(\epsilon_0, 0). \)

Let us now define the truncated elementary observables. Heuristically we will truncate the domains of adjusted particles in the definition of the observables \( \tilde{I}_{s,k,R,\delta} \), \( \tilde{I}_{s,k,R,\delta} \) (see (11.13)-(11.14)).

More precisely, let \( s, n \in \mathbb{N}, \alpha, \epsilon_0, R, \eta, \delta \) be parameters as in (10.6), \( (N, \epsilon) \) in the scaling (5.43) with \( \epsilon \ll \alpha \), \( 1 \leq k \leq n \), \( (J, M) \in \mathcal{U}_{s,k} \) and \( t \in [0, T] \). For \( X_s \in \Delta_s^X(\epsilon_0) \), Proposition 11.2 implies there is a set of velocities \( M_s(X_s) \subseteq B_R^{2d} \) such that
\[ Z_s = (X_s, V_s) \in G_s(\epsilon, \epsilon_0, \delta), \quad \forall V_s \in M_s^c(X_s). \]
Following the reasoning above, we define the BBGKY hierarchy truncated observables as:
\[
J_{s,k,R,\delta}^N(t, J, M)(X_s) := \int_{M_s^c(X_s)} \phi_s(V_s) \int_{J_s(\delta)} T_{t_s}^{t_1} \tilde{c}_{s+2s+2}^{N,R,j_1,m_1} T_{s+2s}^{t_2} \cdots \tilde{c}_{s+2k}^{N,R,j_k,m_k} T_{s+2k}^{t_{k-1}} \int_{0}^{t_{k-1}} (Z_s) dt_k, \ldots \int_{0}^{t_1} dt_1 dV_s,
\]
(11.24)
\[ \tilde{C}^{N,R,j_i,m_i}_{s,2i-2,s+2} := \mathcal{C}^{N,R,j_i,m_i}_{s,2i-2,s+2} \left[ g^{N,s+2i}1_{(\omega_{s+2i-1},\omega_{s+2i},v_{s+2i-1},v_{s+2i}) \in \mathcal{B}_{m_i}(Z_{s+2i}^{\infty})} \right]. \]

In the same spirit, for \( X_s \in \Delta^X_s(\epsilon_0) \), we define the Boltzmann hierarchy truncated elementary observables as:

\[ J^\infty_{s,k,R,\delta}(t,J,M)(X_s) := \int_{\mathcal{N}_s(X_s)} \phi_s(V_s) \int_{\mathcal{T}_s(t)} S^{t-t_1} \mathcal{C}^{\infty,R,j_k,m_k}_{s+2k-2,s+2k} \mathcal{T}^{t_1-t_2}_{s+2k} f^{(s+2k)}(Z_s) dt_k ... dt_1 dV_s, \quad (11.25) \]

where

\[ \mathcal{C}^{\infty,R,j_k,m_k}_{s+2k-2,s+2k} := \mathcal{C}^{\infty,R,j_k,m_k}_{s+2k-2,s+2k} \left[ g^{s+2k}1_{(\omega_{s+2k-1},\omega_{s+2k},v_{s+2k-1},v_{s+2k}) \in \mathcal{B}_{m_k}(Z_{s+2k}^{\infty})} \right]. \]

Recalling the observables \( \tilde{T}^N_{s,k,R,\delta}, \tilde{T}^\infty_{s,k,R,\delta} \) from (11.21), (11.23) and using Proposition 10.7, we obtain:

**Proposition 11.7.** Let \( s, n \in \mathbb{N}, \alpha, \epsilon_0, R, \eta, \delta \) be parameters as in (10.6), \((N, \epsilon)\) in the scaling (5.43) with \( \epsilon << \alpha \) and \( t \in [0,T] \). Then the following estimates hold:

\[
\sum_{k=1}^{n} \sum_{(J,M) \in \mathcal{U}_{s,k}} \| \tilde{T}^N_{s,k,R,\delta}(t,J,M) - J^N_{s,k,R,\delta}(t,J,M) \|_{L^\infty(\Delta^X_s(\epsilon_0))} \leq C_{d,s,\mu_0,T} \| \phi_s \|_{L^\infty\mathcal{V}_s} R^{d(s+3n)} \eta^{d+1} \| F_{N,0} \|_{N,\beta_0,\mu_0},
\]

uniformly in \( N \), and

\[
\sum_{k=1}^{n} \sum_{(J,M) \in \mathcal{U}_{s,k}} \| \tilde{T}^\infty_{s,k,R,\delta}(t,J,M) - J^\infty_{s,k,R,\delta}(t,J,M) \|_{L^\infty(\Delta^X_s(\epsilon_0))} \leq C_{d,s,\mu_0,T} \| \phi_s \|_{L^\infty\mathcal{V}_s} R^{d(s+3n)} \eta^{d+1} \| F_0 \|_{\infty,\beta_0,\mu_0},
\]

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Proof. As usual, it suffices to prove the estimate for the BBGKY hierarchy case and the Boltzmann hierarchy case follows similarly. Fix $k \in \{1, \ldots, n\}$ and $(J, M) \in U_{s,k}$.

We first estimate the difference:

$$\tilde{I}_{s,k,R,\delta}^N(t, J, M)(X_s) - J_{s,k,R,\delta}^N(t, J, M)(X_s).$$

Triangle inequality implies

$$|b(\omega_1, \omega_2, v_1 - v, v_2 - v)| \leq 4R, \quad \forall (\omega_1, \omega_2) \in S_{1}^{2d-1}, \quad \forall v, v_1, v_2 \in B_{R}^{d},$$

so

$$\int_{S_{1}^{2d-1} \times B_{R}^{2d}} |b(\omega_1, \omega_2, v_1 - v, v_2 - v)| d\omega_1 d\omega_2 dv_1 dv_2 \leq C_d R^{2d+1} \leq C_d R^{3d}, \quad \forall v \in B_{R}^{d}.$$  

But in order to estimate the difference (11.26), we integrate at least once over $B_{m_{i}}(Z_{s+2i-2}^{\infty}(t_i^+))$ for some $i \in \{1, \ldots, k\}$. Proposition 10.7 and the expression (11.27) yield the estimate:

$$\int_{B_{m_{i}}(Z_{s+2i-2}^{\infty}(t_i^+))} |b(\omega_1, \omega_2, v_1 - v, v_2 - v)| d\omega_1 d\omega_2 dv_1 dv_2 \leq C_d(s + 2i - 2)R^{2d+1} \eta_{\frac{d-1}{d+2}} \leq C_d(s + 2k)R^{3d} \eta_{\frac{d-1}{d+2}}, \quad \forall v \in B_{R}^{d}.$$  

Moreover, we have the elementary inequalities:

$$\|f_{N,0}^{(s+2k)}\|_{L^{\infty}} \leq e^{-(s+2k)\mu_0} \|F_{N,0}\|_{N,\beta_0,\mu_0},$$

$$\int_{T_{k,\delta}(t)} dt_1 \cdots dt_k = \int_{0}^{t} \int_{0}^{t_1} \cdots \int_{0}^{t_{k-1}} dt_1 \cdots dt_k = \frac{t^k}{k!} \leq \frac{T^k}{k!}.$$
Therefore, (11.28)-(11.31) imply
\[
\left| \tilde{I}_{s,k,R,\delta}^N(t, J, M)(X_s) - J_{s,k,R,\delta}^N(t, J, M)(X_s) \right| \leq \\
\leq \| \phi_s \|_{L^\infty_V} e^{-(s+2k)^{\mu_0}} \| F_{N,0} \|_{N,\beta_0,\mu_0} C_d R^d s C_d^{k-1} R^{3d(k-1)} (s + 2k) C_d R^{2d} \eta^{d-1} \frac{T^k}{k!} \\
\leq C_{d,s,\mu_0,T}^k \| \phi_s \|_{L^\infty_V} \frac{(s + 2k)}{k!} R^{d(s+3k)} \eta^{\frac{d-1}{4}+\frac{1}{2}} \| F_{N,0} \|_{N,\beta_0,\mu_0}.
\]
Adding for all \((J, M) \in U_{s,k}\), we get \(2^k s(s + 2)\ldots(s + 2k - 2)\) contributions, thus
\[
\sum_{(J, M) \in U_{s,k}} \left| \tilde{I}_{s,k,R,\delta}^N(t, J, M) - J_{s,k,R,\delta}^N(t, J, M) \right|_{L^\infty(V_s)} \leq \\
\leq C_{d,s,\mu_0,T}^k \| \phi_s \|_{L^\infty_V} \frac{(s + 2k)^{k+1}}{k!} \eta^{\frac{d-1}{4}+\frac{1}{2}} \| F_{N,0} \|_{N,\beta_0,\mu_0} \\
\leq C_{d,s,\mu_0,T}^k \| \phi_s \|_{L^\infty_V} \frac{(s + 2k)^{k+1}}{k!} \| F_{N,0} \|_{N,\beta_0,\mu_0}.
\]

since
\[
\frac{(s + 2k)^{k+1}}{k!} = \frac{(s + 2k)(s + 2k)^k}{k!} \leq C_s^k,
\]
by the elementary inequality [8.42]. Summing over \(k = 1, \ldots, n\), we obtain the required estimate. \(\square\)

In the next chapter, in order to conclude the convergence proof, we will estimate the differences of the corresponding BBGKY hierarchy and Boltzmann hierarchy truncated elementary observables in the scaled limit.
Chapter 12

Convergence proof

Recall from Section 11.6 that given \( s, n \in \mathbb{N} \), parameters \( \alpha, \epsilon_0, R, \eta, \delta \) as in (10.6), \((N, \epsilon)\) in the scaling (5.43) with \( \epsilon \ll \alpha \) and \( t \in [0, T] \), we have reduced the convergence proof to controlling the differences:

\[
J_{s,k,R,\delta}^N(t, J, M) - J_{s,k,R,\delta}^\infty(t, J, M)
\]

for given \( 1 \leq k \leq n \) and \((J, M) \in \mathcal{U}_{s,k}\), where \( J_{s,k,R,\delta}^N(t, J, M) \), \( J_{s,k,R,\delta}^\infty(t, J, M) \) are given by (11.24)-(11.25). This will be the aim of this chapter.

Throughout this chapter, \( s \in \mathbb{N} \) will be fixed. We also consider \( \beta_0 > 0, \mu_0 \in \mathbb{R}, T > 0 \) and \( F_0 \in X_{\infty, \beta_0, \mu_0} \) as in the statement of Theorem 7.4.1.

12.1 BBGKY hierarchy pseudo-trajectories and proximity to the Boltzmann hierarchy pseudo-trajectories

In the same spirit as in Section 11.5 we may define the BBGKY hierarchy pseudo-trajectory. Consider \( s \in \mathbb{N}, (N, \epsilon) \) in the scaling (5.43), \( k \in \mathbb{N} \) and \( t \in [0, T] \). Let us recall from (8.1) the set

\[
\mathcal{T}_k(t) = \{(t_1, ..., t_k) \in \mathbb{R}^k : 0 = t_{k+1} < t_k < ... < t_1 < t_0 = t\},
\]

where we use the convention \( t_0 = t \) and \( t_{k+1} = 0 \).
Consider \( Z_s = (X_s, V_s) \in \mathbb{R}^{2ds} \), \((t_1, \ldots, t_k) \in J_k(t), J = (j_1, \ldots, j_k), M = (m_1, \ldots, m_k), (J, M) \in U_{s,k} \), and for each \( i = 1, \ldots, k \), we consider
\[
(\omega_{s+2i-1}, \omega_{s+2i}, v_{s+2i-1}, v_{s+2i}) \in S^{2d-1}_1 \times B^2_R.
\]

The construction that we present is similar to the construction of the Boltzmann hierarchy pseudo-trajectory, the main difference being that we take into account the interaction zone of the adjusted particles in each step. This illustrates the fact that added particles are placed in different positions compared to the particle that they are associated with.

More precisely, we inductively construct the BBGKY hierarchy pseudo-trajectory of \( Z_s = (X_s, V_s) \in \mathbb{R}^{2ds} \) as follows:

**Time** \( t_0 = t \): We initially define
\[
Z^N_s(t^-_0) = (x^N_1(t^-_0), \ldots, x^N_s(t^-_0), v^N_1(t^-_0), \ldots, v^N_s(t^-_0)) := Z_s.
\]

**Time** \( t_1 \): We define \( Z^N_s(t^+_1) = (x^N_1(t^+_1), \ldots, x^N_s(t^+_1), v^N_1(t^+_1), \ldots, v^N_s(t^+_1)) \) as follows:
\[
Z^N_s(t^+_1) := (X^N_s(t^-_0) - (t - t_1)V^N_s(t^-_0), V^N_s(t^-_0)) = (X_s - (t - t_1)V_s, V_s).
\]

We also define \( Z^N_{s+2}(t^-_1) = (x^N_1(t^-_1), \ldots, x^N_s(t^-_1), v^N_1(t^-_1), \ldots, v^N_s(t^-_1)) \) as follows:
\[
(x^N_j(t^-_1), v^N_j(t^-_1)) := (x^N_j(t^+_1), v^N_j(t^+_1)), \quad \forall j \in \{1, \ldots, s\} \setminus \{m_1\},
\]
and if \( j_1 = -1 \):
\[
(x^N_{m_1}(t^-_1), v^N_{m_1}(t^-_1)) := (x^N_{m_1}(t^+_1), v^N_{m_1}(t^+_1)),
\]
\[
(x^N_{s+1}(t^-_1), v^N_{s+1}(t^-_1)) := \left(x^N_{m_1}(t^-_1) - \sqrt{2}\epsilon\omega_{s+1}, v_{s+1}\right),
\]
\[
(x^N_{s+2}(t^-_1), v^N_{s+2}(t^-_1)) := \left(x^N_{m_1}(t^-_1) - \sqrt{2}\epsilon\omega_{s+2}, v_{s+2}\right).
\]
while if $j_1 = 1$:
\[
(x_{m_1}^N(t_{i}^-), v_{m_1}^N(t_{i}^-)) = (x_{m_1}^N(t_{i}^+), v_{m_1}^N(t_{i}^+)),
\]
\[
(x_{s+1}^N(t_{i}^-), v_{s+1}^N(t_{i}^-)) = (x_{m_1}^N(t_{i}^+) + \sqrt{2} \epsilon \omega_{s+1}, v_{s+1}^s),
\]
\[
(x_{s+2}^N(t_{i}^-), v_{s+2}^N(t_{i}^-)) = (x_{m_1}^N(t_{i}^+) + \sqrt{2} \epsilon \omega_{s+2}, v_{s+2}^s),
\]
where
\[
(v_{m_1}^N(t_{i}^+), v_{s+1}^s) = T_{\omega_{s+1}, \omega_{s+2}} (v_{m_1}^N(t_{i}^+), v_{s+1}, v_{s+2}).
\]

**Time** $t_i, i \in \{2, ..., k\}$: Consider $i \in \{2, ..., k\}$, and assume we know
\[
Z_{s+2i-2}^N(t_{i-1}^-) = (x_1^N(t_{i-1}^-), ..., x_{s+2i-2}^N(t_{i-1}^-), v_1^N(t_{i-1}^-), ..., v_{s+2i-2}^N(t_{i-1}^-)).
\]
We define $Z_{s+2i-2}^N(t_{i}^+) = (x_1^N(t_{i}^+), ..., x_{s+2i-2}^N(t_{i}^+), v_1^N(t_{i}^+), ..., v_{s+2i-2}^N(t_{i}^+))$ as:
\[
Z_{s+2i-2}^N(t_{i}^+) := (X_{s+2i-2}^N(t_{i-1}^-) - (t_{i-1} - t_i) V_{s+2i-2}^{N^2} t_{i-1}^-), V_{s+2i-2}^{N^2} t_{i-1}^-)
\]
We also define $Z_{s+2i}^N(t_{i}^-) = (x_1^N(t_{i}^-), ..., x_{s+2i}^N(t_{i}^-), v_1^N(t_{i}^-), ..., v_{s+2i}^N(t_{i}^-))$ as:
\[
(x_{j}^N(t_{i}^-), v_{j}^N(t_{i}^-)) := (x_{j}^N(t_{i}^+), v_{j}^N(t_{i}^+)), \quad \forall j \in \{1, ..., s + 2i - 2\} \setminus \{m_i\},
\]
and if $j_i = -1$:
\[
(x_{m_i}^N(t_{i}^-), v_{m_i}^N(t_{i}^-)) := (x_{m_i}^N(t_{i}^+), v_{m_i}^N(t_{i}^+)),
\]
\[
(x_{s+2i-1}^N(t_{i}^-), v_{s+2i-1}^N(t_{i}^-)) := (x_{m_i}^N(t_{i}^+) - \sqrt{2} \epsilon \omega_{s+2i-1}, v_{s+2i-1}^s),
\]
\[
(x_{s+2i}^N(t_{i}^-), v_{s+2i}^N(t_{i}^-)) := (x_{m_i}^N(t_{i}^+) - \sqrt{2} \epsilon \omega_{s+2i}, v_{s+2i}^s),
\]
while if $j_i = 1$:
\[
(x_{m_i}^N(t_{i}^-), v_{m_i}^N(t_{i}^-)) := (x_{m_i}^N(t_{i}^+), v_{m_i}^N(t_{i}^+)),
\]
\[
(x_{s+2i-1}^N(t_{i}^-), v_{s+2i-1}^N(t_{i}^-)) := (x_{m_i}^N(t_{i}^+) + \sqrt{2} \epsilon \omega_{s+2i-1}, v_{s+2i-1}^s),
\]
\[
(x_{s+2i}^N(t_{i}^-), v_{s+2i}^N(t_{i}^-)) := (x_{m_i}^N(t_{i}^+) + \sqrt{2} \epsilon \omega_{s+2i}, v_{s+2i}^s),
\]

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where
\[(v_{mi}^N(t_i^-), v_{si+2i-1}^*, v_{si+2i}^*) = T_{\omega_{s+2i-1}, \omega_{s+2i}}(v_{mi}^N(t_i^+), v_{s+2i-1}, v_{s+2i})\].

**Time** $t_{k+1} = 0$: We finally obtain
\[Z_{s+2k}^N(0^+) = Z_{s+2k}^N(t_{k+1}^+) = (X_{s+2k}^N(t_k^-) - t_k V_{s+2k}^N(t_k^-), V_{s+2k}^N(t_k^-))\].

The construction made is illustrated in the following diagram:

We give the following definition:

**Definition 12.1.** Let $Z_s = (X_s, V_s) \in \mathbb{R}^{2ds}$, $(t_1, ..., t_k) \in \mathcal{T}_k(t)$, $J = (j_1, ..., j_k)$, $M = (m_1, ..., m_k)$, $(J, M) \in \mathcal{U}_{s,k}$ and for each $i = 1, ..., k$, we consider
\[(\omega_{s+2i-1}, \omega_{s+2i}, v_{s+2i-1}, v_{s+2i}) \in \mathbb{S}^{2d-1} \times B_R^{2d}.

The sequence \(\{Z_{s+2i-2}^N(t_i^+)\}_{i=0, ..., k+1}\) constructed above is called the BBGKY hierarchy pseudo-trajectory of $Z_s$.

We now state the following proximity result of the corresponding BBGKY hierarchy and Boltzmann hierarchy pseudo-trajectories under backwards time evolution.
Lemma 12.2. Let \( s, n \in \mathbb{N} \), \((N, \epsilon)\) in the scaling \((5.43), 1 \leq k \leq n\), \((J, M) \in U_{s,k}, t \in [0,T]\) and \((t_1, \ldots, t_k) \in \mathcal{T}_k(t)\). Let us fix \( Z_s = (X_s, V_s) \in \mathbb{R}^{2ds}\). For each \( i = 1, \ldots, k\), consider \((\omega_{s+2i-1}, \omega_{s+2i}, v_{s+2i-1}, v_{s+2i}) \in \mathbb{S}^{2d-1} \times \mathbb{R}^{2d}\). Then for all \( i = 1, \ldots, k+1 \) and \( \ell = 1, \ldots, s+2i-2 \), we have

\[
|x^N_\ell(t^+_i) - x^\infty_\ell(t^+_i)| \leq \sqrt{2}\epsilon(i-1), \quad v^N_\ell(t^+_i) = v^\infty_\ell(t^+_i). \tag{12.1}
\]

In particular, if \( s < n \), there holds:

\[
|X^N_{s+2i-2}(t^+_i) - X^\infty_{s+2i-2}(t^+_i)| \leq \sqrt{6}n^{3/2}\epsilon, \quad \forall i = 1, \ldots, k+1, \tag{12.2}
\]

where we use notation from Definition 11.1 and Definition 12.1.

Proof. We first prove (12.1) by induction on \( i \in \{1, \ldots, k\} \). For \( i = 1 \) the result is trivial since the pseudo-trajectories initially coincide by construction. Assume (12.1) holds for \( i \in \{1, \ldots, k\} \) i.e. for all \( \ell \in \{1, \ldots, s+2i-2\} \) there holds:

\[
|x^N_\ell(t^+_i) - x^\infty_\ell(t^+_i)| \leq \sqrt{2}\epsilon(i-1) \quad \text{and} \quad v^N_\ell(t^+_i) = v^\infty_\ell(t^+_i). \tag{12.3}
\]

We prove (12.1) holds for \((i+1) \in \{2, \ldots, k+1\}\). We need to consider different cases for \( j_i = \pm 1 \). Assume first \( j_i = -1 \). Then for the Boltzmann hierarchy pseudo-trajectory we have

\[
x^\infty_\ell(t^+_{i+1}) = x^\infty_\ell(t^+_i) - (t_i - t_{i+1})v^\infty_\ell(t^+_i), \quad v^\infty_\ell(t^+_{i+1}) = v^\infty_\ell(t^+_i), \quad \forall \ell \leq s+2i-2,
\]

\[
x^\infty_{s+2i-1}(t^+_{i+1}) = x^\infty_{m_i}(t^+_i) - (t_i - t_{i+1})v_{s+2i-1}, \quad v^\infty_{s+2i-1}(t^+_{i+1}) = v_{s+2i-1},
\]

\[
x^\infty_{s+2i}(t^+_{i+1}) = x^\infty_{m_i}(t^+_i) - (t_i - t_{i+1})v_{s+2i}, \quad v^\infty_{s+2i}(t^+_{i+1}) = v_{s+2i},
\]

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while for the BBGKY hierarchy pseudo-trajectory we obtain
\[ x^N_{\ell}(t_{i+1}^+) = x^N_{\ell}(t_i^+) - (t_i - t_{i+1})v^N_{\ell}(t_i^+), \quad v^N_{\ell}(t_{i+1}^+) = v^N_{\ell}(t_i^-), \quad \forall \ell \leq s + 2i - 2; \]
\[ x^N_{s+2i-1}(t_{i+1}^+) = x^N_{m_i}(t_i^+) - (t_i - t_{i+1})v^N_{s+2i-1} - \sqrt{2} \epsilon \omega_{s+2i-1}, \]
\[ v^N_{s+2i-1}(t_{i+1}^+) = v_{s+2i-1}, \]
\[ x^N_{s+2i}(t_{i+1}^+) = x^N_{m_i}(t_i^+) - (t_i - t_{i+1})v^N_{s+2i} - \sqrt{2} \epsilon \omega_{s+2i}, \]
\[ v^N_{s+2i}(t_{i+1}^+) = v_{s+2i}. \]

Thus, for any \( \ell \leq s + 2i \), the induction assumption (12.3) implies
\[ v^N_{\ell}(t_{i+1}^+) = v^N_{\ell}(t_i^+) = v^\infty_{\ell}(t_i^+) = v^\infty_{\ell}(t_{i+1}^+). \]

Moreover, for \( \ell \leq s + 2i - 2 \), we have
\[ |x^N_{\ell}(t_{i+1}^+) - x^\infty_{\ell}(t_{i+1}^+)| = |x^N_{\ell}(t_i^+) - x^\infty_{\ell}(t_i^+)| \leq \sqrt{2} \epsilon (i - 1), \]
for \( \ell = s + 2i - 1 \), we have
\[ |x^N_{s+2i-1}(t_{i+1}^+) - x^\infty_{s+2i-1}(t_{i+1}^+)| \leq |x^N_{m_i}(t_i^+) - x^\infty_{m_i}(t_i^+)| + \sqrt{2} \epsilon |\omega_{s+2i-1}| \]
\[ \leq \sqrt{2} \epsilon (i - 1) + \sqrt{2} \epsilon = \sqrt{2} \epsilon i, \]
while for \( \ell = s + 2i \), we obtain
\[ |x^N_{s+2i}(t_{i+1}^+) - x^\infty_{s+2i}(t_{i+1}^+)| \leq |x^N_{m_i}(t_i^+) - x^\infty_{m_i}(t_i^+)| + \sqrt{2} \epsilon |\omega_{s+2i}| \]
\[ \leq \sqrt{2} \epsilon (i - 1) + \sqrt{2} \epsilon = \sqrt{2} \epsilon i, \]
and (12.1) follows for \( j_i = -1 \).

Assume now \( j_i = 1 \). Then for the Boltzmann hierarchy pseudo-trajectory we
have

\[ x^\infty_\ell(t_{i+1}^+) = x^\infty_\ell(t_i^+) - (t_i - t_{i+1})v^\infty_\ell(t_i^+), \quad v^\infty_\ell(t_{i+1}^+) = v^\infty_\ell(t_i^+), \]
\[ \forall \ell \in \{1, ..., s + 2i - 2\} \setminus \{m_i\}, \]

\[ x^\infty_m(t_{i+1}^+) = x^\infty_m(t_i^+) - (t_i - t_{i+1})v^\infty_m(t_i^+), \quad v^\infty_m(t_{i+1}^+) = v^\infty_m(t_i^+), \]

\[ x^\infty_{s+2i-1}(t_{i+1}^+) = x^\infty_m(t_i^+) - (t_i - t_{i+1})v^\ast_{s+2i-1}, \]

\[ v^\infty_{s+2i-1}(t_{i+1}^+) = v^\ast_{s+2i-1}, \]

\[ x^\infty_{s+2i}(t_{i+1}^+) = x^\infty_m(t_i^+) - (t_i - t_{i+1})v^\ast_{s+2i}, \quad v^\infty_{s+2i}(t_{i+1}^+) = v^\ast_{s+2i}. \]

and for the BBGKY hierarchy pseudo-trajectory we have

\[ x^N_\ell(t_{i+1}^+) = x^N_\ell(t_i^+) - (t_i - t_{i+1})v^N_\ell(t_i^+), \quad v^N_\ell(t_{i+1}^+) = v^N_\ell(t_i^+), \]
\[ \forall \ell \in \{1, ..., s + 2i - 2\} \setminus \{m_i\}, \]

\[ x^N_m(t_{i+1}^+) = x^N_m(t_i^+) - (t_i - t_{i+1})v^{N\ast}_m(t_i^+), \quad v^N_m(t_{i+1}^+) = v^{N\ast}_m(t_i^+), \]

\[ x^N_{s+2i-1}(t_{i+1}^+) = x^N_m(t_i^+) - (t_i - t_{i+1})v^\ast_{s+2i-1} + \sqrt{2}e\omega_{s+2i-1}, \]

\[ v^N_{s+2i-1}(t_{i+1}^+) = v^\ast_{s+2i-1}, \]

\[ x^N_{s+2i}(t_{i+1}^+) = x^N_m(t_i^+) - (t_i - t_{i+1})v^\ast_{s+2i} + \sqrt{2}e\omega_{s+2i}, \]

\[ v^\infty_{s+2i}(t_{i+1}^+) = v^\ast_{s+2i}. \]

For \( \ell \in \{1, ..., s + 2i - 2\} \setminus \{m_i\} \), the induction assumption (12.3) yields

\[ v^N_\ell(t_{i+1}^+) = v^N_\ell(t_i^+) = v^\infty_\ell(t_i^+), \quad \forall \ell \in \{1, ..., s + 2i - 2\} \setminus \{m_i\}, \]

\[ |v^N_\ell(t_{i+1}^+) - v^\infty_\ell(t_{i+1}^+)| = |v^N_\ell(t_i^+) - v^\infty_\ell(t_i^+)| \leq \sqrt{2}e(i - 1). \]

Thus, for \( \ell = m_i \), we have

\[ v^N_{m_i}(t_{i+1}^+) = v^{N\ast}_{m_i}(t_i^+) = v^\infty_{m_i}(t_i^+), \quad \forall \ell \in \{1, ..., s + 2i - 2\} \setminus \{m_i\}, \]

\[ |v^N_{m_i}(t_{i+1}^+) - v^\infty_{m_i}(t_{i+1}^+)| = |v^N_{m_i}(t_i^+) - v^\infty_{m_i}(t_i^+)| \leq \sqrt{2}e(i - 1), \]

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for \( \ell = s + 2i - 1 \), we have
\[
v_N^{s+2i-1}(t_{i+1}^+) = v^*_{s+2i-1} = v^\infty_{s+2i-1}(t_{i+1}^+)
v_N^{s+2i-1}(t_{i+1}^+) - v^\infty_{s+2i-1}(t_{i+1}^+)| \leq |v_N^{s+2i-1}(t_i^+) - v^\infty_{s+2i-1}(t_i^+)| + \sqrt{2}\epsilon|\omega_{s+2i-1}|
\leq \sqrt{2}\epsilon(i - 1) + \sqrt{2}\epsilon = \sqrt{2}\epsilon i,
\]
while for \( \ell = s + 2i \), we have
\[
v_N^{s+2i}(t_{i+1}^+) = v^*_{s+2i} = v^\infty_{s+2i}(t_{i+1}^+)
v_N^{s+2i}(t_{i+1}^+) - v^\infty_{s+2i}(t_{i+1}^+)| \leq |v_N^{s+2i}(t_i^+) - v^\infty_{s+2i}(t_i^+)| + \sqrt{2}\epsilon|\omega_{s+2i}|
\leq \sqrt{2}\epsilon(i - 1) + \sqrt{2}\epsilon = \sqrt{2}\epsilon i,
\]
and (12.1) is proved for \( j_i = 1 \). By induction, (12.1) is proved. To prove (12.2), we use (12.1) to obtain
\[
|X_{s+2i-2}^N(t_i^+)^2 - X_{s+2i-2}^\infty(t_i^+)|^2 = \sum_{\ell=1}^{s+2i-2} |x_N^\ell(t_i^+) - x^\infty_\ell(t_i^+)|^2
\leq 2(s + 2i - 2)i^2(i - 1)^2
\leq 2(s + 2k)i^2k^2
\leq 6\epsilon^2n^3,
\]
since \( 0 \leq i - 1 \leq k \leq n \) and \( s < n \) by assumption. Taking square roots, we obtain (12.2).

12.3 Reformulation in terms of pseudo-trajectories

We will now re-write the Boltzmann hierarchy truncated elementary observables, defined in (11.25), and the BBGKY hierarchy truncated elementary observables, defined in (11.24), in terms of pseudo-trajectories.
Let $s, n \in \mathbb{N}$ with $s < n$, parameters $\alpha, \epsilon_0, R, \eta, \delta$ as in (10.6). For the Boltzmann hierarchy case, there is always free flow between the collision times. Therefore, for $X_s \in \Delta_s^X(\epsilon_0), 1 \leq k \leq n, (J, M) \in U_{s,k}$ and $t \in [0, T]$, the Boltzmann hierarchy truncated elementary observable can be equivalently written as:

$$J_{s,k,R,\delta}^\infty(t, J, M)(X_s) =$$

$$\int_{\mathcal{N}(X_s)} \psi_s(V_s) \int_{\mathcal{T}_k(t)} \int_{B_{m_1}^\infty(Z_{s+2k-2}(t^+_k))} \cdots \int_{B_{m_k}^\infty(Z_{s+2k-2}(t^+_k))} \prod_{i=1}^k b_+ (\omega_{s+2i-1}, \omega_{s+2i}, v_{s+2i-1} - v_{m_i}^\infty (t^+_i), v_{s+2i} - v_{m_i}^\infty (t^+_i)) f_0^{(s+2k)} (Z_{s+2k}^\infty (0^+))$$

$$\times \prod_{i=1}^k (d\omega_{s+2i-1} d\omega_{s+2i} dv_{s+2i-1} dv_{s+2i}) \ dt_k \ldots dt_1 \ dV_s.$$

$$\text{(12.4)}$$

It is not immediate to obtain a similar observation at the BBGKY level because of the possibility of recollisions appearing. However, thanks to Proposition [10.4] and Lemma [12.2], it is possible to obtain an expansion for the BBGKY hierarchy truncated elementary observables for sufficiently large $N$ as well. In what follows, we provide more details.

More precisely, fix $X_s \in \Delta_s^X(\epsilon_0), 1 \leq k \leq n, (J, M) \in U_{s,k}$, $t \in [0, T]$ and $(t_1, \ldots, t_k) \in \mathcal{T}_{k,\delta}(t)$. Consider $(N, \epsilon)$ in the scaling (5.43) with $N$ large enough such that $n^{3/2} \epsilon << \alpha$. By Proposition [11.2], given $V_s \in \mathcal{M}_s^c(X_s)$, we have $Z_s = (X_s, V_s) \in G_s(\epsilon, \epsilon_0, \delta)$. By the definition of the set $G_s(\epsilon, \epsilon_0, \delta)$, see (11.1), we have

$$Z_s \in G_s(\epsilon, \epsilon_0, \delta) \Rightarrow Z_s(\tau) \in \mathcal{D}_{s,\epsilon}, \ \forall \tau \geq 0,$$

thus

$$\Psi_s^{\tau-t_0} Z_s^N (t_0^-) = \Phi_s^{\tau-t_0} Z_s^N (t_0^-), \ \forall \tau \in [t_1, t_0]$$

$$\text{(12.5)}$$
where $\Psi_s$, given in (4.62), denotes the $s$-particle $\epsilon$-interaction zone flow and $\Phi_s$, given in (4.63), denotes the $s$-particle free flow respectively. We also have

$$Z_s = (X_s, V_s) \in G_s(\epsilon, \epsilon_0, \delta) \Rightarrow Z_s^\infty(t_1^+) \in G_s(\epsilon_0, 0). \quad (12.6)$$

For all $i \in \{1, ..., k\}$, we have seen that an inductive application of Proposition 10.4 yields

$$Z_{s+2i}^\infty(t_{i+1}^+) \in G_{s+2i}(\epsilon, 0), \quad \forall (\omega_{s+2i-1}, \omega_{s+2i}, v_{s+2i-1}, v_{s+2i}) \in B^c_{m_i}(Z_{s+2i-2}^\infty(t_i^+)). \quad (12.7)$$

Since we have assumed $s < n$ and $n^{3/2} \epsilon << \alpha$, (12.2) from Lemma 12.2 implies

$$|X_{s+2i-2}^N(t_i^+) - X_{s+2i-2}^\infty(t_i^+)| \leq \frac{\alpha}{2}, \quad \forall i = 1, ..., k. \quad (12.8)$$

Then, Proposition 10.4 yields that for any $i = 1, ..., k$, we have

$$\Psi_{s+2i}^{\tau-t_i} Z_{s+2i}^N(t_i^-) = \Phi_{s+2i}^{\tau-t_i} Z_{s+2i}^N(t_i^-), \quad \forall \tau \in [t_{i+1}, t_i], \quad (12.9)$$

where $\Psi_{s+2i}$, given in (4.62), denotes the $(s+2i)$-particle $\epsilon$-interaction zone flow and $\Phi_{s+2i}$, given in (4.63), denotes the $(s+2i)$-particle free flow respectively. Combining (12.5), (12.9), we conclude that for any $i = 0, ..., k$, the backwards $\epsilon$-interaction zone flow coincides with the free flow in $[t_{i+1}, t_i]$. Moreover, Lemma 12.2 also implies that

$$v_{m_i}^N(t_i^+) = v_{m_i}^\infty(t_i^+), \quad \forall i = 1, ..., k.$$  

Therefore, given $X_s \in \Delta_s^X(\epsilon_0)$ and for $N$ large enough for $n^{3/2} \epsilon << \alpha$ to hold, the
BBGKY hierarchy truncated elementary observable can be equivalently written as:

\[
J^N_{s,k,R,\delta}(t, J, M)(X_s) = A_{N,\epsilon}^{s,k} \int_{M}(V_s) \int_{\mathcal{T}_{k,\delta}(t)} \int_{\mathcal{B}_{m_1}}(Z^\infty_2(t_1^+)) \ldots \int_{\mathcal{B}_{m_k}}(Z^\infty_{s+2k-2}(t_k^+)) \\
\prod_{i=1}^{k} b_+ (\omega_{s+2i-1}, \omega_{s+2i}, v_{s+2i-1} - v^\infty_{m_i}(t_i^+), v_{s+2i} - v^\infty_{m_i}(t_i^+)) J^{(s+2k)}_{N,0}(Z^N_{s+2k}(0^+)) \\
\times \prod_{i=1}^{k} (d\omega_{s+2i-1} d\omega_{s+2i} dv_{s+2i-1} dv_{s+2i}) dt_k \ldots dt_1 dV_s, \tag{12.10}
\]

where, recalling (5.40), we denote

\[
A_{N,\epsilon}^{s,k} = \prod_{i=1}^{k} A_{N,\epsilon,s+2i-2} = 2^{k(d-2)} \epsilon^{k(2d-1)} \prod_{i=1}^{k} (N - s - 2i + 2)(N - s - 2i + 1). \tag{12.11}
\]

Remark 12.1. Notice that for fixed \(s \in \mathbb{N}\), \((N, \epsilon)\) in the scaling (5.43), \(k \in \mathbb{N}\), the scaling (5.43) implies

\[
0 \leq A_{N,\epsilon}^{s,k} \nearrow 1, \text{ as } N \to \infty. \tag{12.12}
\]

Let us approximate the BBGKY hierarchy initial data by Boltzmann hierarchy initial data defining some auxiliary functionals. Let \(s \in \mathbb{N}\) and \(X_s \in \Delta^s_X(\epsilon_0)\). For \(1 \leq k \leq n\), \((J, M) \in \mathcal{U}_{s,k}\) and \(t \in [0, T]\), we define the auxiliary functional \(\tilde{J}^N_{s,k,R,\delta}(t, J, M)\) which differs from \(J^N_{s,k,R,\delta}(t, J, M)\) by the absence of the scaling fac-
tor $A_{N,c}^{s,k}$ and the use of Boltzmann hierarchy initial data:

$$\tilde{J}_{s,k,R,\delta}^N(t, J, M)(X_s) :=$$

$$= \int_{M_c(X_s)} \phi_s(V_s) \int_{J_{k,\delta}(t)} \int_{B_{m_1}^c(Z^\infty_s(t_i^+))} \cdots \int_{B_{m_k}^c(Z^\infty_{s+2k-2}(t_i^+))}$$

$$\prod_{i=1}^{k} b_+ (\omega_{s+2i-1}, \omega_{s+2i}, v_{s+2i-1} - v^\infty_{m_i}(t_i^+), v_{s+2i} - v^\infty_{m_i}(t_i^+)) f_0^{(s+2k)} (Z^N_{s+2k}(0^+))$$

$$\times \prod_{i=1}^{k} (d\omega_{s+2i-1} d\omega_{s+2i} dv_{s+2i-1} dv_{s+2i}) d t_k \cdots d t_1 d V_s.$$

(12.13)

Due to the scaling (5.43) and Proposition 7.2, we conclude that the auxiliary functionals approximate the BBGKY hierarchy truncated elementary observables $J_{s,k,R,\delta}^N$, defined in (12.10).

**Proposition 12.4.** Let $s, n \in \mathbb{N}$, with $s < n$, $\alpha, \epsilon_0, R, \eta, \delta$ be parameters as in (10.6), and $t \in [0, T]$. Then for any $\zeta > 0$, there is $N_1 = N_1(\zeta, n, \alpha, \epsilon_0) \in \mathbb{N}$, such that for all $(N, \epsilon)$ in the scaling (5.43) with $N > N_1$, there holds:

$$\sum_{k=1}^{n} \sum_{(J,M) \in U_{s,k}} \| J_{s,k,R,\delta}^N(t, J, M) - \tilde{J}_{s,k,R,\delta}^N(t, J, M) \|_{L^\infty(\Delta_s^X(\epsilon_0))} \leq C_d(s, \mu_0, T) \| \phi_s \|_{L^\infty_{V_s} R^{d(s+3n)}} \zeta^2.$$

**Proof.** Fix $1 \leq k \leq n$ and $(J,M) \in U_{s,k}$. Consider $(N, \epsilon)$ in the scaling (5.43) with $N$ large enough such that $n^{3/2} \epsilon < < \alpha$. Triangle inequality and the fact that $\Delta_s^X(\epsilon_0) \subseteq \Delta_s^X(\epsilon_0/2)$ yield

$$\| J_{s,k,R,\delta}^N(t, J, M) - \tilde{J}_{s,k,R,\delta}^N(t, J, M) \|_{L^\infty(\Delta_s^X(\epsilon_0))} \leq$$

$$\leq \| J_{s,k,R,\delta}^N(t, J, M) - A_{N,c}^{s,k} \tilde{J}_{s,k,R,\delta}^N(t, J, M) \|_{L^\infty(\Delta_s^X(\epsilon_0/2))} + | A_{N,c}^{s,k} - 1 | \| \tilde{J}_{s,k,R,\delta}^N(t, J, M) \|_{L^\infty(\Delta_s^X(\epsilon_0))}.$$

(12.14)
We estimate each of the terms in (12.14)-(12.15).

**Term (12.14):** Let us fix $(t_1, ..., t_k) \in \mathcal{T}_k(\epsilon)$. Applying (12.7) for $i = k - 1$, we obtain

$$Z_{s+2k-2}^{\infty}(t_+^k) \in G_{s+2k-2}(\epsilon_0, 0).$$

Since $s < n$ and $n^{3/2} \epsilon << \alpha$, (12.2), applied for $i = k$, implies

$$|X_{s+2k-2}^{N}(t_+^k) - X_{s+2k-2}^{\infty}(t_+^k)| \leq \frac{\alpha}{2}.$$ 

Therefore, Proposition 10.4 (precisely expression (10.15) for the pre-collisional case, (10.19) for the post-collisional case) implies

$$Z_{s+2k}^{N}(0^+) \in G_{s+2k}(\epsilon_0/2, 0) \subseteq \Delta_{s+2k}(\epsilon_0/2). \quad (12.16)$$

Thus (11.28), (11.31), (12.10)-(12.13) and crucially (12.16) imply that for $N$ large enough, we have

$$\|J_{s,k,R,\delta}^{N}(t, J, M) - A_{s,k,R,\delta}^{N}(t, J, M)\|_{L^\infty(\Delta_{s+2k}(\epsilon_0/2))} \leq$$

$$\leq \frac{C_{d,s,\mu_0,T}^{k}}{k!} \|\phi_{s}\|_{L_{s+2k}^\infty} R_{d,s+3k} \|f_{s+2k}^{(s+2k)} - f_{0}^{(s+2k)}\|_{L^\infty(\Delta_{s+2k}(\epsilon_0/2))}. \quad (12.17)$$

**Term (12.15):** By definition of the norms, we have

$$\|f_{0}^{(s+2k)}\|_{L^\infty} \leq e^{-(s+2k)\mu_0} \|F_0\|_{\infty,\beta_0,\mu_0}.$$ 

Therefore, using (11.28) and (11.31), we obtain

$$\|\hat{J}_{s,k,R,\delta}^{N}(t, J, M)\|_{L^\infty(\Delta_{s+2k}(\epsilon_0))} \leq \frac{C_{d,s,\mu_0,T}^{k}}{k!} \|\phi_{s}\|_{L_{s+3k}^\infty} R_{d,s+3k} \|F_0\|_{\infty,\beta_0,\mu_0}. \quad (12.18)$$
Adding over all \((J,M) \in U_{s,k}\), \(k = 1,\ldots,n\), using (12.14)-(12.18) and an argument similar to (11.32) to control the summation over \(k = 1,\ldots,n\), for \(N\) large enough, we obtain the estimate

\[
\sum_{k=1}^{n} \sum_{(J,M) \in U_{s,k}} \|J_{s,k,R}^{N}(t, J, M) - \tilde{J}_{s,k,R}^{N}(t, J, M)\|_{L^{\infty}(\Delta X_{s}(\epsilon_{0}))} \leq C_{d,s,p_{0},T}^{n} \|\phi_{s}\|_{L_{V_{s}}^{\infty}}^{*} R_{d}^{(s+3n)}
\]

\[
\times \left( \sup_{k \in \{1,\ldots,n\}} \|f_{N,0}^{(s+2k)} - f_{0}^{(s+2k)}\|_{L^{\infty}(\Delta X_{s}(\epsilon_{0}/2))} + \|F_{0}\|_{L^{\infty},\beta,\mu_{0}} \sup_{k \in \{1,\ldots,n\}} |A_{s,k}^{N,\epsilon} - 1| \right).
\]

But since \(n \in \mathbb{N}\), \(\epsilon_{0} > 0\) are fixed, (7.8) implies

\[
\lim_{N \to \infty} \sup_{k \in \{1,\ldots,n\}} \|f_{N,0}^{(s+2k)} - f_{0}^{(s+2k)}\|_{L^{\infty}(\Delta X_{s}(\epsilon_{0}/2))} = 0.
\]

Moreover, by (12.12), we have

\[
\lim_{N \to \infty} \sup_{k \in \{1,\ldots,n\}} |A_{N,\epsilon}^{s,k} - 1| = 0,
\]

and the result follows.

Due to the proximity Lemma 12.2 and the uniform continuity assumption (7.10) on the Boltzmann hierarchy initial data, we also obtain the following estimate:

**Proposition 12.5.** Let \(s,n \in \mathbb{N}\) with \(s < n\), \(\alpha,\epsilon_{0}, R, \eta, \delta\) be parameters as in (10.6) and \(t \in [0,T]\). Then for any \(\zeta > 0\), there is \(N_{2} = N_{2}(\zeta,n) \in \mathbb{N}\), such that for all \((N,\epsilon)\) in the scaling (5.43) with \(N > N_{2}\), there holds

\[
\sum_{k=1}^{n} \sum_{(J,M) \in U_{s,k}} \|\tilde{J}_{s,k,R}^{N}(t, J, M) - J_{s,k,R}^{\infty}(t, J, M)\|_{L^{\infty}(\Delta X_{s}(\epsilon_{0}))} \leq C_{d,s,p_{0},T}^{n} \|\phi_{s}\|_{L_{V_{s}}^{\infty}}^{*} R_{d}^{(s+3n)}\zeta^{2}.
\]

**Proof.** Let \(\zeta > 0\). Fix \(1 \leq k \leq n\) and \((J,M) \in U_{s,k}\). Since \(s < n\), Lemma 12.2 yields

\[
|Z_{s+2k}^{N}(0^{+}) - Z_{s+2k}^{\infty}(0^{+})| \leq \sqrt{6n^{3/2}} \epsilon, \quad \forall Z_{s} \in \mathbb{R}^{2ds}.
\]
Thus the continuity assumption (7.10) on $F_0$, (12.19) and the scaling (5.43) imply that there exists $N_2 = N_2(\zeta, n) \in \mathbb{N}$, such that for all $N > N_2$, we have

$$|f_0^{(s+2k)}(Z_{s+2k}^N(0^+)) - f_0^{(s+2k)}(Z_{s+2k}^\infty(0^+))| \leq C^{s+2k-1}\zeta^2, \quad \forall Z_s \in \mathbb{R}^{2ds}. \quad (12.20)$$

In the same spirit as in the proof of Proposition 12.4, using (12.20), (11.28), (11.31), and summing over $(J,M) \in \mathcal{U}_{s,k}$, $k = 1, \ldots, n$, we obtain the result. \hfill $\square$

### 12.6 Proof of Theorem 7.4.1

We are now in the position to prove Theorem 7.4.1. Fix $s \in \mathbb{N}$, $\phi_s \in C_c(\mathbb{R}^{ds})$ and $t \in [0,T]$. Consider $n \in \mathbb{N}$ with $s < n$, and assume there exist parameters $\alpha, \epsilon_0, R, \eta, \delta$ satisfying (10.6). Let $\zeta > 0$ small enough. Triangle inequality, Propositions 8.8, 11.3, 11.7, 12.4, 12.5, Remark 11.1 and part (i) of Proposition 7.2 yield that there is $N_0(\zeta, n, \alpha, \epsilon_0) \in \mathbb{N}$ such that for all $N > N_0$, we have

$$\|I^N_s(t) - I^\infty_s(t)\|_{L^\infty(\Delta X_s(\epsilon_0))} \leq$$

$$\leq C \left(2^{-n} + e^{-\frac{\beta_0}{\eta}R^2} + \delta C^n\right)$$

$$+ C^n R^{4dn} \eta^{\frac{d-1}{2}} + C^n R^{4dn} \zeta^2,$$

where

$$C := C_{d,s,\beta_0,\mu_0,T} \|\phi_s\|_{L^\infty_s} \max \{1, \|F_0\|_{L^\infty,\beta_0,\mu_0} \} > 1, \quad \text{(12.22)}$$

is an appropriate constant.

Let us fix $\sigma > 0$. Recall that we have also fixed $s \in \mathbb{N}$ and $\phi_s \in C_c(\mathbb{R}^{ds})$. We will now choose parameters satisfying (10.6), depending only on $\zeta$, such that the right hand side of (12.21) becomes less than $\zeta$. 

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Choice of parameters: We choose $n \in \mathbb{N}$ and the parameters $\delta, \eta, R, \epsilon_0, \alpha$ in the following order:

$$\max \left\{ s, \log_2 (C\zeta^{-1}) \right\} \ll n, \quad \delta \ll \zeta C^{-(n+1)}, \quad \eta \ll \zeta^{\frac{8d+4}{d+4}}, \quad R \ll \zeta^{-1/4dn} C^{-1/4d},$$

$$\max \left\{ 1, \sqrt{3} \beta_0^{-1/2} \ln^{1/2}(C\zeta^{-1}) \right\} \ll R, \quad \epsilon_0 \ll \min \{ \sigma, \eta \delta \}, \quad \alpha \ll \epsilon_0 \min \{ 1, R^{-1} \eta \}.$$

(12.23)

Relations (12.23) imply the parameters chosen satisfy (10.6) and depend only on $\zeta$. Then, (12.21)-(12.23) imply that we may find $N_0(\zeta) \in \mathbb{N}$, such that for all $N > N_0$, there holds:

$$\| I^N_s(t) - I^\infty_s(t) \|_{L^\infty(\Delta X_{\epsilon_0}(t))} < \zeta.$$

But by (12.23), we have $\epsilon_0 < \sigma$, therefore we obtain

$$\| I^N_s(t) - I^\infty_s(t) \|_{L^\infty(\Delta X_{\sigma}(t))} \leq \| I^N_s(t) - I^\infty_s(t) \|_{L^\infty(\Delta X_{\epsilon_0}(t))} < \zeta,$$

and Theorem 7.4.1 is proved.
Part II

Global well-posedness for the binary-ternary Boltzmann equation
Chapter 13

Introduction

As mentioned in Chapter 1, this part of the dissertation focuses on the well-posedness of the binary-ternary Boltzmann equation \((1.10)\). The results obtained in this direction are submitted for publication [4] in a joint work with Gamba, Pavlović and Tasković.

We provide the first rigorous analytical result that shows global in time existence and uniqueness of \textit{mild solutions} near vacuum to the binary-ternary model and the purely ternary model on its own. By mild solutions we mean that the \(x\)-space dependence of the solution is evaluated along the characteristic curves given by the Hamiltonian evolution of the particle system in between collisions (see Definition 14.1 for details). The analytical techniques we use are inspired by the works \([47, 46, 7, 62, 63, 51]\) and the more recent works \([2, 3]\). These techniques rely on finding convergent supersolutions and subsolutions of \((13.1)\) in the associated strong topology of space-velocity Maxwellian weighted in \(L^\infty\)-functions.

The binary-ternary Boltzmann transport equation we focus on is given by

\[
\begin{aligned}
\partial_t f + v \cdot \nabla_x f &= Q_2(f,f) + Q_3(f,f,f), \\
& \quad (t, x, v) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d, \\
f(0) &= f_0, \\
& \quad (x, v) \in \mathbb{R}^d \times \mathbb{R}^d,
\end{aligned}
\tag{13.1}
\]

and describes the evolution of the probability density \(f\) of a dilute gas in non-equilibrium in \(\mathbb{R}^d, \; d \geq 2\), given an initial condition \(f_0: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}\), when both
binary and ternary interactions among particles can occur. The operator $Q_2(f, f)$ is the classical binary collisional operator, which expresses binary elastic interactions between particles, and is of quadratic order, while the operator $Q_3(f, f, f)$ is the ternary collisional operator which expresses ternary interactions among particles, and is of cubic order. For the exact forms of the operators $Q_2(f, f)$, $Q_3(f, f, f)$ considered in this part of the dissertation, see (14.1), (14.14) respectively. We should mention that the purely ternary model, rigorously derived for short times and hard interactions potentials in Part I, is given by

$$\begin{align*}
\partial_t f + v \cdot \nabla_x f &= Q_3(f, f, f), \quad (t, x, v) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d, \\
f(0) &= f_0, \quad (x, v) \in \mathbb{R}^d \times \mathbb{R}^d,
\end{align*}$$

(13.2)

For the classical Boltzmann transport equation

$$\begin{align*}
\partial_t f + v \cdot \nabla_x f &= Q_2(f, f), \quad (t, x, v) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \\
f(0) &= f_0, \quad (x, v) \in \mathbb{R}^d \times \mathbb{R}^d,
\end{align*}$$

(13.3)

one way to obtain global well-posedness near vacuum is by utilizing an iterative scheme which constructs monotone sequences of supersolutions and subsolutions that converge to the global solution of (13.3). This has been carried out for the first time by Illner and Shinbrot [46], who were motivated by the work of Kaniel and Shinbrot [47], who in turn showed local in time well-posedness for (13.3) following this program. Later, this work was extended to include wider range of potentials and to relax assumptions on initial data by Bellomo and Toscani [7], Toscani [62, 63] and Palczewski and Toscani [51]. Alonso and Gamba [3] used Kaniel-Shinbrot iteration to derive distributional and classical solutions to (13.3) for soft potentials for large initial data near two sufficiently close Maxwellians in position and velocity space,
while Alonso [2] used this technique to study the inelastic Boltzmann equation for hard spheres. Strain [61] remarks that the estimates he derives can be combined with the Kaniel-Shinbrot iteration to obtain existence of unique mild solution for the relativistic Boltzmann equation.

Kaniel-Shinbrot iteration is also an important tool for proving non-negativity of solutions, see for example [53, 34, 20]. Also, when initial data has decay in the direction of $x - v$ as opposed to $x$ and $v$ separately, Kaniel-Shinbrot iteration can be used to construct solutions with infinite energy, see for example [50, 69, 68].

Certain problems have been solved by considering modifications of the Kaniel-Shinbrot iteration. For example, Bellomo and Toscani [64] adapted the iteration to the Boltzmann-Enskog equation. Ha, Noh and Yun [40] and Ha and Noh [39] also modified the iteration to prove global existence of mild solutions to the Boltzmann system for gas mixtures in the elastic and the inelastic cases respectively. Also, Wei and Zhang [67] used another modified iteration to obtain eternal solutions for the Boltzmann equation.

The goal of this part of the dissertation is to establish global existence and uniqueness of a mild solution near vacuum to the binary-ternary Boltzmann equation \((13.1)\) in spaces of non-negative functions bounded by a Maxwellian. Moreover, solution of \((13.2)\) follows as a special case. Inspired by [46, 47, 3], we devise an iterative scheme which constructs monotone sequences of supersolutions and subsolutions to \((13.1)\). For small enough initial data, the beginning condition of the iteration holds globally in time and the two sequences can be shown to converge to the same limit, namely a function $f$ which solves equation \((13.1)\) in a mild sense. This strategy re-
quires new ideas given the fact that ternary interactions are also taken into account in (13.1).

In particular, due to the presence of the ternary correction term, one needs to properly adapt the iteration, so that the corresponding supersolutions and subsolutions remain monotone and convergent. One of the main tools is stated in Lemma 15.4 which provides important exponentially weighted convolution estimates. This Lemma not only recovers the estimates developed in [3] for the binary interaction case, but also develops a new approach in order to treat the ternary interaction case. Lemma 15.4 is crucially used to obtain uniform in time, space-velocity $L^1$-bounds that control the ternary gain and loss terms ($L^\infty L^1$ estimates). In fact, using Lemma 15.4 one first obtains asymmetric estimates (see Lemma 15.6) because of the asymmetry introduced by the ternary collisional operator which is not present in the binary case. However, to obtain convergence, it is essential to have symmetry with respect to the inputs of the gain and loss operators. We were able to achieve this symmetrization in Proposition 15.7. Finally, we also use Lemma 15.4 to prove a global estimate for the time average of the gain and loss operators along the characteristics of the Hamiltonian, see Proposition 15.11. With this, we were able to extend the argument for controlling the binary time integrals of both, gain and loss terms, (see [3]), to the ternary case by invoking properties of ternary interactions and a 2d-analog of the time integration bound for a traveling Maxwellian.

With these tools in hand, for small initial data, the constructed iteration scheme is proved to converge to the unique, global in time mild solution of (13.1). For more details see Chapter 16 and Chapter 17.

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Chapter 14

Towards the statement of the main result

The goal of this chapter is to present the precise statement of the main result of this part of the dissertation. In order to do so, we first review the collisional operators and decompose them to gain and loss form in Section 14.1, introduce necessary notation and the notion of a solution in Section 14.2 and then state the main result in Section 14.3 (Theorem 14.3.1).

14.1 Collisional operators

14.1.1 Binary collisional operator

The binary collisional operator is given by

\[ Q_2(f, f)(t, x, v) = \int_{S^{d-1} \times \mathbb{R}^d} B_2(u, \omega) \left( f(t, x, v') f(t, x, v_1') - f(t, x, v) f(t, x, v_1) \right) d\omega dv_1, \]

where

\[ u := v_1 - v, \]

is the relative velocity of a pair of interacting particles centered at \( x, x_1 \in \mathbb{R}^d \), with velocities \( v, v_1 \in \mathbb{R}^d \) before the binary interaction with respect to the impact direction

\[ \omega := \frac{x_1 - x}{|x - x_1|} \in S^{d-1}, \]

(14.3)
and
\[ v' := v + (\omega \cdot u)\omega, \quad v'_1 := v_1 - (\omega \cdot u)\omega, \quad (14.4) \]
are the outgoing velocities after the binary interaction.

One can easily verify the binary energy-momentum conservation system is satisfied:
\[
 v' + v'_1 = v + v_1, \quad (14.5)
\]
\[
 |v'|^2 + |v'_1|^2 = |v|^2 + |v_1|^2. \quad (14.6)
\]
Either (14.4) or (14.5)-(14.6) imply
\[
 |u'| = |u|, \quad \text{where} \ u' := v'_1 - v'. \quad (14.7)
\]
In addition, equation (14.4) yields the specular reflection with respect to the impact direction \( \omega \)
\[
 \omega \cdot u' = -\omega \cdot u. \quad (14.8)
\]
In fact it is not hard to show that, given \( v, v_1 \in \mathbb{R}^d \), expression (14.4) gives the general solution of the system (14.5)-(14.6), parametrized by \( \omega \in S^{d-1}_1 \). The factor \( B_2 \) in the integrand of (14.1) is referred as the binary interaction differential cross-section which depends on relative velocity \( u \) and the impact direction \( \omega \). It expresses the transition probability of binary interactions, and we assume it is of the form:
\[
 B_2(u, \omega) = |u|^{\gamma_2} b_2(\hat{u} \cdot \omega), \quad \gamma_2 \in (-d + 1, 1], \quad (14.9)
\]
where \( \hat{u} = \frac{u}{|u|} \in S^{d-1}_1 \) is the relative velocity direction and \( b_2 : [-1, 1] \to [0, \infty) \) is the binary angular transition probability density. It is worth mentioning that the
case \( \gamma_2 \in (0, 1) \) corresponds to hard potentials, the case \( \gamma_2 \in (-d + 1, 0) \) corresponds to soft potentials and the case \( \gamma_2 = 0 \) corresponds to Maxwell molecules.

We assume that the binary angular transition probability density \( b_2 \) satisfies the following properties:

- \( b_2 : [-1, 1] \rightarrow \mathbb{R} \) is a measurable, non-negative probability density.
- \( b_2 \) is even i.e.
  \[
  b_2(-z) = b_2(z), \quad \forall z \in [-1, 1],
  \]
  which, due to property from (14.8), yields the binary micro-reversibility condition
  \[
  b_2(\hat{u} \cdot \omega) = b_2(\hat{u} \cdot \omega), \quad \forall \omega \in S^{d-1}_1, \quad \forall v, v_1 \in \mathbb{R}^d,
  \]
  where \( \hat{u}' = \frac{u'}{|u|} \in S^{d-1}_1 \) is the scattering direction. In addition, relations (14.7), (14.9) and (14.11) yield
  \[
  B_2(u', \omega) = B_2(u, \omega), \quad \forall \omega \in S^{d-1}_1, \quad \forall v, v_1 \in \mathbb{R}^d.
  \]
- The probability density is integrable on the sphere, i.e.
  \[
  \|b_2\|_{L^1(S^{d-1}_1)} := \sup_{\nu \in S^{d-1}_1} \int_{S^{d-1}_1} b_2(\nu \cdot \omega) \, d\omega < \infty.
  \]
  By a spherical change of variables, this condition is equivalent to
  \[
  b_2(z)(1 - z^2)^{\frac{d-3}{2}} \in L^1([-1, 1]),
  \]
  since
  \[
  \|b_2\|_{L^1(S^{d-1}_1)} = |S^{d-2}_1| \int_{-1}^{1} |b_2(z)| (1 - z^2)^{\frac{d-3}{2}} \, dz < \infty.
  \]
  where \( |S^{d-2}_1| \) is the measure of the \((d-2)\)-dimensional sphere.
Remark 14.1. The integrability condition on \( b_2 \) is weaker than the classical Grad cut-off assumption which assumes \( b_2 \) is a bounded function of \( z = \hat{u} \cdot \omega \). So our results is valid for a broader class of angular transition probability measures.

Remark 14.2. One can see that the usual hard sphere model is a special case of the form \((14.9)\) for
\[
\gamma_2 = 1, \quad b_2(z) = \frac{|z|}{2}.
\]

### 14.1.2 Ternary collisional operator

The ternary collisional operator is given by (see Part I for details):

\[
Q_3(f, f, f)(t, x, v) = \int_{S_1^{2d-1} \times \mathbb{R}^{2d}} B_3(u, \omega)(f(t, x, v, v^*)f(t, x, v, v_1^*)f(t, x, v, v_2^*) - f(t, x, v)f(t, x, v_1)f(t, x, v_2)) \, d\omega_1 \, d\omega_2 \, dv_1 \, dv_2,
\]

where
\[
\begin{align*}
    u &:= \left( \begin{array}{c} v_1 - v \\ v_2 - v \end{array} \right) \in \mathbb{R}^{2d}, \\
\end{align*}
\]

is the relative velocity of some colliding particles with velocities \( v, v_1, v_2 \in \mathbb{R}^d \) before the ternary interaction with respect to the impact directions vector:

\[
\omega := \left( \begin{array}{c} \omega_1 \\ \omega_2 \end{array} \right) \in S_1^{2d-1},
\]

and
\[
\begin{align*}
v^* & = v + \frac{\omega_1 \cdot (v_1 - v) + \omega_2 \cdot (v_2 - v)}{1 + \omega_1 \cdot \omega_2} (\omega_1 + \omega_2), \\
v_1^* & = v_1 + \frac{\omega_1 \cdot (v_1 - v) + \omega_2 \cdot (v_2 - v)}{1 + \omega_1 \cdot \omega_2} (-2\omega_1 + \omega_2), \\
v_2^* & = v_2 + \frac{\omega_1 \cdot (v_1 - v) + \omega_2 \cdot (v_2 - v)}{1 + \omega_1 \cdot \omega_2} (\omega_1 - 2\omega_2),
\end{align*}
\]

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are the outgoing velocities of the particles after the ternary interaction. It has been
seen in Part I that if \( v^*, v_1^*, v_2^* \) are given by (14.16), the ternary energy-momentum
conservation system
\[
v^* + v_1^* + v_2^* = v + v_1 + v_2, \\
|v^*|^2 + |v_1^*|^2 + |v_2^*|^2 = |v|^2 + |v_1|^2 + |v_2|^2,
\]
is satisfied. Expressions (14.17)-(14.18) also imply the ternary velocities conservation
law
\[
|v^* - v_1^*|^2 + |v^* - v_2^*|^2 + |v_1^* - v_2^*|^2 = |v - v_1|^2 + |v - v_2|^2 + |v_1 - v_2|^2,
\]
For the postcollisional relative velocity, we will write
\[
\mathbf{u}^* := \begin{pmatrix} v_1^* - v^* \\ v_2^* - v^* \end{pmatrix},
\]
and let us also define the quantities
\[
|\tilde{u}| := \sqrt{|v - v_1|^2 + |v - v_2|^2 + |v_1 - v_2|^2}, \\
|\tilde{u}^*| := \sqrt{|v^* - v_1^*|^2 + |v^* - v_2^*|^2 + |v_1^* - v_2^*|^2}.
\]
Then (14.19) can be written as
\[
|\tilde{u}| = |\tilde{u}^*|,
\]
which is the ternary analog of the binary expression (14.7). Defining
\[
\bar{u} := \frac{\mathbf{u}}{|\mathbf{u}|}, \quad \bar{u}^* := \frac{\mathbf{u}^*}{|\mathbf{u}|},
\]
equality (14.19) implies \( \bar{u}, \bar{u}^* \in \mathbb{E}^{2d-1}_1 \), where the \((2d - 1)\)-dimensional ellipsoid
\[
\mathbb{E}^{2d-1}_1 := \{ (\nu_1, \nu_2) \in \mathbb{R}^{2d} : |\nu_1|^2 + |\nu_2|^2 + |\nu_1 - \nu_2|^2 = 1 \},
\]
is given by (9.23). The vectors $\vec{u}, \vec{u}^*$ are the ternary analogs of the relative velocity direction and the scattering direction of the binary interaction. Because of the asymmetry of the ternary interaction they are not unit vectors, they lie on the ellipsoid $E^{2d-1}_1$ instead. However, for convenience we will refer to $\vec{u}, \vec{u}^*$ as relative velocity direction and scattering direction respectively.

The collisional formulas (14.16) also imply (see Part I)

$$\omega \cdot \vec{u}^* = -\omega \cdot \vec{u},$$  \hspace{1cm} (14.25)

which is the ternary analog to specular reflection with respect to the impact directions vector $\omega = (\omega_1, \omega_2) \in S^{2d-1}_1$. The term $B_3$ in the integrand of (14.14), depending on the relative velocity $u \in \mathbb{R}^{2d}$ and the impact directions vector $\omega = (\omega_1, \omega_2) \in S^{2d-1}_1$, is the ternary interaction differential cross-section, which describes the transition probability of ternary interactions. Recalling $|\vec{u}|$ from (14.21) and $\vec{u} \in E^{2d-1}_1$ from (14.24), we assume $B_3$ takes the form

$$B_3(u, \omega) = |\vec{u}|^{\gamma_3} b_3 (\vec{u} \cdot \omega, \omega_1 \cdot \omega_2), \quad \gamma_3 \in (-2d + 1, 1], \ \ (14.26)$$

and $b_3 : [-1,1] \times [-\frac{1}{2}, \frac{1}{2}] \to [0, \infty)$ is the ternary angular transition probability density. Since $\omega = (\omega_1, \omega_2) \in S^{2d-1}_1$, Cauchy-Schwartz inequality and (14.21) yield

$$|\vec{u} \cdot \omega| \leq |\vec{u}| \cdot |\omega| = \frac{|u|}{|\vec{u}|} \leq 1.$$

Moreover, for any $\omega = (\omega_1, \omega_2) \in S^{2d-1}_1$, Cauchy-Schwartz inequality followed by Young’s inequality yield

$$|\omega_1 \cdot \omega_2| \leq |\omega_1| \cdot |\omega_2| \leq \frac{|\omega_1|^2 + |\omega_2|^2}{2} = \frac{1}{2}.$$
Therefore, for any \( \omega = (\omega_1, \omega_2) \in S_1^{2d-1} \), the expression \( b_3(\bar{\upsilon} \cdot \omega, \omega_1 \cdot \omega_2) \) is well defined.

In addition, we assume that the ternary angular transition probability density \( b_3 \) satisfies the following properties

- \( b_3 : [-1, 1] \times [-1, 1] \rightarrow \mathbb{R} \) is a measurable, non-negative probability density.
- \( b_3 \) is even with respect to the first argument i.e.
  \[
  b_3(-z, w) = b_3(z, w), \quad \forall (z, w) \in [-1, 1] \times \left[-\frac{1}{2}, \frac{1}{2}\right].
  \]  
  \[
  (14.28)
  \]

In addition, due to (14.25), the ternary micro-reversibility condition holds

\[
  b_3(\bar{\upsilon}^* \cdot \omega, \omega_1 \cdot \omega_2) = b_3(\bar{\upsilon} \cdot \omega, \omega_1 \cdot \omega_2), \quad \forall \omega \in S_1^{2d-1}, \quad \forall v, v_1, v_2 \in \mathbb{R}^d,
\]

\[
  (14.29)
\]

and relations (14.26), (14.24) and (14.29) imply the total ternary collision kernel satisfies

\[
  B_3(u^*, \omega) = B_3(u, \omega), \quad \forall \omega \in S_1^{2d-1}, \quad \forall v, v_1, v_2 \in \mathbb{R}^d.
\]

\[
  (14.30)
\]

- The probability density \( b_3 \) is integrable on \( S_1^{2d-1} \) i.e.
  \[
  \|b_3\|_{L^1(S_1^{2d-1})} := \sup_{\nu \in S_1^{2d-1}} \int_{S_1^{2d-1}} b_3(\nu \cdot \omega, \omega_1 \cdot \omega_2) d\omega < \infty.
  \]
  \[
  (14.31)
  \]

\textbf{Remark 14.3}. One can see that the ternary operator introduced in Part I is a special case of (14.26) for

\[
  \gamma_3 = 1, \quad b_3(z) = \frac{1}{2} \left| \frac{z}{1 + w} \right|.
\]
Remark 14.4. Throughout this part of the dissertation, we assume that at least one of $b_2, b_3$ is not trivially zero; however one of the two could be zero. If $b_3 = 0$, we recover the classical Boltzmann equation (13.3), while if $b_2 = 0$ we recover the ternary Boltzmann equation (13.2). As it will become clear, see for instance (14.77), the dependence on the size of $b_2$ and $b_3$ is additive implying that the two collisional operators can be studied separately.

Remark 14.5. One can assume as well that the cross-section is of the form

$$\tilde{B}_3(\mathbf{u}, \omega) = |\mathbf{u}|^{\gamma_3} \tilde{b}_3(\hat{\mathbf{u}} \cdot \omega, \langle \omega_2, \omega_2 \rangle),$$

with $\tilde{b}_3$ non-negative, even at the first argument and integrable in the sense that

$$\|\tilde{b}_3\|_{L^1(S^{d-1})} := \sup_{\theta \in S^{d-1}} \int_{S^{d-1}} \tilde{b}_3(\hat{\mathbf{v}} \cdot \omega, \omega_1, \omega_2) d\omega < \infty, \quad (14.32)$$

and obtain similar results. This is possible because $|\mathbf{u}|$ is always comparable to the preserved quantity of the ternary interaction $|\tilde{\mathbf{u}}|$ (see (15.12)), thus one can replace (14.30) by the corresponding inequality.

### 14.1.3 Gain and loss operators

It turns out more convenient to study the more general collisional operators

$$Q_2(f, g)(t, x, v) = \int_{S^{d-1} \times \mathbb{R}^d} B_2(u, \omega) \left(f(t, x, v')g(t, x, v_1') - f(t, x, v)g(t, x, v_1)\right) d\omega dv_1, \quad (14.33)$$

$$Q_3(f, g, h)(t, x, v) = \int_{S^{d-1} \times \mathbb{R}^d} B_3(u, \omega) \left(f(t, x, v')g(t, x, v_1')h(t, x, v_2') - f(t, x, v)g(t, x, v_1)h(t, x, v_2)\right) d\omega_1 d\omega_2 dv_1 dv_2, \quad (14.34)$$

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Due to the assumptions (14.13), (14.31), the binary-ternary operator $Q_2(f, g) + Q_3(f, g, h)$ can be decomposed into a gain and a loss term as follows:

$$Q_2(f, g) + Q_3(f, g, h) = G(f, g, h) - L(f, g, h),$$  \hspace{1cm} (14.35)\

where

$$L(f, g, h) = L_2(f, g) + L_3(f, g, h),$$  \hspace{1cm} (14.36)\

$$G(f, g, h) = G_2(f, g) + G_3(f, g, h).$$  \hspace{1cm} (14.37)\

The binary gain and loss operators $G_2, L_2$ are given respectively by

$$G_2(f, g)(t, x, v) = \int_{S^{d-1} \times \mathbb{R}^d} B_2(u, \omega) f(t, x, v') g(t, x, v_1') d\omega dv_1,$$  \hspace{1cm} (14.38)\

$$L_2(f, g)(t, x, v) = \int_{S^{d-1} \times \mathbb{R}^d} B_2(u, \omega) f(t, x, v) g(t, x, v_1) d\omega dv_1,$$  \hspace{1cm} (14.39)\

and are clearly bilinear. The ternary gain and loss operators $L_3, G_3$ are given respectively by

$$G_3(f, g, h)(t, x, v) = \int_{S^{2d-1} \times \mathbb{R}^{2d}} B_3(u, \omega) f(t, x, v^*) g(t, x, v_1^*) h(t, x, v_2^*) d\omega_1 d\omega_2 dv_1 dv_2,$$  \hspace{1cm} (14.41)\

$$L_3(f, g, h)(t, x, v) = \int_{S^{2d-1} \times \mathbb{R}^{2d}} B_3(u, \omega) f(t, x, v) g(t, x, v_1) h(t, x, v_2) d\omega_1 d\omega_2 dv_1 dv_2,$$  \hspace{1cm} (14.42)\

and are clearly trilinear. Notice the loss term can be factorized as

$$L(f, g, h) = fR(g, h),$$  \hspace{1cm} (14.43)
where $R$ is given by

$$R(g, h) := R_2(g) + R_3(g, h), \quad (14.44)$$

$R_2$ is the linear operator

$$R_2(g)(t, x, v) := \int_{S^{d-1} \times \mathbb{R}^d} B_2(u, \omega) g(t, x, v_1) d\omega dv_1, \quad (14.45)$$

and $R_3$ is the bilinear operator

$$R_3(g, h)(t, x, v) := \int_{S^{2d-1} \times \mathbb{R}^{2d}} B_3(u, \omega^1) g(t, x, v_1) h(t, x, v_2) d\omega_1 d\omega_2 dv_1 dv_2. \quad (14.46)$$

### 14.2 Some notation and the notion of a solution

Throughout this part of the dissertation, the dimension $d \geq 2$, the binary and ternary integrability assumptions which imply (14.13), (14.31) respectively, and the cross-section exponents

$$\gamma_2 \in (-d + 1, 1], \quad \gamma_3 \in (-2d + 1, 1], \quad (14.47)$$

appearing respectively in (14.9), (14.26) will be fixed.

#### 14.2.1 Functional spaces

Let us introduce the functional spaces used in this part of the dissertation. First, in order to point out the dependence in positions and velocities, we will use the notation:

$$L^1_{x,v} := L^1(\mathbb{R}^d \times \mathbb{R}^d), \quad (14.48)$$

$$L^\infty_{x,v} := L^\infty(\mathbb{R}^d \times \mathbb{R}^d). \quad (14.49)$$
We also define the sets of space-velocity functions:

$$F_{x,v} := \{ f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}, \text{ such that } f \text{ is measurable} \}. \quad (14.50)$$

$$F_{x,v}^+ := \{ f \in F_{x,v} : f(x,v) \geq 0, \text{ for a.e. } (x,v) \in \mathbb{R}^d \times \mathbb{R}^d \}, \quad (14.51)$$

$$L_{x,v}^{1+} := L_{x,v}^1 \cap F_{x,v}^+. \quad (14.52)$$

In general, for $f, g \in F_{x,v}$, we write $f \geq g$ iff $f(x,v) \geq g(x,v)$ for a.e. $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$. Same notation will hold for equality as well.

Given $\alpha, \beta > 0$, we define the corresponding (non-normalized) Maxwellian $M_{\alpha,\beta} : \mathbb{R}^d \times \mathbb{R}^d \to (0, \infty)$ by:

$$M_{\alpha,\beta}(x,v) := e^{-\alpha|x|^2 - \beta|v|^2}. \quad (14.53)$$

We also define the corresponding Banach space of functions essentially bounded by $M_{\alpha,\beta}$ as:

$$\mathcal{M}_{\alpha,\beta} := \{ f \in F_{x,v} : \|f\|_{M_{\alpha,\beta}} < \infty \}. \quad (14.54)$$

where

$$\|f\|_{M_{\alpha,\beta}} := \|f M_{\alpha,\beta}^{-1}\|_{L_x^\infty}.$$ 

We will write $f_n \overset{\mathcal{M}_{\alpha,\beta}}{\longrightarrow} f$ if

$$f_n \overset{\text{a.e.}}{\longrightarrow} f \quad \text{and} \quad \sup_{n \in \mathbb{N}} \|f_n\|_{\mathcal{M}_{\alpha,\beta}} < \infty. \quad (14.55)$$

It is clear that if $f_n \overset{\mathcal{M}_{\alpha,\beta}}{\longrightarrow} f$ then $f_n \in \mathcal{M}_{\alpha,\beta}$ for all $n \in \mathbb{N}$ and $f \in \mathcal{M}_{\alpha,\beta}$. If $k \in \mathbb{N}$ and $f_{1,n} \overset{\mathcal{M}_{\alpha,\beta}}{\longrightarrow} f_1, f_{2,n} \overset{\mathcal{M}_{\alpha,\beta}}{\longrightarrow} f_2, \ldots, f_{k,n} \overset{\mathcal{M}_{\alpha,\beta}}{\longrightarrow} f_k$, we will write

$$(f_{1,n}, \ldots, f_{k,n}) \overset{\mathcal{M}_{\alpha,\beta}}{\longrightarrow} (f_1, \ldots, f_k).$$
We also define the set of a.e. non-negative functions essentially bounded by $M_{\alpha,\beta}$ as:

\[ M_{\alpha,\beta}^+ := M_{\alpha,\beta} \cap F_{x,v}^+. \tag{14.56} \]

Given $0 < T \leq \infty$, we define the sets of time dependent functions

\[ \mathcal{F}_T := \{ f : [0, T) \to F_{x,v} \}, \tag{14.57} \]
\[ \mathcal{F}_T^+ := \{ f : [0, T) \to F_{x,v}^+ \}. \tag{14.58} \]

and given $f, g \in \mathcal{F}_T$, we will write $f \geq g$ iff $f(t) \geq g(t)$ for all $t \in [0, T)$. Same notation will hold for equalities as well.

Finally, we define the following subsets of functional spaces

\[ C^0([0, T), L^1_{x,v}^+) := C^0([0, T), L^1_{x,v}) \cap \mathcal{F}_T^+, \tag{14.59} \]
\[ L^1_{loc}([0, T), L^1_{x,v}^+) := L^1_{loc}([0, T), L^1_{x,v}) \cap \mathcal{F}_T^+, \tag{14.60} \]
\[ L^\infty([0, T), L^1_{x,v}^+) := L^\infty([0, T), L^1_{x,v}) \cap \mathcal{F}_T^+, \tag{14.61} \]

and given $\alpha, \beta > 0$, we define the Banach space of time dependent functions uniformly essentially bounded by $M_{\alpha,\beta}$

\[ L^\infty([0, T), M_{\alpha,\beta}) := \{ f \in \mathcal{F}_T : |||f|||_\infty < \infty \}, \tag{14.62} \]

with norm

\[ |||f|||_\infty = \sup_{t \in [0,T)} \| f(t) \|_{M_{\alpha,\beta}}. \tag{14.63} \]

Notice that in definition (14.62), the supremum is taken with respect to all $t \in [0, T)$.

We also write

\[ L^\infty([0, T), M_{\alpha,\beta}^+) := L^\infty([0, T), M_{\alpha,\beta}) \cap \mathcal{F}_T^+. \tag{14.64} \]
14.2.2 Transport operator

We now introduce the transport operator which will be crucial to define mild solutions to (13.1). Let us recall from (14.50)-(14.51) the sets of functions:

\[ F_{x,v} := \{ f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}, \text{ such that } f \text{ is measurable} \}, \]

\[ F_{x,v}^+ := \{ f \in F_{x,v} : f(x,v) \geq 0, \text{ for a.e. } (x,v) \in \mathbb{R}^d \times \mathbb{R}^d \}. \]

Consider a positive time \( 0 < T \leq \infty \) (we can have \( T = \infty \)) and recall from (14.57)-(14.58) the sets of time dependent functions

\[ \mathcal{F}_T := \{ f : [0,T) \to F_{x,v} \}, \]

\[ \mathcal{F}_T^+ := \{ f : [0,T) \to F_{x,v}^+ \}. \]

Given \( f \in \mathcal{F}_T \), we define \( f^\# \in \mathcal{F}_T \) by:

\[ f^\#(t,x,v) := f(t,x + tv,v). \quad (14.65) \]

and \( f^\# \in \mathcal{F}_T \) by

\[ f^\#(t,x,v) := f(t,x - tv,v). \]

Clearly, the operators \( \# : \mathcal{F}_T \to \mathcal{F}_T \) and \( -\# : \mathcal{F}_T \to \mathcal{F}_T \) are linear and invertible and in particular

\[ (\#)^{-1} = -\#. \]

Remark 14.6. Let \( f, g \in \mathcal{F}_T \). Since the maps \( (x,v) \to (x + tv,v) \) and \( (x,v) \to (x - tv,v) \) are measure-preserving, for all \( t \in [0,T) \), we have

\[ f \geq g \iff f^\# \geq g^\# \iff f^\# \geq g^\#. \]
In particular
\[ f \in \mathcal{F}_T^+ \iff f^\# \in \mathcal{F}_T^+ \iff f^{-\#} \in \mathcal{F}_T^+ . \] (14.66)

**Remark 14.7.** Let \( f, g \in \mathcal{F}_T \). Since the maps \((x, v) \to (x + tv, v)\) and \((x, v) \to (x - tv, v)\) are measure-preserving, for all \( t \in [0, T) \), we have
\[
\|f^\#(t)\|_{L^1_{x,v}} = \|f(t)\|_{L^1_{x,v}} = \|f^{-\#}(t)\|_{L^1_{x,v}}, \quad \forall t \in [0, T). \tag{14.67}
\]

Relation (14.66)-(14.67) and linearity of the transport operator imply
\[
f \in C^0([0, T), L^1_{x,v}^{1,+}) \iff f^\# \in C^0([0, T), L^1_{x,v}^{1,+}) \iff f^{-\#} \in C^0([0, T), L^1_{x,v}^{1,+}). \tag{14.68}
\]

Throughout this part of the dissertation, we often define \( f^\# \in \mathcal{F}_T \) directly, implying that \( f \) is defined by \( f := (f^\#)^{-\#} \).

### 14.2.3 Transported gain and loss operators

In order to define mild solutions to (13.1), it is important to understand the action of the transport operator on the gain and loss operators. More specifically, given \( f, g, h \in \mathcal{F}_T \), for the gain operators we write
\[
G^\#_2(f, g)(t, x, v) := (G_2(f, g))^\#(t, x, v) \\
= \int_{S^d \times \mathbb{R}^d} B_2(u, \omega) f(t, x + tv, v') g(t, x + tv, v') \, d\omega \, dv,
\]
\[
G^\#_3(f, g, h)(t, x, v) := (G_3(f, g, h))^\#(t, x, v) \\
= \int_{S^{2d-1} \times \mathbb{R}^{2d}} B_3(u, \omega) f(t, x + tv, v^*) g(t, x + tv, v^*) h(t, x + tv, v^*) \, d\omega_1 \, d\omega_2 \, dv_1 \, dv_2,
\]
\[
G^\#(f, g, h)(t, x, v) := G^\#_2(f, g)(t, x, v) + G^\#_3(f, g, h)(t, x, v). \tag{14.69}
\]
and for the loss operators we write

\[ L_2^#(f, g)(t, x, v) := (L_2(f, g))^#(t, x, v) \]
\[ = \int_{S^{d-1} \times \mathbb{R}^d} B_2(u, \omega) f(t, x + tv, v) g(t, x + tv, v_1) \, d\omega \, dv_1, \]

\[ L_3^#(f, g, h)(t, x, v) := (L_3(f, g, h))^#(t, x, v) \]
\[ = \int_{S^{2d-1} \times \mathbb{R}^{2d}} B_3(u, \omega) f(t, x + tv, v) g(t, x + tv, v_1) h(t, x + tv, v_2) \, d\omega_1 \, d\omega_2 \, dv_1 \, dv_2, \]

\[ L^#(f, g, h)(t, x, v) := L_2^#(f, g)(t, x, v) + L_3^#(f, g, h)(t, x, v), \quad (14.70) \]

Under this notation, it is straightforward to verify that

\[ L_2^#(f, g, h)(t) = f^#(t) R_2^#(g)(t), \]
\[ L_3^#(f, g, h)(t) = f^#(t) R_3^#(g, h)(t), \quad (14.71) \]
\[ L^#(f, g, h)(t) = f^#(t) R^#(g, h)(t). \]

where

\[ R_2^#(g)(t, x, v) := (R_2(g))^#(t, x, v) = \int_{S^{d-1} \times \mathbb{R}^d} B_2(u, \omega) g(t, x + tv, v_1) \, d\omega \, dv_1, \]
\[ R_3^#(g, h)(t, x, v) := (R_3(g, h))^#(t, x, v) \]
\[ = \int_{S^{2d-1} \times \mathbb{R}^{2d}} B_3(u, \omega) g(t, x + tv, v_1) h(t, x + tv, v_2) \, d\omega_1 \, d\omega_2 \, dv_1 \, dv_2, \]
\[ R^#(g, h)(t, x, v) := R_2^#(g)(t, x, v) + R_3^#(g, h)(t, x, v). \quad (14.72) \]

14.2.4 Notion of a mild solution

Using (14.35), the binary-ternary Boltzmann equation (13.1) is written as follows:

\[ \begin{align*}
\frac{\partial}{\partial t} f + v \cdot \nabla_x f &= G(f, f, f) - L(f, f, f), \quad (t, x, v) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d, \\
f(0) &= f_0, \quad (x, v) \in \mathbb{R}^d \times \mathbb{R}^d,
\end{align*} \quad (14.73) \]
where the gain term $G(f, f, f)$ and the loss term $L(f, f, f)$ are given by (14.37)-(14.36) respectively.

Here is where the importance of the transport operator will become clear. Indeed, using the chain rule, the initial value problem (14.73) can be formally written as

\[
\begin{aligned}
\partial_t f^# + L^#(f, f, f) &= G^#(f, f, f), \quad (t, x, v) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d, \\
 f^#(0) &= f_0, \quad (x, v) \in \mathbb{R}^d \times \mathbb{R}^d.
\end{aligned}
\]

(14.74)

Motivated by (14.74), we aim to define solutions of (13.1) up to time $0 < T \leq \infty$, with respect to a given Maxwellian $M_{\alpha, \beta}$, where $\alpha, \beta > 0$.

**Definition 14.1.** Let $0 < T \leq \infty$, $\alpha, \beta > 0$ and $f_0 \in M_{\alpha, \beta}^+$. A mild solution to (13.1) in $[0, T)$, with initial data $f_0 \in M_{\alpha, \beta}^+$, is a function $f \in \mathcal{F}_T^+$ such that

(i) $f^# \in C^0([0, T), L_{x,v}^1) \cap L^\infty([0, T), M_{\alpha, \beta}^+)$,

(ii) $L^#(f, f, f), G^#(f, f, f) \in L^\infty([0, T), L_{x,v}^{1+})$,

(iii) $f^#$ is weakly differentiable and satisfies

\[
\begin{aligned}
\frac{df^#}{dt} + L^#(f, f, f) &= G^#(f, f, f), \\
f^#(0) &= f_0.
\end{aligned}
\]

(14.75)

**Remark 14.8.** The differential equation of (14.75) is interpreted as an equality of distributions since all terms involved belong to $L_{loc}^1([0, T), L_{x,v}^{1+})$.

**Remark 14.9.** Remarks 14.6-14.7 imply that a mild solution $f$ to (13.1) belongs to $C^0([0, T), L_{x,v}^{1+})$. 

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14.3 Statement of the main result

Now we are ready to state the main result of this part of the dissertation.

Theorem 14.3.1. Let $0 < T \leq \infty$, $\alpha, \beta > 0$. Then for any initial data $f_0 \in \mathcal{M}_{\alpha,\beta}^+$ with

\[
\|f_0\|_{\mathcal{M}_{\alpha,\beta}} < \frac{\alpha^{1/2}}{48K_\beta(1 + \frac{\alpha^{1/4}}{2\sqrt{6K_\beta}})},
\]

where

\[
K_\beta = C_d \left[ \|b_2\|_{L^1(S^d-1)}(\beta^{-d/2} + \frac{1}{d + \gamma_2 - 1}) + \|b_3\|_{L^1(S^d-1)}(\beta^{-d} + \frac{1}{2d + \gamma_3 - 1}) \right] > 0,
\]

and $C_d$ is an appropriate constant depending on the dimension $d$, the binary-ternary Boltzmann equation (13.1) has a unique mild solution $f$ satisfying the bound

\[
\|f^\#\|_{\infty} \leq \frac{1}{\sqrt{1 - 48K_\beta\alpha^{-1/2}(1 + \frac{\alpha^{1/4}}{2\sqrt{6K_\beta}})\|f_0\|_{\mathcal{M}_{\alpha,\beta}}}} - \frac{24K_\beta\alpha^{-1/2}}{\left(1 + \frac{\alpha^{1/4}}{2\sqrt{6K_\beta}}\right)}.
\]

Remark 14.10. As we will see, the uniqueness claimed above holds in the class of solutions of (13.1) satisfying (14.78).

Remark 14.11. According to the assumptions on $b_2, b_3$ made in Remark 14.4, Theorem 14.3.1 applies as well to the end point cases where either $b_2 = 0$ or $b_3 = 0$ (but not both). In the case $b_3 = 0$, one recovers the solution of the classical Boltzmann equation (13.3) constructed in [46], while in the case $b_2 = 0$, one obtains well-posedness of the ternary Boltzmann equation (13.2), introduced in Part I.
**Remark 14.12.** Similar well-posedness results may be obtained if instead of $Q_3$, one considers the symmetrized ternary operator $\tilde{Q}_3$ as a higher correction term, see Section 5.4 for an analysis of the symmetrized ternary collisional operator. In fact, we have seen that the symmetrized operator $\tilde{Q}_3$ satisfies the same conservation laws as the binary operator $Q_2$, therefore mass, momentum and energy are conserved by the mild solutions. Indeed, assume $f$ is a mild solution of the symmetrized binary-ternary Boltzmann equation:

$$
\begin{cases}
\partial_t f + v \cdot \nabla_x f = Q_2(f, f) + \tilde{Q}_3(f, f, f), \\
f(0) = f_0.
\end{cases}
$$

i.e.

$$
f(t, x, v) = f_0(x - vt, v) + \int_0^t \left( Q_2(f, f) + \tilde{Q}_3(f, f, f) \right) (\tau, x, v) \, d\tau.
$$

(14.79)

Let $\phi(v) \in \{1, v, |v|^2\}$. Integrating (14.79) in $v$, and using Fubini’s Theorem, we obtain

$$
\int_{\mathbb{R}^d} f(t, x, v) \phi(v) \, dv
= \int_{\mathbb{R}^d} f_0(x - tv, v) \phi(v) \, dv + \int_0^t \int_{\mathbb{R}^d} \left( Q_2(f, f) + \tilde{Q}_3(f, f, f) \right) (\tau, x, v) \phi(v) \, dv \, d\tau
= \int_{\mathbb{R}^d} f_0(x - tv, v) \phi(v) \, dv,
$$

by the conservation laws of $Q_2$ and $\tilde{Q}_3$. Integrating in $x$ as well, Fubini’s Theorem and the change of variables $x - tv \to x$ imply

$$
\int_{\mathbb{R}^{2d}} f(t, x, v) \phi(v) \, dv \, dx = \int_{\mathbb{R}^{2d}} f_0(x - tv, v) \phi(v) \, dv \, dx = \int_{\mathbb{R}^{2d}} f_0(x, v) \phi(v) \, dv \, dx,
$$

therefore mass, momentum and energy are conserved by the mild solutions in the symmetrized case.
Chapter 15

Properties of the transported gain and loss operators

In this chapter, we investigate properties of the transported gain and loss operators which will be important for proving global well-posedness of (13.1).

15.1 Monotonicity and $L^1$-norms

As we will see, the transported gain and loss operators are monotone increasing when acting on non-negative functions. These monotonicity properties will allow us to construct monotone sequences of supersolutions and subsolutions to (13.1). Moreover, we show that the $L^1$-norm of the gain is equal to the $L^1$-norm of the loss. This equality will allow us to reduce estimates on the norm of the gain term to estimating the norm of the loss term. In the following, saying that an operator is bilinear/trilinear, we mean it is linear in each argument, and saying it is monotone increasing, we mean it is increasing in each argument.

**Proposition 15.2.** Let $0 < T \leq \infty$. Then the following hold

(i) $R^\#_2 : \mathcal{F}_T^+ \to \mathcal{F}_T^+$ is linear and monotone increasing.

(ii) $L^\#_2, G^\#_2, R^\#_3 : \mathcal{F}_T^+ \times \mathcal{F}_T^+ \to \mathcal{F}_T^+$ are bilinear and monotone increasing.
(iii) $L_3^#, G_3^# : \mathcal{F}_T^+ \times \mathcal{F}_T^+ \times \mathcal{F}_T^+ \rightarrow \mathcal{F}_T^+$ are trilinear and monotone increasing.

(iv) $L^#, G^# : \mathcal{F}_T^+ \times \mathcal{F}_T^+ \times \mathcal{F}_T^+ \rightarrow \mathcal{F}_T^+$ and $R^# : \mathcal{F}_T^+ \times \mathcal{F}_T^+ \rightarrow \mathcal{F}_T^+$ are monotone increasing.

(v) For any $f, g, h \in \mathcal{F}_T^+$, the following identities hold

\[
\|G_2^#(f, g)(t)\|_{L_{1,v}} = \|G_2(f, g)(t)\|_{L_{1,v}}, \quad \forall t \in [0, T),
\]

\[
\|G_3^#(f, g, h)(t)\|_{L_{1,v}} = \|L_3^#(f, g, h)(t)\|_{L_{1,v}}, \quad \forall t \in [0, T),
\]

\[
\|G^#(f, g, h)(t)\|_{L_{1,v}} = \|L^#(f, g, h)(t)\|_{L_{1,v}}, \quad \forall t \in [0, T).
\]

(15.1)

Proof. Parts (i)-(iv) are immediate by linearity of the integral, positivity of the functions considered and relation (14.66).

Let us now prove (v). We first prove (15.1) for the binary case. By (14.67), for any $t \in [0, T)$, we have

\[
\|G_2^#(f, g)(t)\|_{L_{1,v}} = \|G_2(f, g)(t)\|_{L_{1,v}}, \quad \|L_2^#(f, g)(t)\|_{L_{1,v}} = \|L_2(f, g)(t)\|_{L_{1,v}}.
\]

Therefore, for any $t \in [0, T)$, using (14.12) and involutionary substitution $(v', v'_1) \rightarrow (v, v_1)$, we obtain

\[
\|G_2^#(f, g)(t)\|_{L_{1,v}} = \|G_2(f, g)(t)\|_{L_{1,v}} = \int_{\mathbb{R}^d \times \mathbb{S}_1^{d-1}} B_2(u, \omega) f(t, x, v') g(t, x, v'_1) \, d\omega \, dv_1 \, dv \, dx
\]

\[
= \int_{\mathbb{R}^d \times \mathbb{S}_1^{d-1}} B_2(u', \omega) f(t, x, v') g(t, x, v'_1) \, d\omega \, dv_1 \, dv \, dx
\]

\[
= \int_{\mathbb{R}^d \times \mathbb{S}_1^{d-1}} B_2(u, \omega) f(t, x, v) g(t, x, v_1) \, d\omega \, dv_1 \, dv \, dx
\]

\[
= \|G_2(f, g)(t)\|_{L_{1,v}}.
\]

\[
= \|L_2^#(f, g)(t)\|_{L_{1,v}}.
\]
We now prove (15.1) for the ternary case. By (14.67), for all \( t \in [0,T) \), we have
\[
\|G_3^\#(f, g, h)(t)\|_{L_{x,v}^1} = \|G_3(f, g, h)(t)\|_{L_{x,v}^1}, \quad \|L_3^\#(f, g, h)(t)\|_{L_{x,v}^1} = \|L_3(f, g, h)(t)\|_{L_{x,v}^1},
\]
Therefore, for any \( t \in [0,T) \), using (14.30) and the involutory substitution \((v^*, v_1^*, v_2^*) \rightarrow (v, v_1, v_2)\),
we obtain
\[
\|G_3^\#(f, g, h)(t)\|_{L_{x,v}^1} = \|G_3(f, g, h)(t)\|_{L_{x,v}^1}
= \int_{\mathbb{R}^{4d} \times S_{d-1}^{2d}} B_3(u, \omega) f(t, x, v^*) g(t, x, v_1^*) h(t, x, v_2^*) \, dw_1 \, dw_2 \, dv_1 \, dv_2 \, dv \, dx
= \int_{\mathbb{R}^{4d} \times S_{d-1}^{2d}} B_3(u^*, \omega) f(t, x, v^*) g(t, x, v_1^*) h(t, x, v_2^*) \, dw_1 \, dw_2 \, dv_1 \, dv_2 \, dv \, dx
= \|L_3(f, g, h)(t)\|_{L_{x,v}^1},
= \|L_3^\#(f, g, h)(t)\|_{L_{x,v}^1}.
\]
We finally prove (15.1) for the mixed case. By positivity, for any \( t \in [0,T) \), we have
\[
\|G^\#(f, g, h)(t)\|_{L_{x,v}^1} = \|G_2^\#(f, g)(t) + G_3^\#(f, g, h)(t)\|_{L_{x,v}^1}
= \|G_2^\#(f, g)(t)\|_{L_{x,v}^1} + \|G_3^\#(f, g, h)(t)\|_{L_{x,v}^1},
\]
\[
\|L^\#(f, g, h)(t)\|_{L_{x,v}^1} = \|L_2^\#(f, g)(t) + L_3^\#(f, g, h)(t)\|_{L_{x,v}^1}
= \|L_2^\#(f, g)(t)\|_{L_{x,v}^1} + \|L_3^\#(f, g, h)(t)\|_{L_{x,v}^1}.
\]
Equality (15.1) for the mixed case immediately follows the from the corresponding binary and ternary equalities. \( \square \)
15.3 Convolution estimates

We now present a general convolution-type result, which will be essential for the control of the binary and the ternary collisional operators. These estimates will be of fundamental importance in the proof of the $L^\infty L^1$ estimates (see Section 15.5) and the global estimate on the time average of the transported gain and loss operators appearing in Proposition 15.11 which in turn will be crucial for the proof of global well-posedness of (13.1). For the binary case one can find similar convolution estimates in [47, 46, 3]. Here, our contribution is the derivation of these estimates for the ternary case, since this is the first time global well-posedness is studied for such a ternary correction of the Boltzmann equation. The estimates of the ternary term illustrate that consideration of softer potentials is allowed for the ternary collisional operator.

Lemma 15.4. Let $\beta > 0$, $q_2 \in (-d, 1]$ and $q_3 \in (-2d, 1]$. Then the following hold

(i) For any $v \in \mathbb{R}^d$, we have

\[
\int_{\mathbb{R}^d} |u|^{q_2} e^{-\beta |v_1|^2} dv_1 \leq \tilde{K}_{\beta, q_2}^2 (1 + |v|^{q_2^+}),
\]

where $u = v_1 - v$, $q_2^+ := \max\{0, q_2\}$, $\tilde{K}_{\beta, q_2}^2$ is given by

\[
\tilde{K}_{\beta, q_2}^2 = C_d \left[ (1 + \beta^{-d/2} + \beta^{-d+1/2}) 1_{q_2 > 0}(q_2) + (\beta^{-d/2} + \frac{1}{d + q_2^+}) 1_{q_2 \leq 0}(q_2) \right],
\]

and $C_d$ is an appropriate constant depending on the dimension $d$.

(ii) For any $v \in \mathbb{R}^d$, we have

\[
\int_{\mathbb{R}^{2d}} |\tilde{u}|^{q_3} e^{-\beta (|v_1|^2 + |v_2|^2)} dv_1 dv_2 \leq \tilde{K}_{\beta, q_3}^3 (1 + |v|^{q_3^+}),
\]
where $|\tilde{u}|$ is given by (14.21), $q_3^+ := \max\{0, q_3\}$, $\tilde{K}_{\beta, q_3}$ is given by

$$\tilde{K}_{\beta, q_3}^3 = C_d \left[ (1 + \beta - d + \beta^{-\frac{2d+1}{2}})1_{q_3 > 0}(q_3) + (\beta^{-d} + \frac{1}{2d+q_3})1_{q_3 \leq 0}(q_3) \right],$$

and $C_d$ is an appropriate constant depending on the dimension $d$.

Proof. We will rely on the elementary estimate

$$\int_{\mathbb{R}^d} e^{-\beta|v_1|^2} dv_1 \leq C_d \beta^{-d/2},$$

and, given $q \in (0, 1]$, on the estimate

$$\int_{\mathbb{R}^d} |v|^q e^{-\beta|v_1|^2} dv_1 \leq |B_d^q| + \int_{|v_1| > 1} |v|^q e^{-\beta|v_1|^2} dv_1$$

$$\leq |B_d^q| + \int_{|v_1| > 1} |v|^q e^{-\beta|v_1|^2} dv_1$$

$$\leq C_d(1 + \beta^{-\frac{d+1}{2}}),$$

where $|B_d^q|$ denotes the volume of the $d$-dimensional unit ball.

(i) We take separate cases for $q_2 \in (-d, 1]$:

- $q_2 \in (0, 1]$: Since $q_2 \in (0, 1]$, we have

$$|u|^{q_2} = |v - v_1|^{q_2} \leq (|v| + |v_1|)^{q_2} \leq C_{q_3}(|v|^q_2 + |v_1|^{q_2}).$$

Therefore

$$\int_{\mathbb{R}^d} |v - v_1|^{q_2} e^{-\beta|v_1|^2} dv_1 \leq C_{q_2} \int_{\mathbb{R}^d} (|v|^{q_2} + |v_1|^{q_2}) e^{-\beta|v_1|^2} dv_1$$

$$\leq C_{d, q_2}(1 + \beta^{-d/2} + \beta^{-\frac{d+1}{2}})(1 + |v|^{q_2}),$$

where to obtain (15.8), we use the estimates (15.6)-(15.7) for $q = q_2$. 

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\( q_2 \in (-d, 0] \): Since \( q_2 \leq 0 \), estimate (15.6) implies

\[
\int_{\mathbb{R}^d} |v - v_1|^{q_2} e^{-\beta |v_1|^2} \, dv_1 \leq \int_{|v-v_1|>1} e^{-\beta |v_1|^2} \, dv_1 + \int_{|v-v_1|<1} |v - v_1|^{q_2} \, dv_1
\]

\[
= C_d \beta^{-d/2} + \int_{|y|<1} |y|^{q_2} \, dy
\]

\[
= C_d \beta^{-d/2} + C_d \int_0^1 r^{d-1+q_2} \, dr
\]

\[
= C_d \left( \beta^{-d/2} + \frac{1}{d+q_2} \right), \quad (15.9)
\]

since we have assumed \( q_2 > -d \).

\( (ii) \) We take separate cases for \( q_3 \in (-2d, 1] \):

\( q_3 \in (0, 1] \): Since \( q_3 \in (0, 1] \), we have

\[
|\tilde{u}|^{q_3} = (|v - v_1|^2 + |v - v_2|^2 + |v_1 - v_2|^2)^{q_3/2}
\]

\[
\leq 2^{q_3} (|v|^2 + |v_1|^2 + |v_2|^2)^{q_3/2}
\]

\[
\leq C_{q_3} (|v|^{q_3} + |v_1|^{q_3} + |v_2|^{q_3}).
\]

Therefore, Fubini’s Theorem and estimates (15.6)-(15.7) applied for \( q = q_3 \) imply

\[
\int_{\mathbb{R}^{2d}} |\tilde{u}|^{q_3} e^{-\beta (|v_1|^2 + |v_2|^2)} \, dv_1 \, dv_2
\]

\[
\leq C_{q_3} \int_{\mathbb{R}^{2d}} (|v|^{q_3} + |v_1|^{q_3} + |v_2|^{q_3}) e^{-\beta (|v_1|^2 + |v_2|^2)} \, dv_1 \, dv_2
\]

\[
\leq C_{d,q_3} \left( 1 + \beta^{-d} + \beta^{-\frac{2d+1}{2}} \right) (1 + |v|^{q_3}). \quad (15.11)
\]
• $q_3 \in (-2d, 0)$: First, by triangle inequality and Young’s inequality, we observe that

$$|\tilde{u}|^2 = |v - v_1|^2 + |v - v_2|^2 + |v_1 - v_2|^2$$
$$\leq |v - v_1|^2 + |v - v_2|^2 + (|v - v_1| + |v - v_2|)^2$$
$$\leq 3(|v - v_1|^2 + |v - v_2|^2)$$
$$= 3|u|^2. \quad (15.12)$$

Using (15.12), the fact that $q_3 \leq 0$, and Fubini’s Theorem and the estimates (15.6)-(15.7), we obtain

$$\int_{\mathbb{R}^{2d}} |\tilde{u}|^{q_3} e^{-\beta(|v_1|^2 + |v_2|^2)} \, dv_1 \, dv_2 \leq \sqrt{3} \int_{\mathbb{R}^{2d}} |u|^{q_3} e^{-\beta(|v_1|^2 + |v_2|^2)} \, dv_1 \, dv_2$$
$$\leq \sqrt{3} \int_{|u| > 1} e^{-\beta(|v_1|^2 + |v_2|^2)} \, dv_1 \, dv_2 + \sqrt{3} \int_{|u| < 1} |u|^{q_3} \, dv_1 \, dv_2$$
$$\leq C_d \beta^{-d} + \sqrt{3} \int_{|u| < 1} |u|^{q_3} \, dv_1 \, dv_2$$
$$= C_d \beta^{-d} + \sqrt{3} \int_{|y| < 1} |y|^{q_3} \, dy$$
$$= C_d \beta^{-d} + C_d \int_{0}^{1} r^{2d-1+q_3} \, dr$$
$$= C_d \left( \beta^{-d} + \frac{1}{2d + q_3} \right), \quad (15.13)$$

since we have assumed $q_3 > -2d$.

Combining (15.8)-(15.11) and (15.13), we obtain (15.2)-(15.4).
15.5 $L^\infty L^1$ estimates

Here we prove uniform in time, space-velocity $L^1$ estimates on the transported gain and loss operators. These estimates will be of fundamental importance for the convergence of the iteration to the global solution. As we will see, the ternary collisional operator introduces some asymmetry which is not present in the binary case. For this reason, when we use Lemma 15.4, we first obtain estimates in asymmetric form (see Lemma 15.6). However, we will need a symmetric version of this estimate which we derive in Proposition 15.7. To achieve that, we crucially rely on properties of the ternary interactions.

Recall from (14.47) the fixed cross-section exponents $\gamma_2 \in (-d + 1, 1]$ and $\gamma_3 \in (-2d + 1, 1]$. For convenience, we define the function

$$p_{\gamma_2, \gamma_3}(v) = 1 + |v|^\gamma_2^+ + |v|^\gamma_3^+.$$  \hspace{1cm} (15.14)

Notice that, given $\alpha > 0$, $\beta > 0$, we have

$$p_{\gamma_2, \gamma_3} M_{\alpha, \beta} \in L^1_{x,v}.$$ \hspace{1cm} (15.15)

Using Lemma 15.4 for $q_2 = \gamma_2$ and $q_3 = \gamma_3$, we obtain some assymmetric estimates mentioned above.

**Lemma 15.6.** Let $0 < T \leq \infty$ and $\alpha, \beta > 0$. Then there is a constant $C_\beta > 0$ such that the following hold
(i) For any \( g, h \in \mathcal{F}_T^+ \), with \( g^#, h^# \in L^\infty([0,T), \mathcal{M}_{a,\beta}^+) \), and any \( t \in [0,T) \), we have

\[
0 \leq R_2^#(g)(t) \leq C_\beta \| g^# \|_{\infty} p_{\gamma_2,\gamma_3},
\]

(15.16)

\[
0 \leq R_3^#(g,h)(t) \leq C_\beta \| g^# \|_{\infty} \| h^# \|_{\infty} p_{\gamma_2,\gamma_3},
\]

(15.17)

\[
0 \leq R^#(g,h)(t) \leq C_\beta \| g^# \|_{\infty} (1 + \| h^# \|_{\infty}) p_{\gamma_2,\gamma_3}.
\]

(15.18)

(ii) For any \( f, g, h \in \mathcal{F}_T^+ \), with \( f^#, g^#, h^# \in L^\infty([0,T), \mathcal{M}_{a,\beta}^+) \), and \( t \in [0,T) \), we have

\[
\| L_2^#(f,g)(t) \|_{L^1_v}, \| G_2^#(f,g)(t) \|_{L^1_v} \leq C_\beta \| g^# \|_{\infty} \| f^#(t) \|_{p_{\gamma_2,\gamma_3}} \|_{L^1_v},
\]

(15.19)

\[
\| L_3^#(f,g,h)(t) \|_{L^1_v}, \| G_3^#(f,g,h)(t) \|_{L^1_v} \leq C_\beta \| g^# \|_{\infty} \| h^# \|_{\infty} \| f^#(t) \|_{p_{\gamma_2,\gamma_3}} \|_{L^1_v},
\]

(15.20)

\[
\| L^#(f,g,h)(t) \|_{L^1_v}, \| G^#(f,g,h)(t) \|_{L^1_v} \leq C_\beta \| g^# \|_{\infty} (1 + \| h^# \|_{\infty}) \| f^#(t) \|_{p_{\gamma_2,\gamma_3}} \|_{L^1_v}.
\]

(15.21)

Moreover,

\[
L^#(f,g,h), \ G^#(f,g,h) \in L^\infty([0,T), L^1_v^+). \quad (15.22)
\]

Proof.

Proof of (i): Positivity follows immediately by the monotonicity of \( R_2^#, R_3^#, R^# \) on \( \mathcal{F}_T^+ \) (see Proposition 15.2). Since \( g^#, h^# \in L^\infty([0,T), \mathcal{M}_{a,\beta}^+) \), for any \( t \in [0,T) \), we have

\[
0 \leq g(t,x,v) \leq \| g^# \|_{\infty} e^{-\alpha|x-te|^2-\beta|v|^2}, \quad \text{for a.e.} \ (x,v) \in \mathbb{R}^d \times \mathbb{R}^d,
\]

(15.23)

\[
0 \leq h(t,x,v) \leq \| h^# \|_{\infty} e^{-\alpha|x-te|^2-\beta|v|^2}, \quad \text{for a.e.} \ (x,v) \in \mathbb{R}^d \times \mathbb{R}^d.
\]
Recalling the fact that \( R(g, h) = R_2(g) + R_3(g, h) \), it suffices to prove the estimates (15.16)-(15.17).

Let us first prove (15.16). For a.e. \((x, v) \in \mathbb{R}^d\), estimate (15.23) and part (i) of Lemma 15.4, applied for \( q_2 = \gamma_2 \) and \( q_3 = \gamma_3 \), imply

\[
R_2(g)(t, x, v) \leq \|b_2\|_{L^1(S^{d-1})} \int_{\mathbb{R}^d} |u|^{\gamma_2} g(t, x, v_1) \, dv_1
\]
\[
\leq \|b_2\|_{L^1(S^{d-1})} |||g^#|||_{\infty} \int_{\mathbb{R}^d} |u|^\gamma_2 e^{-\beta|v_1|^2} \, dv_1
\]
\[
\leq C_\beta |||g^#|||_{\infty} (1 + |v|^\gamma_2),
\]
\[
\leq C_\beta |||g^#|||_{\infty} p_{\gamma_2, \gamma_3}(v). \tag{15.24}
\]

Since the right hand side of (15.24) does not depend on \( x \), we obtain (15.16).

Let us now prove (15.17). For a.e. \((x, v) \in \mathbb{R}^d\), estimate (15.23) and part (ii) of Lemma 15.4, applied for \( q_2 = \gamma_2 \) and \( q_3 = \gamma_3 \), imply

\[
R_3(g, h)(t, x, v) \leq \|b_3\|_{L^1(S^{d-1})} \int_{\mathbb{R}^d} |\tilde{u}|^{\gamma_3} g(t, x, v_1) h(t, x, v_2) \, dv_1 \, dv_2
\]
\[
\leq \|b_3\|_{L^1(S^{d-1})} |||g^#|||_{\infty} |||h^#|||_{\infty} \int_{\mathbb{R}^d} |\tilde{u}|^{\gamma_3} e^{-\beta(|v_1|^2 + |v_2|^2)} \, dv_1
\]
\[
\leq C_\beta |||g^#|||_{\infty} |||h^#|||_{\infty} (1 + |v|^\gamma_3)
\]
\[
\leq C_\beta |||g^#|||_{\infty} |||h^#|||_{\infty} p_{\gamma_2, \gamma_3}(v). \tag{15.25}
\]

Since the right hand side of (15.25) does not depend on \( x \), we obtain (15.17).

Estimate (15.18) follows by the fact that \( R^#(g, h) = R_2^#(g) + R_3^#(g, h) \).

Proof of (ii): We first prove the claim for the loss operators. Positivity follows immediately from the monotonicity of \( L_2^#, L_3^#, L^# \) on \( \mathcal{F}^+_T \). Estimates (15.19)-(15.21)
follow directly from (14.71) and part (i). Moreover, estimate (15.21) implies (15.22) since \( f^#, g^#, h^# \in L^\infty([0,T), M^{+}_{\alpha,\beta}) \) and \( p_{\gamma_2,\gamma_3} M_{\alpha,\beta} \in L^1_{x,v} \) by (15.15).

For the gain operators, positivity follows immediately from the monotonicity of \( G^2_2, G^3_3, G^# \) on \( F^+_T \). Estimates (15.19)-(15.21) and (15.22) come from (15.1) and the estimates for the loss operators.

Notice that bounds (15.19)-(15.21) are only with respect to the first argument \( f \). Although this is not an issue in the binary case where the gain and loss collisional operators are symmetric with respect to the inputs in the \( L^1 \)-norm, this is not the case for the ternary operators. In order to treat this asymmetry, we need to derive estimates with respect to all three inputs of the ternary gain and loss collisional operators. This is achieved in the following result

**Proposition 15.7.** Let \( 0 < T \leq \infty \) and \( \alpha, \beta > 0 \). Consider \( f_1, f_2, f_3 \in F^+_T \) with \( f^1_1, f^2_2, f^3_3 \in L^\infty([0,T), M^{+}_{\alpha,\beta}). \) Then, there is a constant \( C_\beta > 0 \) such that, for any permutation \( \pi : \{1,2,3\} \to \{1,2,3\} \), the following estimates hold for any \( t \in [0,T) \):

\[
\|L^#_2(f_1, f_2)(t)\|_{L^1_{x,v}} \leq C_\beta \|f^#_2(t)\|_{L^\infty_{x,v}} \|p_{\gamma_2,\gamma_3} f^#(t)\|_{L^1_{x,v},} (15.26)
\]
\[
\|L^#_3(f_1, f_2, f_3)(t)\|_{L^1_{x,v}} \leq C_\beta \|f^#_1(f_1, f_2, f_3)(t)\|_{L^1_{x,v}} \|p_{\gamma_2,\gamma_3} f^#(t)\|_{L^1_{x,v}.} (15.27)
\]
\[
\|L^#(f_1, f_2, f_3)(t)\|_{L^1_{x,v}} \leq C_\beta \|f^#_1(t)\|_{L^\infty_{x,v}} \|f^#_2(t)\|_{L^\infty_{x,v}} \|f^#_3(t)\|_{L^\infty_{x,v}}\|p_{\gamma_2,\gamma_3} f^#(t)\|_{L^1_{x,v}.} (15.28)
\]

**Proof.** By (15.1), triangle inequality and part (ii) of Lemma 15.6 the proof of
for the loss term reduces to showing the following estimates:

\[ \| L^\#_2 (f_1, f_2)(t) \|_{L^1_{x,v}} \leq C_\beta ||| f_1^\# |||_{\infty} \| f_2^\# p_{\gamma_2, \gamma_3} \|_{L^1_{x,v}} \]  \hspace{1cm} (15.29)

\[ \| L^\#_3 (f_1, f_2, f_3)(t) \|_{L^1_{x,v}} \leq C_\beta ||| f_1^\# |||_{\infty} ||| f_3^\# |||_{\infty} \| f_2^\# p_{\gamma_2, \gamma_3} \|_{L^1_{x,v}} \]  \hspace{1cm} (15.30)

\[ \| L^\#_3 (f_1, f_2, f_3)(t) \|_{L^1_{x,v}} \leq C_\beta ||| f_1^\# |||_{\infty} ||| f_2^\# |||_{\infty} \| f_3^\# p_{\gamma_2, \gamma_3} \|_{L^1_{x,v}}. \]  \hspace{1cm} (15.31)

- Proof of (15.29): Performing the involutionary change of variables \((v, v_1) \rightarrow (v_1, v)\) and using (14.10), for any \(t \in [0, T]\), we have

\[ \| L_2(f_1, f_2)(t) \|_{L^1_{x,v}} = \| L_2(f_2, f_1)(t) \|_{L^1_{x,v}} \Rightarrow \| L^\#_2 (f_1, f_2)(t) \|_{L^1_{x,v}} = \| L^\#_2 (f_2, f_1)(t) \|_{L^1_{x,v}}. \]

The claim comes from part (ii) of Lemma 15.6.

- Proof of (15.30): Here the proof is subtler because the inner product \(\bar{u} \cdot \omega\) is not symmetric upon renaming the velocities. However, we will strongly rely on the fact that the expression

\[ |\tilde{u}|^2 = |v - v_1|^2 + |v - v_2|^2 + |v_1 - v_2|^2, \]

given in (14.21) is symmetric with respect to the inputs \(v, v_2, v_2\).

Since \(f_1^\#, f_3^\# \in L^\infty([0, T), M_{\alpha, \beta}^+), \) for any \(t \in [0, T]\) and a.e. \((x, v) \in \mathbb{R}^d \times \mathbb{R}^d,\) we have

\[ 0 \leq f_i(t, x, v) \leq ||| f_i^\# |||_{\infty} e^{-\alpha|v_1|^2 - \beta|v|^2} \leq ||| f_i^\# |||_{\infty} e^{-\beta|v|^2}, \quad \forall i \in \{1, 3\}. \]  \hspace{1cm} (15.32)
Using (14.67), we have

$$\|L^\#_{3}(f, g, h)(t)\|_{L^1_{x,v}} = \|L_3(f, g, h)\|_{L^1_{x,v}}$$

$$\leq \|b_3\|_{L^1_1(S_1^{2d-1})} \int_{\mathbb{R}^{2d}} |\tilde{u}|^{\gamma_3} |f_1(t, x, v)| |f_2(t, x, v_1)| |f_3(t, x, v_2)| \, dv_1 \, dv_2 \, dv \, dx$$

$$= \|b_3\|_{L^1_1(S_1^{2d-1})} \int_{\mathbb{R}^{2d}} (|v - v_1|^2 + |v - v_2|^2 + |v_1 - v_2|^2)^{\gamma_3/2}$$

$$\times |f_1(t, x, v)| |f_2(t, x, v_1)| |f_3(t, x, v_2)| \, dv_1 \, dv_2 \, dv \, dx$$

$$= \|b_3\|_{L^1_1(S_1^{2d-1})} \int_{\mathbb{R}^{2d}} (|v - v_1|^2 + |v - v_2|^2 + |v_1 - v_2|^2)^{\gamma_3/2}$$

$$\times |f_2(t, x, v)| |f_1(t, x, v_1)| |f_3(t, x, v_2)| \, dv_1 \, dv_2 \, dv \, dx$$  \hspace{1cm} (15.33)

$$\leq \|b_3\|_{L^1_1(S_1^{2d-1})} \int_{\mathbb{R}^{2d}} |\tilde{u}|^{\gamma_3} |f_2(t, x, v)| |f_1(t, x, v_1)| |f_3(t, x, v_2)| \, dv_1 \, dv_2 \, dv \, dx$$

$$= \|b_3\|_{L^1_1(S_1^{2d-1})} \int_{\mathbb{R}^{2d}} |f_2(t, x, v)| \int_{\mathbb{R}^{2d}} |\tilde{u}|^{\gamma_3} |f_1(t, x, v_1)| |f_3(t, x, v_2)| \, dv_1 \, dv_2 \, dv \, dx$$

$$\leq \|b_3\|_{L^1_1(S_1^{2d-1})} \|f_1\|_{\infty} \|f_3\|_{\infty} \int_{\mathbb{R}^{2d}} |f_2(t, x, v)| \int_{\mathbb{R}^{2d}} |\tilde{u}|^{\gamma_3} e^{-\beta(|v_1|^2 + |v_2|^2)} \, dv_1 \, dv_2 \, dv \, dx$$  \hspace{1cm} (15.34)

$$\leq C_\beta \|f_1\|_{\infty} \|f_3\|_{\infty} \|f_2(t) p_{\gamma_2, \gamma_3}\|_{L^1_{x,v}}$$  \hspace{1cm} (15.35)

$$= C_\beta \|f_1\|_{\infty} \|f_3\|_{\infty} \|f_2(t) p_{\gamma_2, \gamma_3}\|_{L^1_{x,v}}$$  \hspace{1cm} (15.36)

$$= C_\beta \|f_1\|_{\infty} \|f_3\|_{\infty} \|f_2(t) p_{\gamma_2, \gamma_3}\|_{L^1_{x,v}}$$  \hspace{1cm} (15.37)

where to obtain (15.33) we use the change of variables $(v, v_1) \rightarrow (v_1, v)$, to obtain (15.34) we use (15.32), to obtain (15.35) we use part (ii) of Lemma 15.4, to obtain (15.36) we use (14.67), and to obtain (15.37) we use the fact that $p_{\gamma_2, \gamma_3}$ is invariant.

- Proof of (15.31): Follows in a similar way to the proof of (15.30).

Estimates (15.26)-(15.28) for the loss operators follow. Estimates for the gain
operators follow from (15.1) and the estimates for the loss operators. The proof is complete.

Proposition 15.7 also implies an $L^1$-continuity result for the transported gain and loss operators

**Corollary 15.8.** Let $0 < T \leq \infty$ and $\alpha, \beta > 0$. For $i \in \{1, 2, 3\}$, consider some sequences $(f_{i,n})_n \subseteq F^+_T$ and $f_i \in F^+_T$ such that $f_{i,n}^\#(t) \xrightarrow{M_{\alpha,\beta}} f_i^\#(t)$ for all $t \in [0,T)$. Then, for all $t \in [0,T)$, the following convergence holds as $n \to \infty$

$$
(L^\#(f_{1,n}, f_{2,n}, f_{3,n})(t), G^\#(f_{1,n}, f_{2,n}, f_{3,n})(t)) \xrightarrow{L^1_{x,v}} (L^\#(f_1, f_2, f_3)(t), G^\#(f_1, f_2, f_3)(t)).
$$

(15.38)

**Proof.** Fix $t \in [0,T)$. Since $f_{i,n}^\#(t) \xrightarrow{M_{\alpha,\beta}} f_i^\#(t)$, for any $i \in \{1, 2, 3\}$, we have

$$
f_{i,n}^\#(t) \xrightarrow{a.e.} f_i^\#(t), \quad \sup_{n \in \mathbb{N}} \{|f_{i,n}^\#(t)|, |f_i^\#(t)|\} \leq CM_{\alpha,\beta},
$$

(15.39)

for some constant $C > 0$. Thus

$$
f_{i,n}(t) \xrightarrow{a.e.} f_i(t), \quad \sup_{n \in \mathbb{N}} \{|f_{i,n}(t)|, |f_i(t)|\} \leq CM_{\alpha,\beta}^-(t),
$$

(15.40)

Let us first prove (15.38) for the loss case. By (14.36) and triangle inequality, it suffices to prove

$$
\|L^\#_2(f_{1,n}, f_{2,n})(t) - L^\#_2(f_1, f_2)(t)\|_{L^1_{x,v}} \xrightarrow{n \to \infty} 0,
$$

(15.41)

$$
\|L^\#_3(f_{1,n}, f_{2,n}, f_{3,n})(t) - L^\#_3(f_1, f_2, f_3)(t)\|_{L^1_{x,v}} \xrightarrow{n \to \infty} 0.
$$

(15.42)
• Proof of (15.41): Using bilinearity of $L_2^\#$, bound (15.40) and monotonicity of $L_2^\#$, we have

$$\|L_2^\#(f_{1,n}, f_{2,n})(t) - L_2^\#(f_1, f_2)(t)\|_{L^1_{x,v}} \leq$$

$$\leq \|L_2^\#(f_{1,n} - f_1, f_{2,n})(t)\|_{L^1_{x,v}} + \|L_2^\#(f_1, f_{2,n} - f_2)(t)\|_{L^1_{x,v}}$$

$$\leq C\|L_2^\#(f_{1,n} - f_1, M_{\alpha,\beta}^{-\#})(t)\|_{L^1_{x,v}} + C\|L_2^\#(M_{\alpha,\beta}^{-\#}, f_{2,n} - f_2)(t)\|_{L^1_{x,v}}$$

$$\leq C_{\beta}(\|f_{1,n}(t) - f_1(t)\|_{p_{\gamma_2,\gamma_3}L^1_{x,v}} + C_{\beta}(\|f_{2,n}(t) - f_2(t)\|_{p_{\gamma_2,\gamma_3}L^1_{x,v}}),$$

(15.43)

where to obtain (15.43) we use (15.26) from Proposition 15.7 and the fact that

$$\|M_{\alpha,\beta}^{-\#}(t)\|_{X_{\alpha,\beta}} = \|M_{\alpha,\beta}\|_{X_{\alpha,\beta}} = 1, \quad \forall t \in [0, T)$$

(15.44)

By (15.39) and the Dominated Convergence Theorem, each of the terms in (15.43) goes to zero as $n \to \infty$ and (15.41) is proved.

• Proof of (15.42): Using trilinearity of $L_3^\#$, bound (15.40) and monotonicity of $L_3^\#$, we have

$$\|L_3^\#(f_{1,n}, f_{2,n}, f_{3,n})(t) - L_3^\#(f_1, f_2, f_3)(t)\|_{L^1_{x,v}} \leq$$

$$\leq \|L_3^\#(f_{1,n} - f_1, f_{2,n}, f_{3,n})(t)\|_{L^1_{x,v}} + \|L_3^\#(f_1, f_2, f_{3,n} - f_3)(t)\|_{L^1_{x,v}}$$

$$+ \|L_3^\#(f_1, f_{2,n} - f_2, f_{3,n})(t)\|_{L^1_{x,v}}$$

$$\leq C\|L_3^\#(f_{1,n} - f_1, M_{\alpha,\beta}^{-\#}, M_{\alpha,\beta}^{-\#})(t)\|_{L^1_{x,v}} + C\|L_3^\#(M_{\alpha,\beta}^{-\#}, f_{2,n} - f_2, M_{\alpha,\beta}^{-\#})(t)\|_{L^1_{x,v}}$$

$$+ C\|L_3^\#(M_{\alpha,\beta}^{-\#}, M_{\alpha,\beta}^{-\#}, f_{3,n} - f_3)(t)\|_{L^1_{x,v}}$$

$$\leq C_{\beta}(\|f_{1,n}(t) - f_1(t)\|_{p_{\gamma_2,\gamma_3}L^1_{x,v}} + C_{\beta}(\|f_{2,n}(t) - f_2(t)\|_{p_{\gamma_2,\gamma_3}L^1_{x,v}}$$

$$+ C_{\beta}(\|f_{3,n}(t) - f_3(t)\|_{p_{\gamma_2,\gamma_3}L^1_{x,v}}),$$

(15.45)
where to obtain (15.45) we use (15.27) from Proposition 15.7 and (15.44). By (15.39) and the Dominated Convergence Theorem, each of the terms in (15.43) goes to zero as \( n \to \infty \) and (15.42) is proved. Combining (15.41)-(15.42), we obtain (15.38).

The gain operator convergence follows with a similar argument. \( \square \)

15.9 A global estimate on the time average of the transported gain and loss operators

Here, we prove Proposition 15.11 which provides upper global bounds for the time average of the transported operators. These estimates will be essential to prove that the necessary beginning condition (16.49) for the convergence of the iteration holds globally in time for small enough initial data (see Chapter 17). For the binary case and soft potentials, these bounds were established in [3]. However, the presence of the ternary collisional operator requires new treatment which strongly relies on the properties of ternary interactions.

Before stating Proposition 15.11 we provide the following auxiliary estimate for the time integral of a traveling Maxwellian which will be used in the proof of the result for \( n = d \) in the binary case and \( n = 2d \) in the ternary case.

**Lemma 15.10.** Let \( n \in \mathbb{N} \), \( x_0, u_0 \in \mathbb{R}^n \), with \( u_0 \neq 0 \) and \( \alpha > 0 \). Then, the following estimate holds

\[
\int_0^\infty e^{-\alpha |x_0-\tau u_0|^2} d\tau \leq \frac{\sqrt{\pi}}{2} \alpha^{-1/2} |u_0|^{-1}.
\]

**Proof.** By triangle inequality, we have

\[
\tau |u_0| - |x_0| \leq |x_0 - \tau u_0| \Rightarrow e^{-\alpha |x_0-\tau u_0|^2} \leq e^{-\alpha |u_0||x_0|} \leq e^{-\alpha (|u_0|^2 + |x_0|^2)}, \quad \forall \tau \geq 0.
\]
Therefore integrating in $\tau$, we obtain

$$
\int_0^\infty e^{-\alpha|x_0-\tau u_0|^2} d\tau \leq \int_0^\infty e^{-\alpha(\tau|u_0|-x_0)^2} d\tau \leq \alpha^{-1/2}|u_0|^{-1} \int_0^\infty e^{-\frac{\sqrt{\pi}}{2}\alpha^{-1/2}|u_0|^{-1}},
$$

and the estimate is proved. \hfill \Box

We now state and prove Proposition 15.11. Given $f \in L^\infty([0,T),M_{\alpha,\beta})$, recall from (14.63) the norm

$$
|||f|||_\infty = \sup_{t \in [0,T)} \|f(t)\|_{M_{\alpha,\beta}}.
$$

**Proposition 15.11.** Let $0 < T \leq \infty$ and $\alpha, \beta > 0$. Then, for all $f, g, h \in \mathcal{F}_T$ with $f^#, g^#, h^# \in L^\infty([0,T),M_{\alpha,\beta})$, the following bounds hold for any $t \in [0,T)$

- **For the binary operators:**

  $$
  \int_0^t |L_2^#(f,g)(\tau)| d\tau, \int_0^t |G_2^#(f,g)(\tau)| d\tau \leq K_\beta \alpha^{-1/2}M_{\alpha,\beta}|||f^#|||_\infty|||g^#|||_\infty. \quad (15.46)
  $$

- **For the ternary operators:**

  $$
  \int_0^t |L_3^#(f,g,h)(\tau)| d\tau, \int_0^t |G_3^#(f,g,h)(\tau)| d\tau \\
  \leq K_\beta \alpha^{-1/2}M_{\alpha,\beta}|||f^#|||_\infty|||g^#|||_\infty|||h^#|||_\infty. \quad (15.47)
  $$

- **For the mixed operators:**

  $$
  \int_0^t |L^#(f,g,h)(\tau)| d\tau, \int_0^t |G^#(f,g,h)(\tau)| d\tau \\
  \leq K_\beta \alpha^{-1/2}M_{\alpha,\beta}|||f^#|||_\infty|||g^#|||_\infty|||h^#|||_\infty(1 + |||h^#|||_\infty). \quad (15.48)
  $$

where

$$
K_\beta = C_d \left\| b_2 \right\|_{L^1(S_{\beta^{-d/2}})} \left( \beta^{-d/2} + \frac{1}{d + \gamma_2 - 1} \right) + \left\| b_3 \right\|_{L^1(S_{\beta^{-d}})} \left( \beta^{-d} + \frac{1}{2d + \gamma_3 - 1} \right).
$$

(15.49)
Proof. Recall $\tilde{K}_{\beta,\gamma_2-1}^2$ and $\tilde{K}_{\beta,\gamma_3-1}^3$ from (15.3), (15.5) and $K_\beta$ from (14.77). Since $\gamma_2, \gamma_3 \leq 1$, relations (15.3) imply that

$$K_\beta = \frac{\sqrt{\pi}}{2} (\|b_2\|_{L^1(S_{t_0}^{d-1})} \tilde{K}_{\beta,\gamma_2-1}^2 + \|b_3\|_{L^1(S_{t_0}^{d-1})} \tilde{K}_{\beta,\gamma_3-1}^3)$$

(15.50)

Proof of (15.46): As mentioned above, these bounds were established for the soft potential case in [3]. Here we also treat the hard potential case. Since $L_2^\#, G_2^\#$ are bilinear, we may assume without loss of generality that

$$|||f^\#|||_{\infty} = |||g^\#|||_{\infty} = 1.$$

(15.51)

Let us first prove it for the loss term. For any $t \in [0, T)$ and a.e. $(x, v) \in \mathbb{R}^{2d}$, relation (15.51), followed by an application of Lemma 15.10 for $n = d, x_0 = x, u_0 = u$, an application of part (i) of Lemma 15.4 for $q^2 = \gamma_2 - 1$, and the fact that $\gamma_2 \leq 1$ imply

$$\int_0^t |L_2^\#(f, g)(\tau, x, v)| \, d\tau$$

$$\leq \|b_2\|_{L^1(S_{t_0}^{d-1})} \int_0^t \int_{\mathbb{R}^d} |u|^{\gamma_2} |f(\tau, x + \tau v, v)| |g(\tau, x + \tau (v - v_1), v_1)| \, dv_1 \, d\tau$$

$$= \|b_2\|_{L^1(S_{t_0}^{d-1})} \int_0^t \int_{\mathbb{R}^d} |u|^{\gamma_2} |f^\#(\tau, x, v)| |g^\#(\tau, x + \tau(v - v_1), v_1)| \, d\omega \, dv_1 \, d\tau$$

$$\leq \|b_2\|_{L^1(S_{t_0}^{d-1})} e^{-\alpha|v|^2 - \beta|v_1|^2} \int_0^t \int_{\mathbb{R}^d} |u|^{\gamma_2} e^{-\alpha|x - \tau u|^2 - \beta|v_1|^2} \, dv_1 \, d\tau$$

$$\leq \|b_2\|_{L^1(S_{t_0}^{d-1})} M_{\alpha, \beta}(x, v) \int_{\mathbb{R}^d} |u|^{\gamma_2} e^{-\beta|v_1|^2} \int_0^\infty e^{-\alpha|x - \tau u|^2} \, d\tau \, dv_1$$

$$\leq \|b_2\|_{L^1(S_{t_0}^{d-1})} \frac{\sqrt{\pi}}{2} \alpha^{-1/2} M_{\alpha, \beta}(x, v) \int_{\mathbb{R}^d} |u|^{\gamma_2 - 1} e^{-\beta|v_1|^2} \, dv_1$$

$$\leq \|b_2\|_{L^1(S_{t_0}^{d-1})} \frac{\sqrt{\pi}}{2} \tilde{K}_{\beta,\gamma_2-1}^2 \alpha^{-1/2} M_{\alpha, \beta}(x, v).$$

(15.52)

Notice that we are able to use part (i) of Lemma 15.4 because $\gamma_2 > -d + 1$. Recalling (15.50), estimate (15.46) is proved for the loss term.
To prove (15.46) for the gain term, we will use the identity
\[
|x + \tau(v - v')|^2 + |x + \tau(v - v_1')|^2 = |x|^2 + |x + \tau(v - v_1)|^2, \tag{15.53}
\]
which follows from the binary conservation of momentum and energy:
\[
v' + v_1' = v + v_1, \tag{15.54}
\]
\[
|v'|^2 + |v_1'|^2 = |v|^2 + |v_1|^2.
\]
For any \(t \in [0, T)\) and a.e. \((x, v) \in \mathbb{R}^{2d}\), (15.51) and (15.53)-(15.54) imply
\[
\int_0^t |G^2_2(f, g)(\tau, x, v)| \, d\tau
\leq \|b_2\|_{L^1(S^{d-1})} \int_0^t \int_{\mathbb{R}^d} |u|^\gamma f(\tau, x + \tau v, v') g(\tau, x + \tau v, v') \, dv_1 \, d\tau
\]
\[
= \|b_2\|_{L^1(S^{d-1})} \int_0^t \int_{\mathbb{R}^d} |u|^\gamma f^#(\tau, x + \tau(v - v'), v') g^#(\tau, x + \tau(v - v_1), v_1') \, d\omega \, dv_1 \, d\tau
\]
\[
\leq \|b_2\|_{L^1(S^{d-1})} \int_0^t \int_{\mathbb{R}^d} |u|^\gamma e^{-\alpha(x + \tau v - v')^2 + |x + \tau(v - v_1)|^2} e^{-\beta(|v'|^2 + |v_1'|^2)} \, dv_1 \, d\tau
\]
\[
= \|b_2\|_{L^1(S^{d-1})} e^{-\alpha|x|^2 - \beta|v|^2} \int_0^t \int_{\mathbb{R}^d} |u|^\gamma e^{-\alpha|x + \tau(v - v_1)|^2 - \beta|v_1|^2} \, dv_1 \, d\tau. \tag{15.55}
\]
Combining (15.55) with an identical argument to the one used for the loss term, we obtain
\[
\int_0^t |G^2_2(f, g)(\tau, x, v)| \, d\tau \leq \|b_2\|_{L^1(S^{d-1})} \sqrt{\frac{\pi}{2}} K_{\beta, \gamma_2-1}^{-1/2} M_{\alpha, \beta}(x, v). \tag{15.56}
\]
Recalling (15.50), estimate (15.46) is proved for the gain term as well.

**Proof of (15.47):** Since \(L^#_3, G^#_3\) is trilinear, we may assume without loss of generality that
\[
|||f^#|||_\infty = |||g^#|||_\infty = |||h^#|||_\infty = 1. \tag{15.57}
\]
Let us first prove (15.47) for the loss term. For any $t \in [0, T)$ and a.e. $(x, v) \in \mathbb{R}^{2d}$, (15.57) implies

$$
\int_0^t |L^\#_3(f, g, h)(\tau, x, v)| d\tau \\
\leq \|b_3\|_{L^1(\mathbb{R}^{2d-1})} \int_0^t \int_{\mathbb{R}^{2d}} |\tilde{u}|^{\gamma_3} |f(\tau, x + \tau v, v)||g(\tau, x + \tau v, v_1)| \\
\times |h(\tau, x + \tau v, v_2)| dv_1 dv_2 d\tau \\
= \|b_3\|_{L^1(\mathbb{R}^{2d-1})} \int_0^t \int_{\mathbb{R}^{2d}} |\tilde{u}|^{\gamma_3} |f^\#(\tau, x, v)||g^\#(\tau, x + \tau (v - v_1), v_1)| \\
\times |h^\#(\tau, x + \tau (v - v_2), v_2)| dv_1 dv_2 d\tau \\
\leq \|b_3\|_{L^1(\mathbb{R}^{2d-1})} e^{-\alpha |x|^2 - \beta |v|^2} \int_0^t \int_{\mathbb{R}^{2d}} |\tilde{u}|^{\gamma_3} e^{-\alpha(|x + \tau(v - v_1)|^2 + |x + \tau(v - v_2)|^2)} \\
\times e^{-\beta(|v_1|^2 + |v_2|^2)} dv_1 dv_2 d\tau \\
\leq \|b_3\|_{L^1(\mathbb{R}^{2d-1})} M_{\alpha, \beta}(x,v) \int_{\mathbb{R}^{2d}} |\tilde{u}|^{\gamma_2} e^{-\beta(|v_1|^2 + |v_2|^2)} \int_0^\infty e^{-\alpha |x - \tau u|^2} d\tau dv_1 dv_2, \quad (15.58)
$$

where in (15.58) we use the notation

$$
\mathbf{x} := \begin{pmatrix} x \\ x \end{pmatrix} \in \mathbb{R}^{2d}, \quad \mathbf{u} = \begin{pmatrix} v_1 - v \\ v_2 - v \end{pmatrix} \in \mathbb{R}^{2d}.
$$

Therefore, an application of Lemma [15.10] for $n = 2d$, $x_0 = \mathbf{x}$, $u_0 = \mathbf{u}$, followed by (15.12), an application of part (ii) of Lemma [15.4] for $q_3 = \gamma_3 - 1$ and the fact that
\[ \gamma_3 \leq 1 \]

\[
\int_0^t |L^\#_3(f, g, h)(\tau, x, v)| \, d\tau 
\leq \|b_3\|_{L^1(S^{2d-1})} M_{\alpha, \beta}(x, v) \int_{\mathbb{R}^{2d}} |\tilde{u}|^{\gamma_2-\beta(|v_1|^2+|v_2|^2)} \int_0^{\infty} e^{-\alpha|\tilde{x} - \tau u|^2} \, d\tau \, dv_1 \, dv_2 
\leq \|b_3\|_{L^1(S^{2d-1})} \frac{\sqrt{\pi}}{2} \alpha^{-1/2} M_{\alpha, \beta}(x, v) \int_{\mathbb{R}^{2d}} |\tilde{u}|^{\gamma_2-1} e^{-\beta(|v_1|^2+|v_2|^2)} \, dv_1 \, dv_2 
\leq \|b_3\|_{L^1(S^{2d-1})} \frac{\sqrt{\pi}}{2} \tilde{K}^3_{\beta, \gamma_3-1} \alpha^{-1/2} M_{\alpha, \beta}(x, v), \tag{15.59}
\]

where \(\tilde{K}^3_{\beta, \gamma_3-1}\) is given by (15.5). Notice that we are able to use part (ii) of Lemma 15.4 because \(\gamma_3 > -2d + 1\). Recalling (15.50), estimate (15.47) is proved for the loss term.

To prove (15.47) for the gain term, we will use the identity:

\[
|x + \tau(v-v^*)|^2 + |x + \tau(v-v_1^*)|^2 + |x + \tau(v-v_2^*)|^2 = |x|^2 + |x + \tau(v-v_1)|^2 + |x + \tau(v-v_2)|^2, \tag{15.60}
\]

following from the ternary conservation of momentum and energy:

\[
v^* + v_1^* + v_2^* = v + v_1 + v_2,
\]

\[
|v^*|^2 + |v_1^*|^2 + |v_2^*|^2 = |v|^2 + |v_1|^2 + |v_2|^2. \tag{15.61}
\]
For any \( t \in [0, T) \) and a.e. \((x, v) \in \mathbb{R}^d\), by (15.57) and (15.60)-(15.61), we obtain
\[
\int_0^t |G^\#_3(f, g, h)(\tau)| \, d\tau \\
\leq \|b_3\|_{L^1(S_1^{d-1})} \int_0^t \int_{\mathbb{R}^d} |\tilde{u}|^{\gamma_3} f(\tau, x + \tau v, v^*) g(\tau, x + \tau v, v_1^*) h(\tau, x + \tau v, v_2^*) \, dv_1 \, dv_2 \, d\tau \\
= \|b_3\|_{L^1(S_1^{d-1})} \int_0^t \int_{\mathbb{R}^d} |\tilde{u}|^{\gamma_3} f^\#(\tau, x + \tau(v - v^*), v^*) g^\#(\tau, x + \tau(v - v_1^*), v_1^*) \times h^\#(\tau, x + \tau(v - v_2^*), v_2^*) \, dv_1 \, dv_2 \, d\tau \\
\leq \|b_3\|_{L^1(S_1^{d-1})} \int_0^t \int_{\mathbb{R}^d} |\tilde{u}|^{\gamma_3} e^{-\alpha|x + \tau(v - v^*)|^2 + |x + \tau(v - v_1^*)|^2 + |x + \tau(v - v_2^*)|^2} \, dv_1 \, dv_2 \, d\tau \\
= \|b_3\|_{L^1(S_1^{d-1})} e^{-\alpha|x|^2 - \beta|v|^2} \int_0^t \int_{\mathbb{R}^d} |\tilde{u}|^{\gamma_3} e^{-\alpha|x + \tau(v - v_1)|^2 + |x + \tau(v - v_2)|^2} e^{-\beta(|v_1|^2 + |v_2|^2)} \, dv_1 \, dv_2 \, d\tau \\
\leq \|b_3\|_{L^1(S_1^{d-1})} M_{\alpha, \beta}(x, v) \int_{\mathbb{R}^d} |\tilde{u}|^{\gamma_2} e^{-\beta(|v_1|^2 + |v_2|^2)} \int_0^\infty e^{-\alpha|x - \tau u|^2} \, d\tau \, dv_1 \, dv_2. \tag{15.62}
\]
Combining (15.62) with an identical argument to the one used for the loss case, we obtain
\[
\int_0^t |G^\#_3(f, g, h)(\tau)| \leq \|b_3\|_{L^1(S_1^{d-1})} \sqrt{\pi} \, K_{3, \gamma_3 - 1} / 2 \, \alpha^{-1/2} M_{\alpha, \beta}(x, v). \tag{15.63}
\]
Recalling (15.50), estimate (15.47) is proved for the gain term as well.

**Proof of (15.48):** To obtain (15.48) for the loss term, we use (15.52), (15.59) and (15.50). Similarly, to obtain (15.48) for the gain term, we use (15.56), (15.63) and (15.50).

\[\square\]

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Chapter 16

The Kaniel-Shinbrot iteration scheme and the associated linear problem

In this chapter, we present the Kaniel-Shinbrot iteration scheme which will then be used as the heart of the construction of a global solution in Section 17. This scheme is motivated by the works of [46, 47]. However the presence of the ternary collisional operator, in addition to the binary collisional operator, required a modification of the original construction.

In particular, we outline the construction of the Kaniel-Shinbrot iteration that we will use in this paper. Formally speaking, given an initial data $f_0$, and some initial functions $l_0, u_0$ satisfying $l_0 \leq f_0 \leq u_0$, we construct an increasing sequence $(l_n)_{n \in \mathbb{N}}$ and a decreasing sequence $(u_n)_{n \in \mathbb{N}}$, with $l_n \leq u_n$, through the iteration

$$\frac{dl_n}{dt} + v \cdot \nabla x l_n = G(l_{n-1}, l_{n-1}, l_{n-1}) - L(l_n, u_{n-1}, u_{n-1}),$$

$$l_n(0) = f_0.$$  \hspace{1cm} (16.1)

$$\frac{du_n}{dt} + v \cdot \nabla x u_n = G(u_{n-1}, u_{n-1}, u_{n-1}) - L(u_n, l_{n-1}, l_{n-1}),$$

$$u_n(0) = f_0.$$  \hspace{1cm} (16.2)

We will see that the sequences $l_n$, $u_n$ converge to the same limit, namely a function $f$, which will be the solution of the binary-ternary Boltzmann equation (13.1).
To make things rigorous, we first study an associated linear problem, and then inductively apply these results, together with the estimates derived in Section 15 to establish that the Kaniel-Shinbrot iteration converges to a solution of (13.1), provided that an appropriate beginning condition is satisfied. This solution will be unique in the class of functions uniformly bounded by a Maxwellian.

16.1 The associated linear problem

Here, we prove well-posedness for a linear problem associated to the iteration scheme (16.1)-(16.2). More precisely, given some functions of time $g, h$, we show well-posedness up to time $0 < T \leq \infty$ of the linear problem

$$\begin{align*}
\partial_t f + v \cdot \nabla_x f &= h - L(f, g, g), \quad (t, x, v) \in (0, T) \times \mathbb{R}^d \times \mathbb{R}^d, \\
f(0) &= f_0, \quad (x, v) \in \mathbb{R}^d \times \mathbb{R}^d,
\end{align*}$$

(16.3)

**Definition 16.1.** Let $0 < T \leq \infty$, $\alpha, \beta > 0$, $f_0 \in L_{x,v}^{1,+}$, $g^\# \in L^\infty([0, T), M_{\alpha, \beta}^+)$ and $h^\# \in L_{loc}^1([0, T), L_{x,v}^{1,+})$. We say that a function $f \in \mathcal{F}_T^+$ with

(i) $f^\# \in C^0([0, T), L_{x,v}^{1,+})$,

(ii) $L^\#(f, g, g) \in L_{loc}^1([0, T), L_{x,v}^{1,+})$,

(iii) $f^\#$ is weakly differentiable and satisfies

$$\begin{align*}
\frac{df^\#}{dt} + L^\#(f, g, g) &= h^\#, \\
f^\#(0) &= f_0,
\end{align*}$$

(16.4)

is a mild solution of (16.3) in $[0, T)$ with initial data $f_0 \in L_{x,v}^{1,+}$.  

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Remark 16.1. The differential equation of (16.4) is interpreted as an equality of distributions since all terms involved belong to \( L^1_{loc}([0, T), L^1_{x,v}) \).

Remark 16.2. Remarks 14.6-14.7 imply that a mild solution \( f \) to (16.3) belongs to \( C^0([0, T), L^1_{x,v}) \).

For technical reasons, we first prove well-posedness of (16.3) under the additional assumptions

\[
f_0 \in \mathcal{M}^{+}_{\alpha, \beta}, \quad 0 \leq h^\#(t) \leq C e^{-t^2} M_{\alpha, \beta}, \quad \forall t \in [0, T),
\]

(16.5)

for some constant \( C > 0 \). Clearly if (16.5) holds, then \( f_0 \in L^1_{x,v} \) and \( h^\# \in L^1_{loc}([0, T), L^1_{x,v}) \), thus (16.5) is a stronger assumption than those appearing in Definition 16.1. This additional assumption will be removed later using an approximation argument.

**Lemma 16.2.** Let \( 0 < T \leq \infty \) and \( \alpha, \beta > 0 \). Consider \( f_0, h \) satisfying (16.5) and \( g^\# \in L^{\infty}([0, T), \mathcal{M}^{+}_{\alpha, \beta}) \). Then, there exists a mild solution \( f \) of (16.3) with \( f^\# \in L^{\infty}([0, T), \mathcal{M}^{+}_{\alpha, \beta}) \). Moreover, \( \|f^\#(\cdot)\|_{L^1_{x,v}} \) is absolutely continuous and satisfies

\[
\|f^\#(t)\|_{L^1_{x,v}} + \int_0^t \|L^\#(f, g, g)(\tau)\|_{L^1_{x,v}} d\tau = \|f_0\|_{L^1_{x,v}} + \int_0^t \|h^\#(\tau)\|_{L^1_{x,v}} d\tau, \quad \forall t \in [0, T).
\]

(16.6)

**Proof.** Since \( g^\# \in L^{\infty}([0, T), \mathcal{M}^{+}_{\alpha, \beta}) \), part (i) of Lemma 15.6 implies

\[
0 \leq R^\#(g, g)(t) \leq C_g ||g||_\infty (1 + ||g||_\infty) p_{\gamma_2, \gamma_3}, \quad \forall t \in [0, T),
\]

(16.7)

for some constant \( C_g > 0 \) depending on \( g \). We define \( f \) by

\[
f^\#(t) := f_0 \exp \left( - \int_0^t R^\#(g, g)(\sigma) d\sigma \right) + \int_0^t h^\#(\tau) \exp \left( - \int_\tau^t R^\#(g, g)(\sigma) d\sigma \right) d\tau.
\]

(16.8)
By (16.5), (16.7), and the fact $f_0 \in M^{+}_{\alpha, \beta}$, $f^\#$ is well-defined and satisfies the bound
\[
0 \leq f^\#(t) \leq f_0 + \int_0^t h^\#(\tau) \, d\tau \leq \left(\|f_0\|_{M^{+}_{\alpha, \beta}} + C \frac{\sqrt{\pi}}{2} \right) M^{+}_{\alpha, \beta}, \quad \forall t \in [0, T), \tag{16.9}
\]
thus $f \geq 0$ and
\[
f^\# \in L^\infty([0, T), M^{+}_{\alpha, \beta}). \tag{16.10}
\]
Let us now show that $f^\# \in C^0([0, T), L^{1, \alpha}_{t,v})$. For any $t, s \in [0, T)$, expression (16.8) yields
\[
|f^\#(t) - f^\#(s)| = \left[ f_0 \exp \left( - \int_0^s R^\#(g, g)(\sigma) \, d\sigma \right) + \int_0^s h^\#(\tau) \exp \left( - \int_\tau^s R^\#(g, g)(\sigma) \right) \, d\tau \
\times \left[ 1 - \exp \left( - \int_\tau^t R^\#(g, g)(\sigma) \right) \right] - \int_0^t h(\tau) \exp \left( - \int_\tau^t R^\#(g, g)(\sigma) \right) \, d\tau,
\]
therefore by (16.5), (16.7), we may find a positive constants $C_{f_0, g, h} > 0$ such that
\[
|f^\#(t) - f^\#(s)| \leq C_{f_0, g, h} M^{\alpha, \beta}_{\gamma_2, \gamma_3} (1 - e^{-C_{f_0, g, h}|t-s|\beta_{\gamma_2, \gamma_3}}) + C_{f_0, g, h} |t-s| M^{\alpha, \beta}, \quad \forall t \in [0, T). \tag{16.11}
\]
Using the elementary inequality
\[
1 - e^{-x} \leq x, \quad \forall x \geq 0,
\]
we obtain
\[
|f^\#(t) - f^\#(s)| \leq 2C_{f_0, g, h} |t-s| M^{\alpha, \beta}_{\gamma_2, \gamma_3}, \quad \forall t \in [0, T), \tag{16.12}
\]
Integrating (16.12), we obtain
\[
\|f^\#(t) - f^\#(s)\|_{L^1_{t,v}} \leq 2C_{f_0, g, h} |t-s|, \quad \forall t, s \in [0, T), \tag{16.13}
\]
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since \( p_{\gamma_2, \gamma_3} M_{\alpha, \beta} \in L^{1,+}_{x,v} \). We conclude that \( f^\# \in C^0([0,T), L^{1,+}_{x,v}) \), therefore \( f \in C^0([0,T), L^{1,+}_{x,v}) \). In particular, bound (16.13) implies that \( f \) is actually Lipschitz continuous.

Since \( f^\#, g^\# \in L^\infty([0,T), M^{+}_{\alpha, \beta}) \), part (ii) of Lemma 15.6 implies

\[
L^\#(f, g, g) \in L^\infty([0,T), L^{1,+}_{x,v}) \subseteq L^1_{loc}([0,T), L^{1,+}_{x,v}).
\]  

(16.14)

Finally, by (16.5), (16.14), representation (16.8) and the Dominated Convergence Theorem, we conclude that \( f^\# \) is weakly differentiable and satisfies

\[
\begin{cases}
\frac{df^\#}{dt} + L^\#(f, g, g) = h^#, \\
f^\#(0) = f_0,
\end{cases}
\]  

(16.15)

thus it is a mild solution of (16.3).

Integrating (16.15), the Fundamental Theorem of Calculus and the fact that \( f^\# \in C^0([0,T), L^{1,+}_{x,v}) \), \( L^\#(f, g, g) \) and \( h^\# \in L^1_{loc}([0,T), L^{1,+}_{x,v}) \), imply

\[
f^\#(t) + \int_0^t L^\#(f, g, g)(\tau) d\tau = f_0 + \int_0^t h^\#(\tau) d\tau, \quad \forall t \in [0,T).
\]  

(16.16)

Using non-negativity of all terms involved in (16.16) and Fubini’s Theorem, we obtain (16.6) and absolute continuity of \( \|f(t)\|_{L^{1}_{x,v}} \) follows. The proof is complete.

Since the gain operator does not satisfy (16.5), it will be convenient to relax assumption (16.5) to \( f_0 \in L^{1,+}_{x,v}, \ h^\# \in L^1_{loc}([0,T), L^{1,+}_{x,v}) \). As in [47], the idea is to approximate \( f_0, h^\# \) in the \( L^{1}_{x,v} \)-norm with a monotone sequence of solutions occurring from a repeated application of Lemma 16.2. We obtain the following well-posedness result
Proposition 16.3. Let $0 < T \leq \infty$ and $\alpha, \beta > 0$. Consider $f_0 \in L^1_{x,v}$, $g^\# \in L^\infty([0,T), M_{\alpha,\beta}^+)$ and $h^\# \in L^1_{\text{loc}}([0,T), L^1_{x,v})$. Then, there exists a unique mild solution $f$ of (16.3). In particular $f^\#$ is given by

$$f^\#(t) := f_0 \exp \left( - \int_0^t R^\#(g,g)(\sigma) \, d\sigma \right) + \int_0^t h^\#(\tau) \exp \left( - \int_\tau^t R^\#(g,g)(\sigma) \, d\sigma \right) \, d\tau.$$  \hspace{1cm} (16.17)

Proof. Existence: Given $n \in \mathbb{N}$, let us define

$$f_{0,n} := \begin{cases} f_0, & \text{if } f_0 \leq nM_{\alpha,\beta}, \\ nM_{\alpha,\beta}, & \text{if } f_0 > nM_{\alpha,\beta}. \end{cases} \hspace{1cm} (16.18)$$

and

$$h^\#_n(t) := \begin{cases} h^\#(t), & \text{if } h^\#(t) \leq ne^{-t^2}M_{\alpha,\beta}, \\ ne^{-t^2}M_{\alpha,\beta}, & \text{if } h^\#(t) > ne^{-t^2}M_{\alpha,\beta}. \end{cases} \hspace{1cm} (16.19)$$

It is clear that $f_{0,n}, h_n$ satisfy condition (16.5) for all $n \in \mathbb{N}$ and that

$$0 \leq f_{0,n} \nearrow f_0 \quad \text{as } n \to \infty, \hspace{1cm} (16.20)$$

$$\forall t \in [0,T) : \quad 0 \leq h^\#_n(t) \nearrow h^\#(t) \quad \text{as } n \to \infty. \hspace{1cm} (16.21)$$

Then the Monotone Convergence Theorem yields that

$$\|f_{0,n}\|_{L^1_{x,v}} \nearrow \|f_0\|_{L^1_{x,v}} \quad \text{as } n \to \infty, \hspace{1cm} (16.22)$$

$$\forall t \in [0,T) : \quad \|h^\#_n(t)\|_{L^1_{x,v}} \nearrow \|h^\#(t)\|_{L^1_{x,v}} \quad \text{as } n \to \infty. \hspace{1cm} (16.23)$$

Moreover, since $f_0 \in L^1_{x,v}$ and $h^\# \in L^1_{\text{loc}}([0,T), L^1_{x,v})$, relations (16.20)-(16.21) and the Dominated Convergence Theorem yield

$$f_{0,n} \xrightarrow{L^1_{x,v}} f_0, \quad \text{as } n \to \infty, \hspace{1cm} (16.24)$$

for a.e. $t \in [0,T)$:

$$h^\#_n(t) \xrightarrow{L^1_{x,v}} h^\#(t), \quad \text{as } n \to \infty, \hspace{1cm} (16.25)$$

$$\forall t \in [0,T) : \quad \int_0^t h^\#_n(\tau) \, d\tau \xrightarrow{L^1_{x,v}} \int_0^t h^\#(\tau) \, d\tau, \quad \text{as } n \to \infty. \hspace{1cm} (16.26)$$
Let $f_n^# \in C^0([0, T), L_{x,v}^1) \cap L^\infty([0, T), M_{a,b}^+)$ be the mild solution to the problem

$$\begin{align*}
\begin{cases}
\frac{df_n}{dt} + v \cdot \nabla_x f_n = h_n - L(f_n, g, g),
\end{cases}
\quad (16.27)
\end{align*}$$

constructed in Lemma 16.2. Let us note that Lemma 16.2 is applicable for all $n \in \mathbb{N}$ since $f_{0,n}, h_n$ satisfy (16.5). Hence, $f_n^#$ satisfies

$$\begin{align*}
\begin{cases}
\frac{df_n^#}{dt} + L^#(f_n, g, g) = h_n^#,
\end{cases}
\quad (16.28)
\end{align*}$$

and is given by the formula

$$f_n^#(t) := f_{0,n} \exp \left( - \int_0^t R^#(g, g)(\sigma) \, d\sigma \right) + \int_0^t h_n^#(\tau) \exp \left( - \int_\tau^t R^#(g, g)(\sigma) \, d\sigma \right) \, d\tau. \quad (16.29)$$

Also by (16.6), given $t \in [0, T)$, we have the bound

$$\sup_{n \in \mathbb{N}} \| f_n^#(t) \|_{L_{x,v}^1} \leq \sup_{n \in \mathbb{N}} \left( \| f_{0,n} \|_{L_{x,v}^1} + \int_0^t \| h_n^#(\tau) \|_{L_{x,v}^1} \, d\tau \right)$$

$$\leq \| f_0 \|_{L_{x,v}^1} + \int_0^t \| h^#(\tau) \|_{L_{x,v}^1} \, d\tau$$

$$< \infty, \quad (16.30)$$

where to obtain the last bound we use (16.22)-(16.23), the fact that $R^#(g, g) \geq 0$ (by monotonicity of $R^#$ and $g \geq 0$), $f_0 \in L_{x,v}^1$ and $h^# \in L_{x,v}^1_{loc}([0, T), L_{x,v}^{1+}).$

Since the sequences $(f_{0,n})_n$, $(h_n^#(t))_n$ are increasing and non-negative for all $t \in [0, T)$, formula (16.29) implies that the sequence $(f_n^#(t))_n$ is increasing for all $t \in [0, T)$. Let us define

$$f^#(t) := \lim_{n \to \infty} f_n^#(t).$$
Clearly \( f \geq 0 \). By the Monotone Convergence Theorem and bound (16.30) we obtain that \( f^\#(t) \in L_{x,v}^{1,+}, \quad \forall t \in [0, T) \). Then, the Dominated Convergence Theorem implies
\[
\forall t \in [0, T) : \quad f_n^\#(t) \xrightarrow{L_{x,v}^1} f^\#(t), \quad \text{as } n \to \infty. \tag{16.31}
\]
Moreover, as \( n \to \infty \), we have
\[
L^\#(f_n, g, g)(t) = f_n^\#(t)R^\#(g, g)(t) \nearrow f^\#(t)R^\#(g, g)(t) = L^\#(f, g, g)(t), \quad \forall t \in [0, T), \tag{16.32}
\]
since \( R^\#(g, g)(t) \geq 0 \) by monotonicity of \( R^\# \) and the fact that \( g \geq 0 \). By the Monotone Convergence Theorem, we obtain
\[
\forall t \in [0, T) : \quad \int_0^t \|L^\#(f_n, g, g)(\tau)\|_{L_{x,v}^1} d\tau \nearrow \int_0^t \|L^\#(f, g, g)(\tau)\|_{L_{x,v}^1} d\tau, \quad \text{as } n \to \infty. \tag{16.33}
\]
Therefore, for any \( t \in [0, T) \), equation (16.6), implies
\[
\int_0^t \|L^\#(f, g, g)(\tau)\|_{L_{x,v}^1} d\tau = \sup_{n \in \mathbb{N}} \int_0^t \|L^\#(f_n, g, g)(\tau)\|_{L_{x,v}^1} d\tau \tag{16.34}
\]
\[
\leq \sup_{n \in \mathbb{N}} \left( \|f_0\|_{L_{x,v}^1} + \int_0^t \|h_n^\#(\tau)\|_{L_{x,v}^1} d\tau \right)
\]
\[
\leq \|f_0\|_{L_{x,v}^1} + \int_0^t \|h^\#(\tau)\|_{L_{x,v}^1} d\tau < \infty, \tag{16.35}
\]
since \( f_0 \in L_{x,v}^1 \), and \( h^\# \in L_{loc}^1([0, T), L_{x,v}^{1,+}) \), thus
\[
L^\#(f, g, g)(t) \in L_{x,v}^1, \quad \text{for a.e. } t \in [0, T), \tag{16.36}
\]
\[
L^\#(f, g, g) \in L_{loc}^1([0, T), L_{x,v}^{1,+}). \tag{16.37}
\]
By (16.32), (16.36) and the Dominated Convergence Theorem, for a.e. \( t \in [0, T) \), we have
\[
L^\#(f_n, g, g)(t) \xrightarrow{L_{x,v}^1} L^\#(f, g, g)(t), \quad \text{as } n \to \infty, \tag{16.38}
\]
and by (16.37) and another application of the Dominated Convergence Theorem, we obtain
\[
\int_0^t L^\#(f_n, g, g)(\tau) d\tau \xrightarrow{L^1_{x,v}} \int_0^t L^\#(f, g, g)(\tau) d\tau, \quad \forall t \in [0, T).
\] (16.39)
Since \( f^\#_n \) satisfies (16.28), the Fundamental Theorem of Calculus and the fact that
\( f^\#_n \in C^0([0, T), L^1_{x,v}) \), \( L^\#(f_n, g, g) \) and \( h^\#_n \in L^1_{loc}([0, T), L^1_{x,v}) \) imply
\[
f^\#_n(t) + \int_0^t L^\#(f_n, g, g)(\tau) d\tau = f^\#_0 + \int_0^t h^\#_n(\tau) d\tau, \quad \forall t \in [0, T), \quad \forall n \in \mathbb{N}.
\] (16.40)
Using (16.31), (16.39), (16.24), and (16.26), we let \( n \to \infty \) in (16.40) to obtain
\[
f^\#(t) + \int_0^t L^\#(f_n, g, g)(\tau) d\tau = f^\#_0 + \int_0^t h^\#(\tau) d\tau, \quad \forall t \in [0, T), \quad \forall n \in \mathbb{N},
\] (16.41)
thus \( f^\# \in C^0([0, T), L^1_{x,v}) \), \( f^\# \) is weakly differentiable and satisfies (16.4). We conclude that \( f \) is a mild solution of (16.3). Moreover, since \( g \geq 0 \), we may take the limit as \( n \to \infty \) in both sides of (16.29) to obtain (16.17).

**Uniqueness:** Since the problem is linear it suffices to show that if \( f \) is a solution of (16.3) with \( f^\#_0 = 0 \) and \( h = 0 \), then \( f = 0 \).

Assume \( f \) is a mild solution of (16.3) with \( f^\#_0 = 0 \) and \( h = 0 \) i.e. \( f \geq 0 \), \( f^\# \in C^0([0, T), L^1_{x,v}) \), \( L^\#(f, g, g) \in L^1_{loc}([0, T), L^1_{x,v}) \) and \( f^\# \) is weakly differentiable and satisfies
\[
\begin{cases}
\frac{df^\#}{dt} + L^\#(f, g, g) = 0, \\ f^\#(0) = 0.
\end{cases}
\] (16.42)
Then (16.42), the Fundamental Theorem of Calculus and the facts \( f^\# \in C^0([0, T), L^1_{x,v}) \), \( L^\#(f, g, g) \in L^1_{loc}([0, T), L^1_{x,v}) \) imply
\[
f^\#(t) = -\int_0^t L^\#(f, g, g)(\tau) d\tau = -\int_0^t f^\#(\tau)R^\#(g, g)(\tau) d\tau, \quad \forall t \in [0, T). \] (16.43)
We claim the following

**Claim:** For any compact set \( K \subseteq \mathbb{R}^d \times \mathbb{R}^d \), we have \( \| f^#(t) \|_{L^1_{x,v}(K)} = 0 \), \( \forall t \in [0, T) \).

**Proof of the claim:** Fix any compact set \( K \subseteq \mathbb{R}^d \times \mathbb{R}^d \). By (16.43), Fubini's Theorem, part (i) of Lemma 15.6 and the fact that \( p_{\gamma_2,\gamma_3} \) is continuous, we obtain

\[
\| f^#(t) \|_{L^1_{x,v}(K)} \leq \int_0^t \| f^#(\tau) R^#(g,g)(\tau) \|_{L^1_{x,v}(K)} d\tau \\
\leq C_\beta \| g \|_\infty (1 + \| g \|_\infty) \int_0^t \| p_{\gamma_2,\gamma_3} f^#(\tau) \|_{L^1_{x,v}(K)} d\tau \\
\leq C_{K,\beta} \| g \|_\infty (1 + \| g \|_\infty) \int_0^t \| f^#(\tau) \|_{L^1_{x,v}(K)} d\tau. \tag{16.44}
\]

Since \( f^# \in C^0([0, T), L^1_{x,v}) \), the map \( t \in [0, T) \to \| f^#(t) \|_{L^1_{x,v}(K)} \in [0, \infty) \) is continuous, thus (16.44) and Gronwall's inequality imply that

\[
\| f^#(t) \|_{L^1_{x,v}(K)} = 0, \quad \forall t \in [0, T).
\]

The claim is proved.

Consider now a sequence of compact sets \( (K_m)_m \nearrow \mathbb{R}^d \times \mathbb{R}^d \). By the claim above, and the Monotone Convergence Theorem, we have

\[
\| f^#(t) \|_{L^1_{x,v}} = \lim_{m \to \infty} \| f^#(t) \|_{L^1_{x,v}(K_m)} = 0, \quad \forall t \in [0, T).
\]

Since \( f^# \geq 0 \), we obtain \( f^# = 0 \) and hence \( f = 0 \). Uniqueness is proved. \( \square \)

The following comparison Corollary comes immediately by the monotonicity of \( R^# \) and representation (16.17).
Corollary 16.4. Let $0 < T \leq \infty$ and $\alpha, \beta > 0$. Consider $f_{0,1}, f_{0,2} \in L^{1,+}_{x,v}$, $g_1, g_2 \in L^\infty([0,T), \mathcal{M}^+_{\alpha,\beta})$ and $h_1, h_2 \in L^1_{\text{loc}}([0,T), L^{1,+}_{x,v})$ with

$$f_{0,1} \leq f_{0,2}, \quad g_1^# \geq g_2^#, \quad h_1^# \leq h_2^#.$$ 

Let $f_i, i \in \{1,2\}$ be the corresponding unique solution of (16.3) with $f_0 := f_{0,i}$, $g := g_i$ and $h := h_i$. Then $f_1 \leq f_2$.

Proof. We have $g_1^# \geq g_2^# \Rightarrow g_1 \geq g_2$. By monotonicity of $R^#$ we obtain $R^#(g_1, g_1) \geq R^#(g_2, g_2)$. The claim then comes immediately by representation (16.17).  

16.5 The Kaniel-Shinbrot iteration

Now, we will use well-posedness of the associated linear problem and the estimates developed in Section 15 to prove convergence of the Kaniel-Shinbrot iteration to the unique solution of (13.1) in the class of functions bounded by a Maxwellian, if an appropriate beginning condition is satisfied.

Let $0 < T \leq \infty$ and $\alpha, \beta > 0$. Consider a function $f_0 \in \mathcal{M}^+_{\alpha,\beta}$ and a pair of functions $(l_0^#, u_0^#) \in \mathcal{M}^+_{\alpha,\beta} \times \mathcal{M}^+_{\alpha,\beta}$. By part (iii) of Lemma 15.6 we have that $G^#(l_0, l_0, l_0), G^#(u_0, u_0, u_0) \in L^\infty([0,T), L^{1,+}_{x,v})$. Applying Proposition 16.3 with $h$ being either $G(l_0, l_0, l_0)$ or $G(u_0, u_0, u_0)$, we find unique functions $l_1, u_1$ such that $l_1$ is the mild solution of

$$\frac{dl_1}{dt} + v \cdot \nabla_x l_1 = G(l_0, l_0, l_0) - L(l_1, u_0, u_0),$$

(l1) \quad l_1(0) = f_0,$$
and $u_1$ is the mild solution of
\[
\frac{du_1}{dt} + v \cdot \nabla_x u_1 = G(u_0, u_0, u_0) - L(u_1, l_0, l_0),
\]
\[
u_1(0) = f_0. \tag{16.46}
\]

We obtain the following result

**Theorem 16.5.1.** Let $0 < T \leq \infty$, $\alpha, \beta > 0$ and
\[
K_\beta = C_d \left[ \|b_2\|_{L^1([\beta^{d-1}])} (\beta^{-d/2} + \frac{1}{d + \gamma_2 - 1}) + \|b_3\|_{L^1([\beta^{2d-1}])} (\beta^{-d} + \frac{1}{2d + \gamma_3 - 1}) \right],
\]
be the constant given in (15.49). Consider $f_0 \in M_{\alpha, \beta}^+$ and $(l_0^\#, u_0^\#) \in M_{\alpha, \beta}^+ \times M_{\alpha, \beta}^+$ with
\[
\|u_0^\#\|_{M_{\alpha, \beta}} < \lambda_{\alpha, \beta}, \tag{16.47}
\]
where
\[
\lambda_{\alpha, \beta} = \min \left\{ \frac{\alpha^{1/4}}{24K_\beta}, \frac{\alpha^{1/4}}{2\sqrt{6}K_\beta} \right\}. \tag{16.48}
\]

Let $l_1, u_1$ be the mild solutions to (16.45) and (16.46) respectively, and assume that the following beginning condition holds
\[
0 \leq l_0^\# \leq l_1^\#(t) \leq u_1^\#(t) \leq u_0^\#, \quad \forall t \in [0, T]. \tag{16.49}
\]

Then we conclude the following

(i) There are unique sequences $(l_n)_n$, $(u_n)_n$ such that, for any $n \in \mathbb{N}$, $l_n, u_n$ are the mild solution to (16.1), (16.2) respectively. Moreover, for any $n \in \mathbb{N}$, we have
\[
0 \leq l_0^\# \leq l_1^\#(t) \leq ... \leq l_n^\#(t) \leq u_n^\#(t) \leq ... \leq u_1^\#(t) \leq u_0^\#, \quad \forall t \in [0, T]. \tag{16.50}
\]
(ii) For all \( t \in [0, T) \), the sequences \( (l_n(t))_n, (u_n(t))_n \) converge in \( M_{\alpha,\beta} \). Let us define

\[
l^\#(t) := \lim_{n \to \infty} l_n(t), \quad u^\#(t) := \lim_{n \to \infty} u_n(t), \quad t \in [0, T).
\]

Then, we conclude that

\[
l^\#, u^\# \in C^0([0, T), L_{x,v}^{1,+}) \cap L^\infty([0, T), M_{\alpha,\beta}^+),
\]

\[
L^\#(l, u, u), L^\#(u, l, l), G^\#(l, l, l), G^\#(u, u, u) \in L^\infty([0, T), L_{x,v}^{1,+}),
\]

and the following integral equations are satisfied

\[
l^\#(t) + \int_0^t L^\#(l, u, u)(\tau) \, d\tau = f_0 + \int_0^t G^\#(l, l, l)(\tau) \, d\tau, \quad \forall t \in [0, T),
\]

\[
u^\#(t) + \int_0^t L^\#(u, l, l)(\tau) \, d\tau = f_0 + \int_0^t G^\#(u, u, u)(\tau) \, d\tau, \quad \forall t \in [0, T).
\]

(iii) The limits \( l, u \) coincide i.e. \( u = l \).

(iv) Let us define \( f := l = u \). Then \( f \) is the unique mild solution of the binary-ternary Boltzmann equation \((13.1)\) in \([0, T)\), with initial data \( f_0 \in M_{\alpha,\beta}^+ \) satisfying

\[
\| f^\# \|_\infty \leq \| u_0^\# \|_{M_{\alpha,\beta}^+}.
\]

Remark 16.3. The uniqueness claimed above holds in the class of solutions satisfying \((16.53)\).

Proof. (i): We will construct sequences \( (l_n)_n, (u_n)_n \) satisfying \((16.1)-(16.50)\) inductively.

- \( n = 1 \): \( l_1, u_1 \) satisfy \((16.1)\) for \( k = 1 \) by assumption. Moreover \((16.50)\) reduces for \( k = 1 \) to the assumption \((16.49)\).
• Assume we have constructed \( l_1, \ldots, l_{n-1}, u_1, \ldots, u_{n-1} \) satisfying (16.1) and

\[
l^\#_0 \leq l^\#_1(t) \leq \ldots \leq l^\#_{n-1}(t) \leq u^\#_{n-1}(t) \leq u^\#_1(t) \leq u^\#_0, \quad \forall t \in [0, T).
\]

(16.54)

Let \( l_n, u_n \) be the mild solutions of (16.1), (16.2) for \( k = n \) respectively, given by Proposition 16.3. Having in mind (16.54), in order to prove (16.50), it suffices to show

\[
l^\#_{n-1}(t) \leq l^\#_n(t) \leq u^\#_n(t) \leq u^\#_{n-1}(t), \quad \forall t \in [0, T).
\]

(16.55)

Fix any \( t \in [0, T) \). Then (16.54) and Proposition 15.2, which gives monotonicity of \( G^\# \), yield that for any \( t \in [0, T) \), we have

\[
G^\#(l_{n-2}, l_{n-2}, l_{n-2})(t) \leq \ldots \leq G^\#(u_{n-1}, u_{n-1}, u_{n-1})(t) \leq G^\#(u_{n-2}, u_{n-2}, u_{n-2})(t).
\]

(16.56)

Using (16.54), (16.56) and Corollary 16.4 with

\[
g_1^\# = u_{n-2}^\#, \quad g_2^\# = u_{n-1}^\#, \quad h_1^\# = G^\#(l_{n-2}, l_{n-2}, l_{n-2}), \quad h_2^\# = G^\#(l_{n-1}, l_{n-1}, l_{n-1}),
\]

we obtain \( l^\#_{n-1} \leq l^\#_n \). Similarly, using Corollary 16.4 for \( g_1^\# = u_{n-1}^\#, \quad g_2^\# = l_{n-1}^\#, \quad h_1^\# = G^\#(l_{n-1}, l_{n-1}, l_{n-1}), \quad h_2^\# = G^\#(u_{n-1}, u_{n-1}, u_{n-1}) \), we obtain \( l^\#_n \leq u^\#_n \), and using it for \( g_1^\# = l_{n}^\#, \quad g_2^\# = l_{n-1}^\#, \quad h_1^\# = G^\#(u_{n-1}, u_{n-1}, u_{n-1}), \quad h_2^\# = G^\#(u_{n-2}, u_{n-2}, u_{n-2}) \), we obtain \( u^\#_n \leq u^\#_{n-1} \). Condition (16.55) is proved and the claim follows.

(ii): To prove convergence, notice that (16.50) implies that, for any \( t \in [0, T) \), the sequence \((l^\#_n(t))_n\) is increasing and upper bounded and the sequence \((u^\#_n(t))_n\) is
decreasing and lower bounded, thus they are convergent. Let us define
\[ l^\#(t) := \lim_{n \to \infty} l^\#_n(t), \quad u^\#(t) := \lim_{n \to \infty} u^\#_n(t), \quad t \in [0, T). \]

Since \( u^\#_0 \in M^+_{\alpha,\beta} \), estimate (16.50) actually implies that the convergence takes place in \( M_{\alpha,\beta} \) and that \( l^\#, u^\# \in L^\infty([0, T), M^+_{\alpha,\beta}) \). Thus relations (15.22) from Lemma 15.6 imply that
\[ L^\#(l, u, u), \quad L^\#(u, l, l), \quad G^\#(l, l, l), \quad G^\#(u, u, u) \in L^\infty([0, T), L^1_{x,v}). \quad (16.57) \]

Moreover, since for any \( t \in [0, T) \) we have
\[ (l^\#_n, u^\#_{n-1}, u^\#_{n-1})(t) \xrightarrow{M_{\alpha,\beta}} (l^\#, u^\#, u^\#)(t), \quad (u^\#_n, l^\#_{n-1}, l^\#_{n-1})(t) \xrightarrow{M_{\alpha,\beta}} (u^\#, l^\#, l^\#)(t), \]
as \( n \to \infty \), Corollary 15.8 implies that for any \( t \in [0, T) \), we have
\[ L^\#(l_n, u_{n-1}, u_{n-1})(t) \xrightarrow{L^1_{x,v}} L^\#(l, u, u), \quad L^\#(u_n, l_{n-1}, l_{n-1})(t) \xrightarrow{L^1_{x,v}} L^\#(u, l, l), \quad (16.58) \]

Similarly, for any \( t \in [0, T) \), we obtain
\[ G^\#(l_{n-1}, l_{n-1}, l_{n-1})(t) \xrightarrow{L^1_{x,v}} G^\#(l, l, l), \quad G^\#(u_{n-1}, u_{n-1}, u_{n-1})(t) \xrightarrow{L^1_{x,v}} G^\#(u, u, u), \]
\[ (16.59) \]
Moreover, by relation (16.50), monotonicity of \( L^\#, G^\# \), and the fact that \( u^\#_0 \in M^+_{\alpha,\beta} \), Lemma 15.6 implies
\[ L^\#(l_n, u_{n-1}, u_{n-1}), \quad G^\#(l_{n-1}, l_{n-1}, l_{n-1}) \in L^\infty([0, T), L^1_{x,v}), \quad \forall n \in \mathbb{N}, \]
\[ L^\#(u_n, l_{n-1}, l_{n-1}), \quad G^\#(u_{n-1}, u_{n-1}, u_{n-1}) \in L^\infty([0, T), L^1_{x,v}), \quad \forall n \in \mathbb{N}. \quad (16.60) \]
Recalling Definition 14.1, the initial value problems (16.1), (16.2) and the Fundamental Theorem of Calculus imply that for all \( n \in \mathbb{N} \) we have

\[
\begin{align*}
\int_0^t L^\#(l_n, u_{n-1}, u_{n-1})(\tau) \, d\tau &= f_0 + \int_0^t G^\#(l_{n-1}, l_{n-1}, l_{n-1})(\tau) \, d\tau, \\
\int_0^t L^\#(u_n, l_{n-1}, l_{n-1})(\tau) \, d\tau &= f_0 + \int_0^t G^\#(u_{n-1}, u_{n-1}, u_{n-1})(\tau) \, d\tau.
\end{align*}
\]  

Letting \( n \to \infty \) and using the Dominated Convergence Theorem, we obtain (16.51)-(16.52). The fact that of \( l^\#, u^\# \in C^0([0, T), L^1_{x,v}) \) easily follows from (16.51)-(16.52).

(iii): Since \( l^\#_n \leq u^\#_n \) by (16.50), letting \( n \to \infty \), we obtain

\[
0 \leq t^\#_0 \leq l^\#(t) \leq u^\#(t) \leq u^\#_0, \quad \forall \, t \in [0, T).
\]  

Subtracting (16.51) from (16.52) and using (16.63) and the triangle inequality, we obtain

\[
|u^\#(t) - l^\#(t)|
\leq \int_0^t |G^\#(u, u, u)(\tau) - G^\#(l, l, l)(\tau)| + |L^\#(l, u, u)(\tau) - L^\#(u, l, l)(\tau)| \, d\tau. 
\]  

Let us estimate the right hand side of (16.64). Recalling (14.37) triangle inequality yields

\[
\int_0^t |G^\#(u, u, u)(\tau) - G^\#(l, l, l)(\tau)| \, d\tau
\leq \int_0^t |G_2^\#(u, u)(\tau) - G_2^\#(l, l)(\tau)| + |G_3^\#(u, u, u)(\tau) - G_3^\#(l, l, l)(\tau)| \, d\tau. 
\]  

Bilinearity of \( G_2^\# \), triangle inequality, bound (15.46) from Proposition 15.11, and the
right hand side inequality of (16.63) yield

\[
\int_0^t \left| G_2^\#(u, u)(\tau) - G_2^\#(l, l)(\tau) \right| \, d\tau \leq \int_0^t \left| G_2^\#(u - l, u)(\tau) \right| + \left| G_2^\#(l, u - l)(\tau) \right| \, d\tau \\
\leq K_2^{-1/2} M_{\alpha, \beta} \left\| u^\# - l^\# \right\|_{L^\infty([0, T], M_{\alpha, \beta})} \left( \left\| u^\# \right\|_\infty + \left\| l^\# \right\|_\infty \right) \\
\leq 2K_2^{-1/2} M_{\alpha, \beta} \left\| u^\#_0 \right\|_{M_{\alpha, \beta}} \left\| u^\# - l^\# \right\|_\infty. 
\] (16.66)

Trilinearity of $G_3^\#$, triangle inequality, bound (15.47) from Proposition 15.11, and the right hand side of (16.63) yield

\[
\int_0^t \left| G_3^\#(u, u, u)(\tau) - G_3^\#(l, l, l)(\tau) \right| \, d\tau \\
\leq \int_0^t \left| G_3^\#(u - l, u, u)(\tau) \right| + \left| G_3^\#(l, u - l, u)(\tau) \right| + \left| G_3^\#(l, l, u - l)(\tau) \right| \, d\tau \\
\leq K_3^{-1/2} M_{\alpha, \beta} \left\| u^\# - l^\# \right\|_{L^\infty([0, T], M_{\alpha, \beta})} \left( \left\| u^\# \right\|_\infty + \left\| l^\# \right\|_\infty + \left\| (l^\#)^2 \right\|_\infty \right) \\
\leq 3K_3^{-1/2} M_{\alpha, \beta} \left\| u^\#_0 \right\|_{M_{\alpha, \beta}}^2 \left\| u^\# - l^\# \right\|_\infty. 
\] (16.67)

Then estimates (16.66)-(16.67) yield

\[
\int_0^t \left| G^\#(u, u, u)(\tau) - G^\#(l, l, l)(\tau) \right| \, d\tau \\
\leq 6K_3^{-1/2} M_{\alpha, \beta} \left( \left\| u^\#_0 \right\|_{M_{\alpha, \beta}} + \left\| u^\#_0 \right\|_{M_{\alpha, \beta}}^2 \right) \left\| u^\# - l^\# \right\|_\infty. 
\] (16.68)

By a similar argument, using (15.46), (15.47) instead, we also have

\[
\int_0^t \left| L^\#(l, u, u)(\tau) - L^\#(u, u, l)(\tau) \right| \, d\tau \\
\leq 6K_3^{-1/2} M_{\alpha, \beta} \left( \left\| u^\#_0 \right\|_{M_{\alpha, \beta}} + \left\| u^\#_0 \right\|_{M_{\alpha, \beta}}^2 \right) \left\| u^\# - l^\# \right\|_\infty. 
\]

Combining (16.64), (16.68)-(16.69), we obtain

\[
|u^\#(t) - l^\#(t)| \leq 12K_3^{-1/2} M_{\alpha, \beta} \left( \left\| u^\#_0 \right\|_{M_{\alpha, \beta}} + \left\| u^\#_0 \right\|_{M_{\alpha, \beta}}^2 \right) \left\| u^\# - l^\# \right\|_\infty, \quad \forall t \in [0, T),
\]

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which is equivalent to

\[ |||u^\# - l^\#|||_\infty \leq 12K_\beta \alpha^{-1/2}(||u_0^\#||_{\mathcal{M}_{\alpha,\beta}} + ||u_0^\#||_{\mathcal{M}_{\alpha,\beta}}^2)|||u^\# - l^\#|||_\infty. \]  

(16.69)

Notice though that (16.47)-(16.48) yield

\[ 12K_\beta \alpha^{-1/2}(||u_0^\#||_{\mathcal{M}_{\alpha,\beta}} + ||u_0^\#||_{\mathcal{M}_{\alpha,\beta}}^2) < 1, \]

hence (16.69) yields \( u = l \).

(iv): To prove existence, let us define \( f \) by \( f^\#: = l^\# = u^\# \in C^0([0, T), L_{x,v}^1) \cap L^\infty([0, T), \mathcal{M}_{\alpha,\beta}^+) \). Then, either (16.51) or (16.52) implies

\[ f^\#(t) + \int_0^t L^\#(f, f, f)(\tau) \, d\tau = f_0 + \int_0^t G^\#(f, f, f)(\tau) \, d\tau, \quad \forall t \in [0, T), \]

denoted as

\[ \begin{cases} \frac{df^\#}{dt} + L^\#(f, f, f) = G^\#(f, f, f), \\ f^\#(0) = f_0. \end{cases} \]

Recalling Definition 14.1, we conclude that \( f \) is a mild solution to the binary-ternary Boltzmann equation (13.1) with initial data \( f_0 \). Bound (16.53) directly follows from (16.63).

Uniqueness of solutions satisfying (16.53) follows similarly to the proof of (iii) using a bilinearity-trilinearity argument and Proposition 15.11. Clearly, condition (16.53) is needed to have a contraction. \( \square \)
Chapter 17

Global well-posedness near vacuum

In this final chapter, we prove the main result of this part of the dissertation, stated in Theorem 14.3.1, which gives global well-posedness of (13.1) near vacuum in the interval \([0, T]\), where \(0 < T \leq \infty\). To prove this result we will rely on the time average bound of the gain term from Proposition 15.11.

Proof of Theorem 14.3.1

Consider \(f_0 \in \mathcal{M}_{\alpha,\beta}^{+}\) satisfying (14.76) and let us define \(t_0^# = 0\), \(u_0^# = C_{out}\mathcal{M}_{\alpha,\beta}\), where

\[
C_{out} = \frac{1}{24K_{\beta}\alpha^{-1/2}\left(1 + \frac{\alpha^{1/4}}{2\sqrt{6K_{\beta}}}\right)} \left(1 - \sqrt{1 - 48K_{\beta}\alpha^{-1/2}\left(1 + \frac{\alpha^{1/4}}{2\sqrt{6K_{\beta}}}\right)}\|f_0\|_{\mathcal{M}_{\alpha,\beta}}\right), 
\]

(17.1)

\(K_{\beta}\) is given by (14.77). The reasoning behind defining \(C_{out}\) will become clear in (17.8). Notice that due to (14.76) \(u_0^#\) is well defined. In order to conclude the proof, we will use Theorem 16.5.1. Recalling

\[
\lambda_{\alpha,\beta} = \min\left\{\frac{\alpha^{1/2}}{24K_{\beta}}, \frac{\alpha^{1/4}}{2\sqrt{6K_{\beta}}}\right\},
\]

from (16.48), (17.1) and (14.76) yield

\[
\|u_0^#\|_{\mathcal{M}_{\alpha,\beta}} = C_{out} < \lambda_{\alpha,\beta},
\]

(17.2)
thus the conditions of Theorem 16.5.1 are satisfied. By Theorem 16.5.1, it suffices to prove that the beginning condition (16.49) for the approximating sequences generated by \( f_0 \in M_{\alpha,\beta}^+ \) and the pair of functions \((l_0^\#, u_0^\#) \in M_{\alpha,\beta}^+ \times M_{\alpha,\beta}^+\) is satisfied. Indeed, by the iteration scheme (16.45)-(16.46), we have

\[
\frac{dl_1^\#}{dt} + l_1^\# R^\#(u_0, u_0) = 0, \\
\frac{du_1^\#}{dt} = G^\#(u_0, u_0, u_0), \\
u_1^\#(0) = l_1^\#(0) = f_0,
\]

therefore, we obtain

\[
l_1^\#(t) = f_0 \exp \left( - \int_0^t R^\#(u_0, u_0)(\tau) \, d\tau \right), \quad t \in [0, T), \quad (17.3)
\]

\[
u_1^\#(t) = f_0 + \int_0^t G^\#(u_0, u_0, u_0)(\tau) \, d\tau, \quad t \in [0, T). \quad (17.4)
\]

Since \( u_0 \geq 0 \), formulas (17.3)-(17.4) together with Proposition 15.2 imply

\[
0 = l_0^\# \leq l_1^\#(t) \leq u_1^\#(t), \quad \forall t \in [0, T). \quad (17.5)
\]

It remains to prove that

\[
u_1^\#(t) \leq u_0^\#, \quad \forall t \in [0, T). \quad (17.6)
\]

By representation (17.4) and (15.48) from Proposition 15.11, we obtain

\[
u_1^\#(t) \leq \|f_0\|_{M_{\alpha,\beta}} M_{\alpha,\beta} + K_{\beta} \alpha^{-1/2} M_{\alpha,\beta} \|u_0^\#\|_{M_{\alpha,\beta}}^2 (1 + \|u_0^\#\|_{M_{\alpha,\beta}}) \]

\[
\leq M_{\alpha,\beta} \left[ \|f_0\|_{M_{\alpha,\beta}} + K_{\beta} \alpha^{-1/2} \left( 1 + \frac{\alpha^{1/4}}{2\sqrt{6K_{\beta}}} \right) C_{\text{out}}^2 \right], \quad (17.7)
\]

where to obtain (17.7), we use the fact that \( u_0^\# = C_{\text{out}} M_{\alpha,\beta} \) and (17.2). Recalling (17.1), we notice that \( C_{\text{out}} \) satisfies the equation

\[
\|f_0\|_{M_{\alpha,\beta}} + 12K_{\beta} \alpha^{-1/2} \left( 1 + \frac{\alpha^{1/4}}{2\sqrt{6K_{\beta}}} \right) C_{\text{out}}^2 = C_{\text{out}}. \quad (17.8)
\]
thus \((17.7)\) implies

\[
u_1^\#(t) \leq C_{out} M_{\alpha,\beta} = u_0^\#, \quad \forall t \in [0, T).
\]

Estimate \((17.6)\) is proved and the claim of Theorem 14.3.1 follows.
Appendix
Appendix A

Some auxiliary results

In this appendix, we present some general, auxiliary results which are used throughout the dissertation.

We first state and prove the following elementary Linear Algebra result which is crucially used for the calculation of the Jacobian of the transition map in Proposition 9.7.

**Lemma A.1.** Let $n \in \mathbb{N}$, $\lambda \neq 0$ and $w, u \in \mathbb{R}^n$. Then

$$\det(\lambda I_n + wu^T) = \lambda^n(1 + \lambda^{-1}\langle w, u \rangle),$$

where $I_n$ is the $n \times n$ identity matrix.

**Proof.** Without loss of generality, we may assume $\lambda = 1$. We use the block matrix identity

$$\begin{pmatrix} I_n & 0 \\ u^T & 1 \end{pmatrix} \begin{pmatrix} I_n + wu^T & w \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I_n & 0 \\ -u^T & 1 \end{pmatrix} = \begin{pmatrix} I_n & w \\ 0 & 1 + w^Tu \end{pmatrix}. \tag{A.1}$$

Taking determinants in both sides of (A.1), and using the fact that all the block matrices are triangular we obtain the result. $\square$

Now, we derive a general change of variables formula for surface integrals given by the level sets of a smooth function. This result is useful for performing change of variables on surface integrals. We will use notation from (1.14)-(1.16).

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Lemma A.2. Let \( n \in \mathbb{N}, \Psi : \mathbb{R}^n \to \mathbb{R} \) be a \( C^1 \) function and \( \gamma \in \mathbb{R} \). Assume there is \( \delta > 0 \) with \( \nabla \Psi(\omega) \neq 0 \) for \( \omega \in [\gamma - \delta < \Psi < \gamma + \delta] \). Let \( \Omega \subseteq \mathbb{R}^n \) be a domain and consider a \( C^1 \) map \( F : \Omega \to \mathbb{R}^n \) of non-zero Jacobian in \( \Omega \). Then, for any measurable \( g : \mathbb{R}^n \to [0, +\infty) \) or \( g : \mathbb{R}^n \to [-\infty, +\infty] \) integrable, there holds the change of variables formula:

\[
\int_{\Psi = \gamma} g(\nu) \mathcal{N}_F(\nu, [\Psi \circ F = \gamma]) \, d\sigma(\nu) = \int_{[\Psi \circ F = \gamma]} (g \circ F)(\omega) \left| \text{Jac} \, F(\omega) \right| \left| \frac{\nabla \Psi(F(\omega))}{\nabla (\Psi \circ F)(\omega)} \right| \, d\sigma(\omega), \quad (A.2)
\]

where \( d\sigma(\nu), d\sigma(\omega) \) denote the surface measures on the corresponding hypersurfaces and given \( \nu \in \mathbb{R}^n \) and \( A \subseteq \Omega \), \( \mathcal{N}_F(\nu, A) \) denotes the Banach indicatrix:

\[
\mathcal{N}_F(\nu, A) = \text{card}(\{ \omega \in A : F(\omega) = \nu \}). \quad (A.3)
\]

Proof. Notice that since \( \nabla \Psi(\omega) \neq 0 \), \( \forall \omega \in [\gamma - \delta < \Psi < \gamma + \delta] \), we have

\[
\nabla \Psi(F(\omega)) \neq 0, \quad \forall \omega \in [\gamma - \delta < \Psi \circ F < \gamma + \delta]. \quad (A.4)
\]

Moreover since \( \text{Jac} \, F(\omega) \neq 0, \quad \forall \omega \in \Omega \), the chain rule implies

\[
\nabla (\Psi \circ F)(\omega) = D^T F(\omega) \nabla \Psi(F(\omega)) \neq 0, \quad \forall \omega \in [\gamma - \delta < \Psi \circ F < \gamma + \delta]. \quad (A.5)
\]
Consider $\epsilon < \delta$. Then the co-area formula implies

\[
\int_{\gamma - \epsilon}^{\gamma + \epsilon} \int_{[\Psi = s]} g(\nu) N_F(\nu, [\Psi \circ F = s]) \, d\sigma(\nu) \, ds = 
\]

\[
= \frac{1}{2\epsilon} \int_{[\gamma - \epsilon < \Psi < \gamma + \epsilon]} g(y) N_F(\nu, [\gamma - \epsilon < \Psi \circ F < \gamma + \epsilon]) |\nabla \Psi(\nu)| \, d\nu 
\]

\[
= \frac{1}{2\epsilon} \int_{[\gamma - \epsilon < \Psi \circ F < \gamma + \epsilon]} (g \circ F)(\omega) |\nabla \Psi(F(\omega))| |\text{Jac} \, F(\omega)| \, d\omega 
\]

\[
= \frac{1}{2\epsilon} \int_{[\gamma - \epsilon < \Psi \circ F < \gamma + \epsilon]} (g \circ F)(\omega) |\text{Jac} \, F(\omega)| \frac{|\nabla \Psi(F(\omega))|}{|\nabla (\Psi \circ F)(\omega)|} \frac{|\nabla (\Psi \circ F)(\omega)|}{|\nabla (\Psi \circ F)(\omega)|} \, d\omega 
\]

\[
= \int_{\gamma - \epsilon}^{\gamma + \epsilon} \int_{[\Psi_F = s]} (g \circ F)(\omega) |\text{Jac} \, F(\omega)| \frac{|\nabla \Psi(F(\omega))|}{|\nabla (\Psi \circ F)(\omega)|} \, d\sigma(\omega) \, ds, 
\]

where to obtain (A.7) we make the substitution $\nu = F(\omega)$ in $\Omega \subseteq \mathbb{R}^n$, to obtain (A.8) we use (A.5), and to obtain (A.9) we use the co-area formula again. Letting $\epsilon \to 0^+$, using Lebesgue’s Differentiation Theorem and the continuity of the surface integral with respect to $s$, we obtain (A.2). \hfill \square
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Vita

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This dissertation was typeset with \LaTeX\ by the author.

\LaTeX is a document preparation system developed by Leslie Lamport as a special version of Donald Knuth’s \TeX Program.