Research Statement

1 Introduction

My research is primarily concerned with foundational aspects of equivariant homotopy theory (homotopy theory in the presence of a group action). The subject first achieved fame in the 1980’s due to Carlson’s proof of the Segal conjecture [11] and Miller’s proof of the Sullivan conjecture [26], each of which concerns group cohomology and the relation between classifying spaces and spheres. A recurrence of interest began in the 2000’s when Hill, Hopkins, and Ravenel solved the Kervaire invariant one problem on exotic spheres [20], constructing an exotic $\mathbb{Z}/8$-equivariant cohomology theory and taking advantage of rich algebraic structure borrowed from representation theory and Witt vectors [10].

In this way, computations in equivariant homotopy theory have proven useful in many of the greatest successes of algebraic topology in the last 30 years. However, the theoretical foundations of the subject have remained opaque, even while recent developments suggest that a better conceptual understanding of equivariant homotopy theory could lead to developments in algebraic K-theory.

Algebraic K-theory can be approximated by topological Hochschild homology (THH) via a trace map, generalizing the trace of linear algebra [27]. These trace methods utilize a natural circle-action on THH and have led to recent successes computing K-theory. Questions in algebraic K-theory are therefore closely related to questions in circle-equivariant stable homotopy theory.

The chromatic redshift conjecture roughly states that iteratively applying algebraic K-theory to a ring produces progressively better approximations to the homotopy groups of spheres [1]. There is a corresponding phenomenon in equivariant homotopy theory which is not yet well-understood: equivariant norm maps (made famous by Hill, Hopkins, and Ravenel) appear to exhibit similar redshift behavior. A long-term goal is to understand these phenomena and their interactions with each other.

1.1 My work

One approach to the foundations of equivariant homotopy theory is being developed by Barwick and his students [4], who rebuild the combinatorics of higher category theory from scratch in an equivariant setting. This approach is powerful but theoretically daunting. I am pursuing an alternative approach to set equivariant homotopy theory into the already existing framework of higher category theory and higher algebra, which has been developed extensively over the past decade. There are three steps, the first of which is mostly complete, and the others in progress:

1. categorify tools from commutative algebra to study commutative semiring categories;

2. identify equivariant homotopy theory with commutative algebra over the ring category $\text{FinSet}_G$;

3. equivariant homotopy theory arises combinatorially from ‘reparametrization’ $\text{FinSet} \to \text{FinSet}_G$.

I found [5] that the categorified commutative algebra from step (1) is well suited both to the study of equivariant homotopy theory and algebraic K-theory. It is also closely connected to work on derived algebraic geometry and chromatic redshift [2]. Thus there is hope that it can serve as a foundation for equivariant homotopy theory which permits the desired applications to K-theory.
Much of my work fits into the common theme of derived algebraic geometry (DAG) and higher category theory, technology which has recently revolutionized algebraic topology, and whose reverberations are being felt in algebraic geometry, symplectic geometry, and mathematical physics. Over the long term, I am eager to use the foundation I am establishing in DAG and higher category theory to explore some of these interdisciplinary connections.

2 Equivariant homotopy theory

2.1 Background

Homotopy theory studies spaces only up to homotopy equivalence. While this approach loses some topological data, it retains all information about algebraic invariants: homology and cohomology, as well as homotopy groups. It also comes with tremendous theoretical benefits, two of which are:

1. There are purely combinatorial models for topological spaces (via simplicial sets), but only \( \text{up to homotopy} \) [23]. These models allow us to import algebraic (rather than geometric) tools into topology.

2. Topological abelian groups approximately correspond to cohomology theories, but only up to homotopy. Thus, exotic cohomology theories can be studied using the tools of \textit{higher algebra} (‘commutative algebra up to homotopy’). This algebraic approach dates back to May’s work on infinite loop space machines [16] and in recent years has been refined by Lurie [22]. It is an approach which plays a central role in my work.

The basic problem in equivariant homotopy theory is that group actions do not respect homotopies. When a group acts on a space, it interacts crucially with the point-set data. This means we should not expect equivariant analogues of either of the statements above!

A natural question then is: \textbf{How can we model equivariant spaces and spectra} (cohomology theories) in a way which is \textit{homotopy invariant}? For spaces, a solution was presented in Elmendorf’s 1983 thesis. The solution for spectra is much more recent (2013).

\textbf{Theorem 2.1} (Elmendorf [15]). \(G\)-spaces are equivalent (up to homotopy) to product-preserving functors from the category of finite \(G\)-sets:

\[ \text{FinSet}_G^{\text{op}} \rightarrow \text{Top}. \]

\textbf{Theorem 2.2} (Guillou, May [18]). \(G\)-(connective) spectra are equivalent (up to homotopy) to product-preserving functors from the Burnside category (virtual spans of finite \(G\)-sets)

\[ \text{Burn}_G \rightarrow \text{Top}. \]

A product-preserving functor \(X: \text{FinSet}_G^{\text{op}} \rightarrow \text{Top}\) is determined on objects by where it sends each orbit \(G/H\). If \(X\) is a \(G\)-space, we define \(X(G/H)\) to be the \(H\)-fixed points of \(X\).

\textbf{The content of Elmendorf’s theorem is this}: If \(X\) is a \(G\)-space, the fixed point spaces of \(X\) are not homotopy invariant, but if we explicitly remember them by assembling them into a functor \(\text{FinSet}_G^{\text{op}} \rightarrow \text{Top}\), this functor \textit{is} homotopy invariant.

The theorem of Guillou and May is less intuitive but has a long history. It has long been known that the homotopy groups of equivariant spectra come with the rich structure of a \textit{Mackey functor} [13] (by definition a product-preserving functor \(\text{Burn}_G \rightarrow \text{Set}\)). The theorem of Guillou and May explained decades later why this is: equivariant spectra are \textit{themselves} topological Mackey functors!
However, these theorems leave many questions open. If an equivariant spectrum comes with additional structure, how is this structure manifested in the corresponding Mackey functor? To solve the Kervaire invariant problem, Hill, Hopkins, and Ravenel construct a $\mathbb{Z}/8$-ring spectrum $E$ and use an extremely rich Tambara functor structure on $\pi_0 E$ [20]. Notably, Tambara functors [10] can also be described as product-preserving functors (Poly$_G \rightarrow \text{Set}$), inspiring a conjecture:

**Conjecture 2.3.** $G$-(connective) commutative ring spectra are equivalent (up to homotopy) to product-preserving functors (topological Tambara functors)

\[ \text{Poly}_G \rightarrow \text{Top}. \]

The theorems of Elmendorf and Guillou and May are important because they are the only known models for equivariant structure which are combinatorial, rather than geometric. Thus they offer an important source of examples of equivariant spaces and spectra coming from algebra.

We hope that a solution to this conjecture would help us construct expected commutative ring structures on equivariant spectra, making available the computational tools of [20]. A priori, this can be difficult. Even proving an ordinary spectrum has such a commutative ring structure is very difficult, and was the subject of some of the most substantial work of the 1990’s in homotopy theory [16].

### 2.2 My work

I will describe a program for proving Conjecture 2.3. The classical description of Poly$_G$ is ad hoc and cumbersome. However, in order to work with Fun$^x$(Poly$_G$, Top), it is first necessary to establish a universal property for Poly$_G$.

For comparison, in order to more cleanly formulate Guillou and May’s theorem, Barwick provided a universal property for Burn$_G$ [3]. Namely, the span construction is a sort of partial right adjoint to the twisted arrow construction (which is classical), and Burn$_G = \text{Span}(\text{FinSet}_G)$.

In an upcoming paper [7], I generalize this construction. If $I$ and $D$ are two categories, and some objects of $I$ are labeled ‘twisted’, I define a twisted functor category TwFun$_{I,D}$. Then Span$_I(\cdot)$ is a partial right adjoint to TwFun($I, \cdot$).

**Theorem 2.4** (B. [7]). Span$_R$, Span$_{T \to \bar{R}}$, and Span$_{T \to N \to \bar{R}}$ recover FinSet$^\text{op}_G$, Burn$_G$, and Poly$_G$.

Notably, $T$, $N$, and $R$ induce transfers, norms, and restrictions in Mackey and Tambara functors.

My Span$_{TNR}$ construction also allows for the first construction of Poly$_G$ as an $\infty$-category. This can be viewed as a technicality, yet it is necessary even to state Conjecture 2.3 rigorously.

Each of the constructions listed above (Span$_R$, Span$_{TR}$, Span$_{TNR}$) is functorial from locally cartesian closed disjunctive (LCCD) categories to cocartesian monoidal categories, LCCD $\rightarrow$ CocartMon. In [7], I will find that such a functor gives rise formally to an equivariant category.

There is a global Burnside category $\text{Burn}_{\text{glo}}$ defined via a span construction on finite groupoids. It encodes families of $G$-equivariant spectra, as $G$ varies compatibly over all finite groups, with each $G$-equivariant object being recovered by restriction along a subcategory inclusion Burn$_G \subseteq \text{Burn}_{\text{glo}}$.

**Theorem 2.5** (B. [7]). LCCD has a closed symmetric monoidal structure. The category Fin$_G$ of finite (right) $G$-sets is dualizable in LCCD; its dual is the category $G\text{Free}$ of finitely generated (left) free $G$-sets.

**Corollary 2.6.** $\text{Burn}_{\text{glo}}$ is the full subcategory of LCCD spanned by $G\text{Free}$ ($G$ varies over finite groups).
Using these observations, virtually every category of interest in equivariant homotopy theory arises from a functor \( \text{LCCD} \to \text{Cat} \), restricted to \( \text{Burn}^{\text{glo}} \) (and from there to \( \text{Burn}_G \)). Moreover:

**Remark 2.7.** For every \( C \in \text{LCCD} \), there is a theory of group cohomology

\[
C_\ast(G) = K_\ast \text{Fun}_{\text{LCCD}(G \text{Free}, C)}.
\]

This recovers \( E^\ast(BG) \) for any spectrum \( E \), as well as the equivariant sphere spectrum.

### 2.3 Future work

I anticipate Theorem 2.5 and its corollary will have applications to Conjecture 2.3 and similar descriptions of equivariant objects. Most of this section is dedicated to those applications.

More tentatively, there may be connections to noncommutative algebraic geometry and algebraic K-theory. Specifically, the enlargement of the global Burnside category (really a 2-category) to \( \text{LCCD} \) (locally cartesian closed disjunctive categories) suggests: **locally cartesian closed disjunctive categories are motives for group cohomology theories.** There is a striking parallel to the study of noncommutative cohomology theories in algebraic geometry (which include algebraic K-theory and Hochschild homology). In that setting, the Morita 2-category is enlarged to **dg-categories, which are motives for noncommutative cohomology theories** [28]. The analogue of the duality in Theorem 2.5 is the classical Eilenberg-Watts Theorem [14]: for any algebra \( A \), \( \text{Mod}_A \) and \( A \text{Mod} \) are dual in \( \text{dgCat} \). I hope to explore these connections in greater detail.

We now return to Conjecture 2.3. When there is no group action (that is, \( G = 0 \)), the theorems of Elmendorf and Guillou-May, as well as Conjecture 2.3, assert that spaces (respectively spectra and commutative ring spectra) can be described as product-preserving functors from a small combinatorial category \( \mathcal{L} \) (\( \text{FinSet}^{\text{op}}, \text{Burn}, \text{Poly} \)) to spaces. Such an \( \mathcal{L} \) is a **Lawvere theory**, and a product-preserving functor is an \( \mathcal{L} \)-**algebra**.

Lawvere theories have extensively been studied by category theorists [21] in set-theoretic, rather than homotopy-theoretic, contexts. **Homotopical** Lawvere theories have only begun to be studied in earnest following developments in higher category theory in the past few years [5] [12] [17].

Theorem 2.4 asserts that the Lawvere theories \( \text{FinSet}^{\text{op}}, \text{Burn}, \text{Poly} \) can be constructed combinatorially by applying a **Lawvere machine** to finite sets: \( \mathcal{L} = \text{Span}_I(\text{FinSet}) \).

<table>
<thead>
<tr>
<th>Algebra ( \mathcal{A} )</th>
<th>Lawvere theory ( T )</th>
<th>Lawvere machine ( \text{Span}_I(__) )</th>
<th>Twisted category ( I )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Space</td>
<td>( \text{FinSet}^{\text{op}} )</td>
<td>opposite category construction</td>
<td>( \hat{R} )</td>
</tr>
<tr>
<td>Spectrum</td>
<td>Burn</td>
<td>span construction</td>
<td>( T \to \hat{R} )</td>
</tr>
<tr>
<td>Ring spectrum</td>
<td>Poly</td>
<td>bispan construction</td>
<td>( T \to N \to \hat{R} )</td>
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I expect Theorems 2.1 and 2.2 and Conjecture 2.3 each to be instances of the following principle:

**Principle 2.8.** If \( \mathcal{M} \) is a Lawvere machine for the homotopical structure \( \mathcal{A} \), then a \( G \)-equivariant object of \( \mathcal{A} \) is a product-preserving functor \( \mathcal{M}(\text{FinSet}_G) \to \text{Top} \).

That is, **equivariance arises from reparametrizing Lawvere theories over \( G \)-sets.** I believe this is the first explanation of the connection between Theorems 2.1 and 2.2 and Conjecture 2.3.

In order to formulate and prove the principle, I hope to address: What **precisely** is a Lawvere machine? I believe that it can be described via a symmetric monoidal adjunction

\[
\text{CocartMon} \cong \text{LCCD},
\]
where the left adjoint is a twisted functor construction, and the right adjoint is a span construction. If so, this would provide a clean universal property for $\text{Poly}_G$ – a major step towards Conjecture 2.3 and more general questions about equivariant structures.

3 Algebraic K-theory and higher category theory

3.1 My work

Many categories $\mathcal{R}$ that arise in nature come with a natural semiring structure. For example, when $R$ is a commutative ring, $\text{Mod}_R$ has two symmetric monoidal operations ($\otimes$ distributing over $\oplus$).

It is too much to ask that $\oplus$ have additive inverses (that $\mathcal{R}$ be a ring). However, we can group-complete $\mathcal{R}$ to produce a ring. If we are very careful in doing so, we produce a ring spectrum $K(\mathcal{R})$. This is the algebraic K-theory of $\mathcal{R}$ – by analogy with topological K-theory which is constructed via such a group-completion operation on vector bundles [25].

That algebraic K-theory inherits ring structures has been known for a long time, but there have historically been two technical questions to overcome:

1. What is a ‘commutative semiring’ category?
2. What is the ‘group completion’ of a symmetric monoidal category?

The most successful solution to (1) has been via bipermutative categories [24]. Unfortunately, these are highly technical, as working with them requires rigidifying the semiring categories we are interested in, a very nontrivial process.

A second option, due to Gepner, Groth, and Nikolaus [17], is made possible by recent work on higher algebra [22]. Commutative rings can be defined in an element-free way as monoids in the category of abelian groups. Categorifying this definition does not work for ordinary categories, but crucially does work as a definition for semiring $\infty$-categories (‘categories up to homotopy’). This is a general principle which plays a significant role in my research: $\infty$-categories have better algebraic properties than ordinary categories. Fortunately, K-theory is typically defined on homotopical structures or derived categories of modules, both of which are naturally $\infty$-categories, rather than regular categories.

Since this construction of semiring $\infty$-categories is formally the same as a classical construction of rings, tools from commutative algebra become available in the categorified setting. My work takes advantage of these tools to provide a new answer to (2) above.

**Theorem 3.1 (B. [5]).** There is a semiring $\infty$-category for which $S$-modules are spectra $\text{Mod}_S \cong \text{Sp}$. Given a symmetric monoidal $\infty$-groupoid $\mathcal{C}$, there is an equivalence of spectra $K(\mathcal{C}) \cong S \otimes \mathcal{C}$.

The language of modules over semiring categories is also well-suited to the study of algebraic theories (Lawvere theories and operads). We expect Lawvere theories to be useful in equivariant homotopy theory (section 2.3). Operads, on the other hand, are omnipresent in homotopy theory, especially in the program since the 1990’s to better understand structured ring spectra [16].

**Theorem 3.2 (B. [5]).** Lawvere theories are cyclic modules over $\text{FinSet}^{op}$.

**Conjecture 3.3 (B. [5]).** Operads are $\text{FinSet}^{op}$-modules with trivialization over $\text{Burn}$.

By setting the study of Lawvere theories into a commutative algebraic framework, Theorem 3.2 has allowed me to simplify key results on Lawvere theories, including some which are known in the ordinary Lawvere theory literature but have never before been extended to higher Lawvere theories. This is the subject of an upcoming paper [6]. It also allows for generalization to equivariant algebraic theories.
Definition 3.4 (B. [5]). Equivariant Lawvere theories are cyclic modules over $\text{FinSet}_G^{op}$. Equivariant operads are $\text{FinSet}_G^{op}$-modules with trivialization over $\text{Burn}_G$.

With this definition (the first of its kind), $\text{Burn}_G$ and $\text{Poly}_G$ are indeed equivariant Lawvere theories.

Many homotopy theorists, including Hill and Hopkins [19] and Barwick et al. [4] have struggled with how to understand the category theory at work in equivariant homotopy theory. Building off of the ideas of Hill and Hopkins, my work (section 2.2) suggests that the key to equivariant category theory is to understand equivariant Lawvere theories. And my work on higher category theory (this section) suggests that these can be approached algebraically, as finite-$G$-set-modules. There are promising signs that this approach can be unified with Barwick’s powerful combinatorial approach to equivariant higher category theory, and hopefully also with my conjectural ideas on Lawvere machines (section 2.3), in order to develop the foundations of equivariant homotopy theory and solve conjectures like 2.3.

3.2 Future work

Theorem 3.1 is a refinement of the group-completion of Baas, Dundas, Richter, and Rognes [2], which they use along with a theory of 2-vector bundles to study iterated K-theory, offering the most compelling evidence to date in favor of the chromatic redshift conjecture. Conjecture 3.3 would naturally identify operads with 2-vector bundles over $\text{FinSet}^{op}$, setting them within a theory of categorified algebraic geometry.

I hope to investigate the geometric properties of operadic Lawvere theories. If they are projective or flat (as some evidence suggests), we could explicitly compute algebraic theories for operadic bialgebras, which are tensor products $\text{Env}(O) \otimes \text{Env}(O')^{op}$ of symmetric monoidal categories – for the same reasons that tensor products of projective modules are computeable. Flatness may also help prove to Conjecture 3.3.

The geometry of a ring begins with its maximal ideals. The universal categorical ring is $\text{FinSet}^{iso}$, and I am aware of three objects which deserve to be called ‘maximal ideals’. These three objects are related to the formation of commutative monoids, cocommutative comonoids, and algebraic K-theory; the corresponding ‘local rings’ are $\text{FinSet}$, $\text{FinSet}^{op}$, and $S$. If these are indeed the only ‘maximal ideals’, a powerful descent story would allow us to reconstruct any symmetric monoidal category from two categories (without extra structure) and the algebraic K-theory spectrum. This work is in an early stage, but would explain many of the phenomena and conjectures in this section and make available new computational tools.

References Cited


