A new covering lemma and its application in 3D incompressible Navier-Stokes equations

Ph.D. Candidacy Talk

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Table of Contents

Review on Navier-Stokes Equations

3D Incompressible Navier-Stokes Equations

- Velocity $u(t, x) : [0, T) \times \mathbb{R}^3 \to \mathbb{R}^3$.
- Pressure $P(t, x) : [0, T) \times \mathbb{R}^3 \to \mathbb{R}$.

$$\begin{aligned} &\partial_t u + u \cdot \nabla u + \nabla P = \Delta u, \\ &\operatorname{div} u = 0, \\ &u\big|_{t=0} = u_0. \end{aligned}$$

- Weak solution: $u \in \mathscr{D}'$, s.t. $\forall \varphi \in C^\infty([0, T) \times \mathbb{R}^3)$, $\operatorname{div} \varphi = 0$, $\operatorname{supp} \varphi \subset\subset \mathbb{R}^3 \times [0, T)$,

$$\int_0^T \int_{\mathbb{R}^3} -\partial_t \varphi \cdot u - (u \cdot \nabla \varphi) \cdot u + \nabla \varphi \cdot \nabla u \, \mathrm{d}x \, \mathrm{d}t = \int_{\mathbb{R}^3} u_0 \cdot \varphi\big|_{t=0} \, \mathrm{d}x.$$

3D Incompressible Navier-Stokes Equations

- Leray-Hopf solution: a weak solution

$$u \in L^2(0,T;H^1(\mathbb{R}^3)) \cap C([0,T];L^2_w(\mathbb{R}^3)),$$

with energy inequality $\forall \tau \in (0,T)$,

$$\frac{1}{2}\int_{\mathbb{R}^3} |u(\tau,x)|^2 \,\mathrm{d}x + \int_0^\tau \int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t \le \frac{1}{2}\int_{\mathbb{R}^3} |u_0|^2 \,\mathrm{d}x.$$

- Suitable weak solution: a Leray-Hopf solution with generalized energy inequality in the sense of distribution, $\forall t \in (0,T)$ a.e.,

$$\partial_t \frac{|u|^2}{2} + \operatorname{div}\left[u\left(\frac{|u|^2}{2} + P\right)\right] + |\nabla u|^2 - \Delta \frac{|u|^2}{2} \le 0.$$

Known Results for Navier-Stokes Equations

- Weak solution
  - non-uniqueness (Buckmaster & Vicol, 2019)
- Leray-Hopf solution
  - Global-in-time existance (Leray, 1934)
  - Smoothness criteria: $L_t^\infty L_x^3 \sim L_t^2 L_x^\infty$ implies smoothness and uniqueness (Ladyženskaja, Prodi & Serrin, 1959-1967)
  - Limit case $L_t^\infty L_x^3$ (Iskauriaza, Serëgin & Shverak, 2003)
  - Partial regularity: $\mathscr{H}^{\frac{5}{3}}(\mathrm{Sing}(u)) < \infty$ (Scheffer, 1976)
- Suitable weak solution
  - Global-in-time existance (Caffarelli, Kohn & Nirenberg, 1982)
  - Partial regularity: $\mathscr{H}^1(\mathrm{Sing}(u)) = 0$ (C-K-N, 1982)
  - Second derivative estimate: $\nabla^2 u \in L_{t,x}^{\frac{4}{3}-\varepsilon}$ (Constantin, 1990)
    $\nabla^2 u \in L_{t,x}^{\frac{4}{3},\infty}$ (Lions, 1996)
  - Higher derivative estimate: $\nabla^\alpha u \in L_{\mathrm{loc}}^{\frac{4}{1+\alpha},\infty}$ (Choi & Vasseur, 2014)

Blow-up Technique along Trajectories

Scaling and Dimension Analysis

- Scaling: $(u_\varepsilon, P_\varepsilon)$ is also a solution to

$$\partial_t u_\varepsilon + u_\varepsilon \cdot \nabla u_\varepsilon + \nabla P_\varepsilon = \Delta u_\varepsilon$$

where

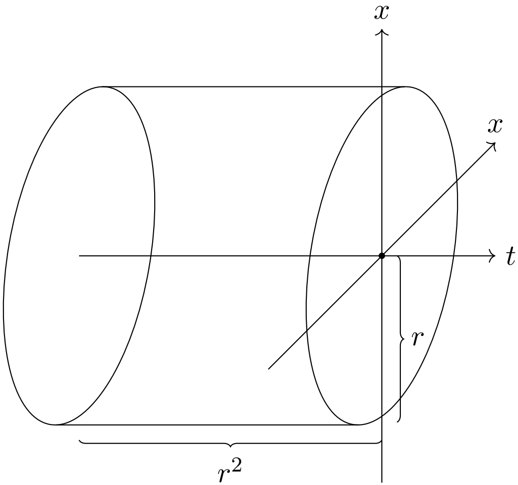
$$u_\varepsilon(t,x) = \varepsilon u(\varepsilon^2 t, \varepsilon x), \qquad P_\varepsilon(t,x) = \varepsilon^2 P(\varepsilon^2 t, \varepsilon x).$$

- Dimension analysis:

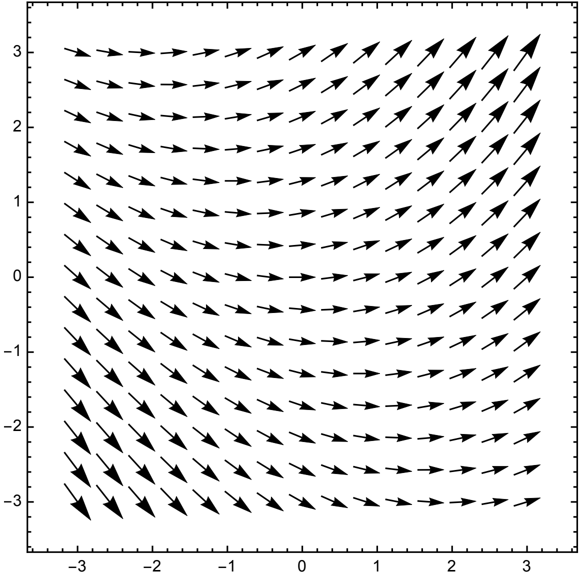
$t : 2$	$\mathrm{d}t : 2$	$\partial_t : -2$
$x : 1$	$\mathrm{d}x : 3$	$\nabla : -1$
$u : -1$	$P : -2$	$\frac{D}{Dt} : -2$
$\int \lvert u\rvert^{\frac{10}{3}} \,\mathrm{d}x\,\mathrm{d}t : \frac{5}{3}$	$\int \lvert\nabla u\rvert^2 \,\mathrm{d}x\,\mathrm{d}t : 1$	$\int \lvert\Delta u\rvert^{\frac{4}{3}} \,\mathrm{d}x\,\mathrm{d}t : 1$

Parabolic Cylinders

$$Q_r = \left\{(t,x) : t \in (-r^2, 0), x \in B_r(0) \subset \mathbb{R}^3\right\}.$$



Parabolic Cylinders along Trajectories



Parabolic Cylinders along Mollified Flow

- Mollified flow: fix a spatial mollifier $\varphi \in C_c^\infty(B_1)$, $\int \varphi = 1$, $\varphi_\varepsilon(x) = \varepsilon^{-3}\varphi\big(\varepsilon^{-1}x\big)$, $\tilde{u}_\varepsilon = u *_x \varphi_\varepsilon$, and let $X_\varepsilon(t_0, x_0; \cdot)$ solve

$$\frac{\mathrm{d}}{\mathrm{d}t} X_\varepsilon(t_0, x_0; t) = \tilde{u}_\varepsilon(t, X_\varepsilon(t_0, x_0; t)),$$
$$X_\varepsilon(t_0, x_0; t_0) = x_0.$$

- Parabolic cylinders along X_ε: given (t_0, x_0), define
  - Starting and terminal time: $S = t_0 - \varepsilon^2$, $T = t_0 + \varepsilon^2$.
  - Central streamline: $X(t) = X_\varepsilon(t_0, x_0; t)$.
  - ε-neighborhood of Central streamline: $B(t) = B_\varepsilon(X(t))$.
  - Curved parabolic cylinder:

$$Q_\varepsilon(t_0, x_0) = \big\{(t, x) \in \mathbb{R} \times \mathbb{R}^3 : S < t < T, x \in B(t)\big\}.$$

Covering Lemma

Admissible Cylinder

- Fix $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^3)$.
- Fix a small universal $\eta > 0$, $Q_\varepsilon(t_0, x_0)$ is an η-admissible cylinder if

$$\fint_{Q_\varepsilon(t_0,x_0)} \mathcal{M}_x(|\nabla u|)\,\mathrm{d}x\,\mathrm{d}t \le \eta\varepsilon^{-2}.$$

  Here $\mathcal{M}_x$ is the spatial maximal function.
- Denote $\mathcal{Q}_\eta$ to be the set of all η-admissible cylinders.
- Admissibility ensures that nearby flows are close.

Statement of Covering Lemma

Theorem (Covering Lemma, Y., 2019)

Let η be small enough. There exists a universal constant $C > 0$ such that the following is true.

- *Let Λ be an index set. Let $\{Q^\alpha\}_{\alpha\in\Lambda}$ be a family of η-admissible cylinders, where*

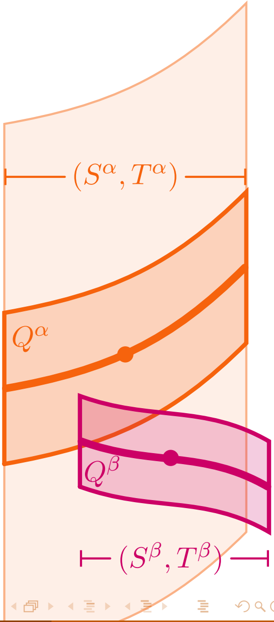
$$Q^\alpha = Q_{\varepsilon^\alpha}(t^\alpha, x^\alpha).$$

- *Assume $\mu(\bigcup_\alpha Q^\alpha) < \infty$. Then we can find a pairwise disjoint subcollection $\{Q^{\alpha_i}\}_{i=1}^I$ such that*

$$\sum_{i=1}^I \mu(Q^{\alpha_i}) \geq \frac{1}{C}\mu\left(\bigcup_\alpha Q^\alpha\right).$$

Claim: Closeness of Intersecting Cylinders

Let η be small enough. Assume $Q^\alpha \cap Q^\beta \neq \varnothing$, $\varepsilon^\beta \le 2\varepsilon^\alpha$. Then for all $t \in (S^\alpha, T^\alpha) \cap (S^\beta, T^\beta)$, $B^\beta(t) \subset 9B^\alpha(t)$.



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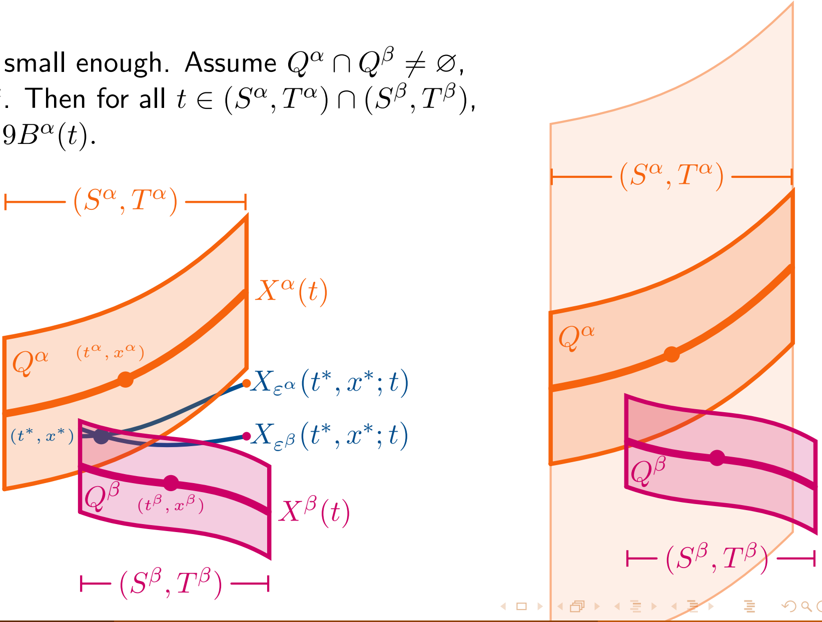


Proof of Covering Lemma

- **Key step**: ensure that the measure of

$$Q_*^\alpha = \bigcup_{\substack{Q^\beta \cap Q^\alpha \neq \varnothing \\ \varepsilon^\beta < 2\varepsilon^\alpha}} Q^\beta$$

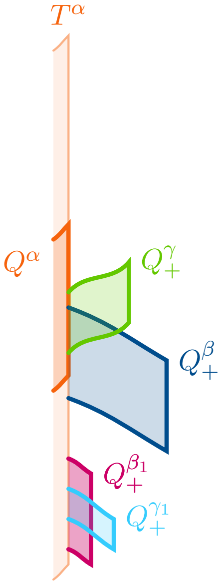
  is comparable to the measure of Q^α.
- Split every Q^β into three parts

$$\begin{aligned} Q_-^\beta &= Q^\beta \cap \{t \le S^\alpha\}, \\ Q_\circ^\beta &= Q^\beta \cap \{S^\alpha < t < T^\alpha\}, \\ Q_+^\beta &= Q^\beta \cap \{t \ge T^\alpha\}. \end{aligned}$$

- Can control $\bigcup_\beta Q_\circ^\beta$ by $9Q^\alpha$, but cannnot control Q_+^β or Q_-^β.

Proof of Covering Lemma

- To control Q_+^β, we make sure they are close to each other.
- Want to pick a subcollection from Q^β that occupies enough space with summable measure.
- Which ones to pick? Q_+^β controls Q_+^γ if Q_+^β
  - is relatively larger.
  - lasts longer.
- Dilemma: $Q_+^{\beta_1}$ and $Q_+^{\gamma_1}$.

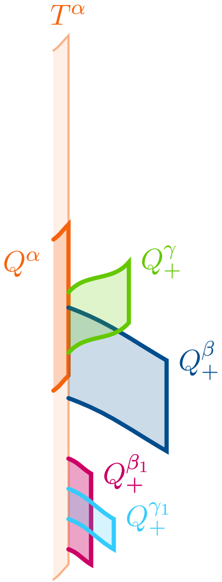


Proof of Covering Lemma

- Solution: Group by size, then sort by length.
- Suppose

$$
\begin{array}{ll}
2\varepsilon^\alpha > \varepsilon^{\beta_1}, \dots, \varepsilon^{\beta_n} \geq \varepsilon^\alpha & T^{\beta_1} \geq \cdots \geq T^{\beta_n} \\
\varepsilon^\alpha > \varepsilon^{\gamma_1}, \dots, \varepsilon^{\gamma_m} \geq \frac{1}{2}\varepsilon^\alpha & T^{\gamma_1} \geq \cdots \geq T^{\gamma_m} \\
\frac{1}{2}\varepsilon^\alpha > \varepsilon^{\delta_1}, \dots, \varepsilon^{\delta_l} \geq \frac{1}{4}\varepsilon^\alpha & T^{\delta_1} \geq \cdots \geq T^{\delta_l} \\
\dots & \dots
\end{array}
$$

- Inside each group, cylinders all have comparable size.
- Select a disjoint subcollection in each group by a Vitali argument according to length.



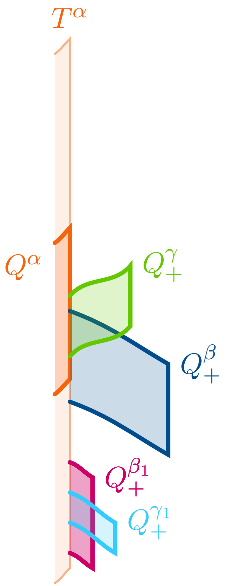
Proof of Covering Lemma

- Let $\left\{ Q_+^{\beta_{j_k}} \right\}$ be a pairwise disjoint selection.
- Their dilation covers all Q_+^{β} in this group.

$$\mu\left(\bigcup_{j=1}^{n} Q_+^{\beta_j}\right) \le \sum_k \mu\left(9 Q_+^{\beta_{j_k}}\right)$$

- Section volume of $\left\{ Q_+^{\beta_{j_k}} \right\}$ is less than $9B^\alpha$.
- Length is less than $2 \cdot (2\varepsilon^\alpha)^2$.

$$\mu\left(\bigcup_{j=1}^{n} Q_+^{\beta_j}\right) \le 9^3 \sum_k \mu\left(Q_+^{\beta_{j_k}}\right) \le 9^3 |9B^\alpha| \cdot 2 \cdot 4(\varepsilon^\alpha)^2 \le 4 \cdot 9^6 \mu(Q^\alpha).$$



- Because the maximal length is shorter for the next group,

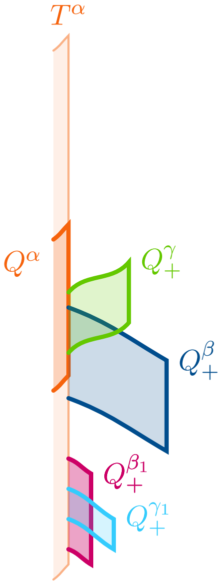
$$\mu\left(\bigcup_{j=1}^{n} Q_+^{\beta_j}\right) \leq 4 \cdot 9^6 \mu(Q^\alpha),$$

$$\mu\left(\bigcup_{j=1}^{m} Q_+^{\gamma_j}\right) \leq 1 \cdot 9^6 \mu(Q^\alpha),$$

$$\mu\left(\bigcup_{j=1}^{l} Q_+^{\delta_j}\right) \leq \frac{1}{4} \cdot 9^6 \mu(Q^\alpha),$$

$$\dots$$

- $\Rightarrow \mu\left(\bigcup_\beta Q_+^\beta\right) \leq C\mu(Q^\alpha).$



Proof of Covering Lemma

- Proof of **Key step**.

$$\mu\Big(\bigcup_\beta Q_+^\beta\Big) \le C\mu(Q^\alpha),$$
$$\mu\Big(\bigcup_\beta Q_\circ^\beta\Big) \le C\mu(Q^\alpha),$$
$$\mu\Big(\bigcup_\beta Q_-^\beta\Big) \le C\mu(Q^\alpha),$$
$$\Rightarrow \mu(Q_*^\alpha) = \mu\Bigg(\bigcup_{\substack{Q^\beta\cap Q^\alpha\neq\varnothing \\ \varepsilon^\beta<2\varepsilon^\alpha}} Q^\beta\Bigg) \le C\mu(Q^\alpha).$$

  So Q_*^α has measure comparable to the measure of Q^α.
- By Vitali, this finishes the proof of the covering lemma.

Consequences of Covering Lemma

- Let $f \in L^1_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^3)$. Define

$$\mathcal{M}_{\mathcal{Q}}(f)(t,x) = \sup_{\varepsilon : Q_\varepsilon(t,x) \in \mathcal{Q}_\eta} \fint_{Q_\varepsilon(t,x)} |f(s,y)| \,\mathrm{d}s \,\mathrm{d}y$$

  to be largest possible average among all admissible cylinders centered at (t,x).
- If u is also divergence free, then
  - $\mathcal{M}_{\mathcal{Q}}$ is of weak type $(1,1)$.
  - $\mathcal{M}_{\mathcal{Q}}$ is of strong type (∞,∞).
  - $\mathcal{M}_{\mathcal{Q}}$ is of strong type (p,p) for all $p > 1$.
  - Almost every $(t,x) \in \mathbb{R} \times \mathbb{R}^3$ are $\mathcal{Q}$-Lebesgue points, i.e.

$$\lim_{\varepsilon \to 0} \fint_{Q_\varepsilon(t,x)} |f(s,y) - f(t,x)| \,\mathrm{d}s \,\mathrm{d}y = 0.$$

Application to Navier-Stokes Equations

Local Theorem

Lemma (Local Theorem, Vasseur, Y., 2019)

If u is a suitable weak solution,

$$\int \varphi(x) u(t,x)\,\mathrm{d}x = 0, \qquad \textit{a.e. } t \in (-2,0),$$
$$\int_{(-2,0)\times B_2} |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t \le \eta,$$

then $|\Delta u| \le 1$ in $(-1,0)\times B_1$.

Putting it back into global coordinate, it means if

$$⨍_{Q_\varepsilon(t,x)} |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t \le \eta\varepsilon^{-4},$$

then $|\Delta u(t,x)| \le \varepsilon^{-3}$.

Second Derivative Estimate

Assume u is a suitable weak solution in $(0,T)$.

- For each $(t,x) \in (0,T) \times \mathbb{R}^3$, select $\varepsilon(t,x)$ such that either

$$\fint_{Q_{\varepsilon(t,x)}(t,x)} |\mathcal{M}_x(|\nabla u|)|^2 \,\mathrm{d}x\,\mathrm{d}t = \eta[\varepsilon(t,x)]^{-4}.$$

  or $\varepsilon(t,x) = \sqrt{t}$, and above $=$ is replaced by $<$.
- In either case, $|\Delta u| \le \varepsilon^{-3}$ by local theorem.
- ε^{-4} is either bounded by $\frac{1}{\eta}\mathcal{M}_{\mathcal{Q}}[\mathcal{M}_x(|\nabla u|)^2]$ or t^{-2}.
- $|\Delta u|\mathbf{1}_{\left\{|\Delta u|>t^{-\frac{3}{2}}\right\}} \in L^{\frac{4}{3},\infty}$.

Hypothesis (Improvement of Local Theorem)

If we can weaken the requirement to for some $p < 2$,

$$\int \varphi(x)u(t,x)\,\mathrm{d}x = 0, \qquad \text{a.e. } t \in (-2,0),$$

$$\int_{(-2,0)\times B_2} |\nabla u|^p \,\mathrm{d}x\,\mathrm{d}t \le \eta,$$

then $|\Delta u| \le 1$ in $(-1,0)\times B_1$.

Putting it back into global coordinate, it means if

$$⨍_{Q_\varepsilon(t,x)} |\nabla u|^p \,\mathrm{d}x\,\mathrm{d}t \le \eta\varepsilon^{-2p},$$

then $|\Delta u(t,x)| \le \varepsilon^{-3}$.

Improvement of Second Derivative Estimate

Assume u is a suitable weak solution in $(0,T)$.

- For each $(t,x) \in (0,T) \times \mathbb{R}^3$, select $\varepsilon(t,x)$ such that either

$$\fint_{Q_{\varepsilon(t,x)}(t,x)} |\mathcal{M}_x(|\nabla u|)|^p \,\mathrm{d}x\,\mathrm{d}t = \eta[\varepsilon(t,x)]^{-2p}.$$

  or $\varepsilon(t,x) = \sqrt{t}$, and above $=$ is replaced by $<$.
- In either case, $|\Delta u| \le \varepsilon^{-3}$ by local theorem.
- ε^{-2p} is either bounded by $\frac{1}{\eta}\mathcal{M}_{\mathcal{Q}}[\mathcal{M}_x(|\nabla u|)^p]$ or t^{-p}.
- $|\Delta u|\mathbf{1}_{\left\{|\Delta u|>t^{-\frac{3}{2}}\right\}} \in L^{\frac{4}{3}}$.

Difficulty of Improvement

- The proof of the local theorem relies on Grönwall and De Giorgi.

$$\partial_t u + u \cdot \nabla u + \nabla P = \Delta u.$$

- Estimating quadratic term $u \cdot \nabla u$ becomes substantially more difficult because it is less than L^1 in time.
- Cannot work with pressure: $\nabla^2 P \in L^1_t \mathcal{H}^1_x$ by Compensated compactness (Coifman, Lions, Meyer & Semmes, 1993).

Thank you for your attention!