26. Find the limit or show that it does not exist for:

\[ \lim_{x \to -\infty} x + \sqrt{x^2 + 2x} \]

Solution:

\[
\lim_{x \to -\infty} x + \sqrt{x^2 + 2x} = \lim_{x \to -\infty} \frac{x + \sqrt{x^2 + 2x}}{1} = \lim_{x \to -\infty} \frac{x^2 - (x^2 + 2x)}{x - \sqrt{x^2 + 2x}} = \lim_{x \to -\infty} \frac{0}{0} = \text{DNE}
\]
\[
\lim_{x \to -\infty} x + \sqrt{x^2 + 2x} = \lim_{x \to -\infty} \frac{-2x}{x - \sqrt{x^2 + 2x}}
\]

In taking the limit one must be very careful about signs. For example, note that \(\sqrt{x^2} = |x| = -x\) as \(x < 0\). The denominator inside the limit may be written as follows:

\[
x - \sqrt{x^2 + 2x} = x - |x| \sqrt{1 + \frac{2}{x}} = x + x \sqrt{1 + \frac{2}{x}}
\]

Hence,

\[
\lim_{x \to -\infty} x + \sqrt{x^2 + 2x} = \lim_{x \to -\infty} \frac{-2x}{x + x \sqrt{1 + \frac{2}{x}}}
\]

\[
= \lim_{x \to -\infty} \frac{-2}{1 + \sqrt{1 + \frac{2}{x}}}
\]

\[
= -1.
\]

where the last step follows because \(\lim_{x \to -\infty} \frac{2}{x} = 0\).

Alternative method: To avoid confusion about signs, substitute \(y = -x\). Then \(x \to -\infty\) corresponds with \(y \to \infty\).

\[
\lim_{x \to -\infty} x + \sqrt{x^2 + 2x} = \lim_{y \to \infty} -y + \sqrt{y^2 - 2y}
\]

\[
= \lim_{y \to \infty} \frac{-y + \sqrt{y^2 - 2y}}{1} \cdot \frac{-y - \sqrt{y^2 - 2y}}{-y - \sqrt{y^2 - 2y}}
\]

\[
= \lim_{y \to \infty} \frac{2y}{-y - \sqrt{y^2 - 2y}}
\]

\[
= \lim_{y \to \infty} \frac{2}{y(-1 - \sqrt{1 - \frac{2}{y}})}
\]

\[
= \lim_{y \to \infty} \frac{2}{-1 - \sqrt{1 - \frac{2}{y}}}
\]

\[
= -1.
\]

62. (a) A tank contains 5000 L of pure water. Brine that contains 30 g of salt per liter of water is pumped into the tank at a rate of 25 L/min. Show that the concentration of salt after \(t\) minutes (in grams per liter) is:

\[
C(t) = \frac{30t}{200 + t}
\]

Solution: The concentration is given by the total mass of salt in the tank divided by the volume of water in the tank. The total mass of salt as a function of time will be \(M(t) = \rho ft\) where \(\rho\) is the density of salt (in g/L) of the brine water, \(f\) is the flow rate of brine water into the tank (in L/min), and \(t\) is the time (in min). The volume in the tank will be given by \(V(t) = 5000 + ft\) where \(f\) and \(t\) are defined above. The concentration \(C(t)\) is then given by:

\[
C(t) = \frac{M(t)}{V(t)} = \frac{\rho ft}{5000 + ft} = \frac{30 \frac{g}{L} \cdot 25 \frac{L}{min} \cdot t \min}{5000 \ L + 25 \frac{L}{min} \cdot t \ min \ + \ 25 \frac{L}{min}} = \frac{25}{25} \frac{30t}{200 + t} \frac{g}{L} = \frac{30t}{200 + t} \frac{g}{L}
\]
(b) What is the concentration as $t \to \infty$?

**Solution:**

\[
\lim_{t \to \infty} C(t) = \lim_{t \to \infty} \frac{30t}{200 + t} = \lim_{t \to \infty} \frac{30t}{t\left(\frac{200}{t} + 1\right)} = \lim_{t \to \infty} \frac{30}{\frac{200}{t} + 1} = \lim_{t \to \infty} \frac{30}{0 + 1} = 30
\]
Section 2.8

52. (a) If \( g(x) = x^{2/3} \) show that \( g'(x) \) does not exist.

**Solution:** The derivative at 0 of \( g(x) \) is:

\[
g'(0) = \lim_{h \to 0} \frac{g(0 + h) - g(0)}{h}
\]

This only exists if the left and right side limits exist and are equal. So check:

\[
\lim_{h \to 0^-} \frac{g(0 + h) - g(0)}{h} = \lim_{h \to 0^+} \frac{g(0 + h) - g(0)}{h}
\]

Start with the left limit:

\[
\lim_{h \to 0^-} \frac{g(0 + h) - g(0)}{h} = \lim_{h \to 0^-} \frac{(0 + h)^{2/3} - 0^{2/3}}{h} = \lim_{h \to 0^-} \frac{h^{2/3}}{h} = \lim_{h \to 0^-} 1\frac{1}{h^{1/3}} = -\infty
\]

Since the left limit does not exist, the limit does not exist, so \( g'(0) \) does not exist. It is also informative to check the right limit (though unnecessary for part a):

\[
\lim_{h \to 0^+} \frac{g(0 + h) - g(0)}{h} = \lim_{h \to 0^+} \frac{(0 + h)^{2/3} - 0^{2/3}}{h} = \lim_{h \to 0^+} \frac{h^{2/3}}{h} = \lim_{h \to 0^+} 1\frac{1}{h^{1/3}} = \infty
\]

\( \square \)

(b) If \( a \neq 0 \), find \( g'(a) \).

**Solution:** The derivative at \( a \) is given by:

\[
g'(a) = \lim_{h \to 0} \frac{g(a + h) - g(a)}{h} = \lim_{h \to 0} \frac{(a + h)^{2/3} - a^{2/3}}{h}
\]

We use difference of cubes trick to deal with the fractional powers of 1/3. Difference of cubes formula:

\[
\]

Apply the formula with \( A = (a + h)^{2/3}, B = a^{2/3} \).

\[
g'(a) = \lim_{h \to 0} \frac{(a + h)^{2/3} - a^{2/3}}{h} = \lim_{h \to 0} \frac{(a + h)^{4/3} + (a + h)^{2/3}a^{2/3} + a^{4/3}}{(a + h)^{2/3} + (a + h)^{2/3}a^{2/3} + a^{4/3}}
\]

\[
= \lim_{h \to 0} \frac{(a + h)^2 - a^2}{h ((a + h)^{4/3} + (a + h)^{2/3}a^{2/3} + a^{4/3})}
\]

\[
= \frac{2a + h}{h ((a + h)^{4/3} + (a + h)^{2/3}a^{2/3} + a^{4/3})}
\]

\[
= \lim_{h \to 0} \frac{2a + h}{3a^{4/3}} = \frac{2}{3}a^{-1/3}
\]

\( \square \)

(c) Show that \( y = x^{2/3} \) has a vertical tangent line at (0,0).

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Solution: Recall from part (a) that $g'(0^+) = \infty$, while $g'(0^-) = -\infty$. The derivative is infinite, but the function is continuous at 0. So, the graph of $y = g(x)$ has a vertical tangent. Alternatively, note from the answer to (b) that the derivative $g'(a) \to \infty$ as $a \to 0^+$, and $g'(a) \to -\infty$ as $a \to 0^-$. This means the slope of the tangent line is becoming more and more vertical as $a \to 0$.

(d) Illustrate part c by graphing $y = x^{2/3}$

Solution: See the graph.

Problem A

(a) Solution: Here is a more intuitive interpretation of BADLIM #1: A function $f(x)$ satisfies BADLIM #1 if there are positive $x$ values that make $f(x)$ arbitrarily large. In order to see what is wrong with this definition consider the following example:

$$f(x) = \frac{1}{(x-2)^2}.$$ 

$f$ has a vertical asymptote at $x = 2$. So $f(x)$ can be made arbitrarily large by choosing $x$ values very close to $x = 2$. Hence $f$ satisfies BADLIM #1. However, by the true definition $\lim_{x \to \infty} f(x) \neq \infty$ (in fact the limit is zero). This is because taking arbitrarily large values of $x$ does not yield arbitrarily large values of $f(x)$. For example, whenever $x > 3$, we have that $f(x) < 1$.

This example exposes one problem with the definition BADLIM #1, namely that BADLIM #1 is satisfied by functions that go to infinity somewhere, but not necessarily when $x \to 0$.

Another possible example is

$$g(x) = x \cos \pi x.$$
Note that
\[ g(n) = \begin{cases} 
  n & \text{if } n \text{ is an even integer} \\
  -n & \text{if } n \text{ is an odd integer}
\end{cases} \]

So there are values of \( x \) making \( g \) arbitrarily large, namely the even integers. So \( g \) satisfies BADLIM #1. However there are arbitrarily large values of \( x \) that make \( g(x) \) negative, for example the odd integers. So, \( g \) does not satisfy the true definition. That is \( \lim_{x \to \infty} g(x) \neq \infty \).

These two examples (and there are many others) show that BADLIM #1 is too weak. It is satisfied by more functions than the true definition, including many functions (like \( f \) and \( g \)) intuitively do not approach infinity as \( x \to \infty \).

(b) Solution: BADLIM #2 has the opposite problem. It is too strong. There are many functions that satisfy the true definition, but do not satisfy BADLIM #2.

An intuitive interpretation of BADLIM #2 is: A function \( f(x) \) satisfies BADLIM #2 if there exists one linear function \( L(x) = Kx \) so that \( f(x) \) is always bigger than \( L(x) \) for big enough \( x \)-values.

An example of a function that does satisfy BADLIM #2 is \( f(x) = x^2 \). This is because \( x^2 > x \) for all \( x > 1 \) (i.e. take \( K = 1, N = 1 \) in the definition). We already know that \( \lim_{x \to \infty} x^2 = \infty \). So this example does not distinguish BADLIM #2 from the true definition (i.e. this is not an answer to this question).

Here is an example of a function that does not satisfy BADLIM #2: Let \( g(x) = \sqrt{x} \). In order to show that \( g(x) \) does not satisfy BADLIM #2, we need to show the following: For every choice of the constants \( K, N > 0 \), there is at least one point \( x > N \), so that \( g(x) \leq Kx \). To show this, let \( K > 0 \) be any constant. Note that
\[
\sqrt{x} \leq Kx \quad \text{if and only if} \quad x \leq K^2 x^2 \\
K^2 x^2 - x \geq 0 \quad \text{if and only if} \quad x(K^2 x - 1) \geq 0 \\
x \geq 1/K^2.
\]

Hence, given any \( K > 0 \) and any \( N > 0 \), we can choose a (very large) \( x > N \) that makes \( \sqrt{x} \leq Kx \) (just choose an \( x \) value bigger than both \( N \) and \( 1/K^2 \)). We showed \( g(x) = \sqrt{x} \) does not satisfy BADLIM #2. However, \( g(x) \) does satisfy the true definition, that is \( \lim_{x \to \infty} \sqrt{x} = \infty \). Here is a proof (using the true definition given above). Given any \( M > 0 \), we must show that there exists a constant \( N > 0 \), so that \( \sqrt{x} > M \) whenever \( x > N \). Let \( M > 0 \) be any positive constant. Then
\[
\sqrt{x} > M \\
x > M^2
\]
So, choose \( N = M^2 \). Then whenever \( x > N = M^2 \), we have that \( \sqrt{x} > M \). This proves that \( \lim_{x \to \infty} \sqrt{x} = \infty \).

\( g(x) = \sqrt{x} \) is an example of a function that demonstrates that BADLIM #2 is too strong: There are functions (\( \sqrt{x} \) being one of them) that satisfy the true definition, but do not satisfy BADLIM #2. Other such functions include \( \log x \) and \( x^{1/3} \) (and there are many others).

(c) Don’t forget the correct definition!

Definition: \( \lim_{x \to \infty} f(x) = \infty \) if for every \( M > 0 \) there exists an \( N > 0 \) so that \( f(x) > M \) whenever \( x > N \).