Exotic real projective Dehn surgery space

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Let $N =$ 3-manifold. A *real projective structure* ($\mathbb{RP}^3$-structure) on N is locally modeled on real projective geometry.

$$\begin{array}{lcl} \text{Real projective geometry} & \supset & \text{hyperbolic geometry} \\ \mathbb{RP}^3 = (\mathbb{R}^4 - 0)/\mathbb{R}^* & \supset & \mathbb{H}^3 = \mathbb{P}\{x_1^2 + x_2^2 + x_3^2 - x_4^2 < 0\} \\ \mathrm{G} = \mathrm{PGL}_4\mathbb{R} = \mathrm{GL}_4\mathbb{R}/\mathbb{R}^* & \supset & \mathbb{G} = \mathrm{PO}(3,1) \end{array}$$

Basic Question

Assume N is a closed 3-manifold which admits an $\mathbb{H}^3$-structure.
Does N admit other $\mathbb{RP}^3$ structures?

dim $= 2$:	$N = \Sigma_g, g \geq 2$:	$\exists$ continuous deformations Hitchin component $\cong \mathbb{R}^{16g-16}$
dim $= 3$:	Cooper-Long-Thistlethwaite Heusener-Porti	continuous deformations seem to be *rare*.

Let $M =$ complete, finite volume, orientable, $\mathbb{H}^3$-manifold, with one cusp.

Thurston's hyperbolic Dehn filling theorem

All but finitely many *Dehn fillings* $M_{p/q}$ of M admit a hyperbolic structure.

Main Theorem (Ballas-D-Lee-Marquis)

Assume:

- M is infinitesimally projectively rigid rel cusp (IPR).
- M does not have the constant unipotent slope property.

} satisfied e.g. by M = m003, m004, m007, m019

Then there exists $I \subset \mathbb{P}^1\mathbb{Q}$ so that for all but finitely many filling slopes $p/q \in I$, $M_{p/q}$ admits an exotic $\mathbb{RP}^3$ structure.

Further these structures are *convex*.

$\implies$ get discrete, faithful, Zariski dense representations $\pi_1 M_{p/q} \to \mathrm{SL}_4\mathbb{R}$

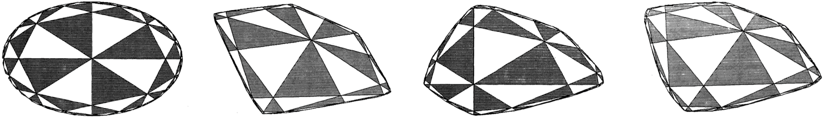
An $\mathbb{RP}^n$ structure on N is *convex* if:

$\widetilde{N} \cong \Omega \subset \mathbb{RP}^n$ open, properly convex:

- $\Omega =$ projection of sharp convex cone in $\mathbb{R}^{n+1}$

deck action: $\rho : \pi_1 N \to \mathrm{Aut}(\Omega) < G = \mathrm{PGL}(n+1, \mathbb{R})$ discrete faithful.

some 2-d convex sets (Goldman):



Theorem (Benoist, Koszul)

If N closed, the convex $\mathbb{RP}^n$ structures on N make up a union of connected components of $X(\pi_1 N, G)$.

Corollary of Main Theorem

There exist closed manifolds whose deformation space of convex $\mathbb{RP}^n$ structures has multiple components.

$M =$ complete, finite volume, orientable $\mathbb{H}^3$-manifold with one cusp.

$$X(\pi_1 M, \mathbb{G}) := \mathrm{Hom}(\pi_1 M, \mathbb{G}) /\!\!/ \mathbb{G}$$

$\mathbb{G}$-character variety
$\mathbb{G} = \mathrm{Isom}(\mathbb{H}^3)$

- basepoint: ρ_{hyp} unique discrete faithful.
- $\dim_{\mathbb{R}} X = 2$: near ρ_{hyp}, X parameterizes deformations of the hyperbolic structure on M (incomplete): *hyperbolic Dehn filling space*.

Let $\rho \in X(\pi_1 M, \mathbb{G})$ near ρ_{hyp}.

- Restrict to the peripheral subgroup $\pi_1 \partial M = \langle \alpha, \beta \rangle \cong \mathbb{Z} \times \mathbb{Z}$.
- If $\rho \neq \rho_{hyp}$, $\rho(\pi_1 \partial M) =$ *loxodromic*:
  - translations and rotations around a geodesic $\mathscr{A} \subset \mathbb{H}^3$.

Generalized Dehn filling parameters.

Solve for $(p, q) \in \mathbb{R}^2$:

$$\rho(\alpha)^p \rho(\beta)^q = \text{Rotation by } 2\pi$$

If $(p, q) =$ relatively prime integers, then

$$\rho(\pi_1 \partial M) \backslash \mathcal{N}_R(\mathscr{A}) \quad \cup \quad (\text{incomplete } \mathbb{H}^3 \text{ structure on } M) \quad = \quad M_{p/q}$$

$M =$ complete, finite volume, orientable $\mathbb{H}^3$-manifold with one cusp.
Deformations of M as an $\mathbb{RP}^3$ structure parametrized by:

$$X(\pi_1 M, \mathrm{G}) := \mathrm{Hom}(\pi_1 M, \mathrm{G}) /\!\!/ \mathrm{G}$$

G-character variety
$\mathrm{G} = \mathrm{PGL}_4\mathbb{R}$

Let $\rho \in X(\pi_1 M, G)$ near ρ_{hyp}. Restrict to $\pi_1 \partial M = \langle \alpha, \beta \rangle$. Solve for (p, q):

$$\rho(\alpha)^p \rho(\beta)^q = \text{Rotation by } 2\pi$$

Proposition

If (p, q) a relatively prime integer solution and ρ sufficiently close to ρ_{hyp}, then: $\mathbb{RP}^3$ structure on M fills in to a convex $\mathbb{RP}^3$ structure on $M_{p/q}$.

Issues

1. Which (p, q) are achieved?
2. For typical ρ, can not solve for (p, q)!

restriction map: $\iota^* : \mathrm{Hom}(\pi_1 M, \mathrm{G}) \to \mathrm{Hom}(\pi_1 \partial M, \mathrm{G})$

Question

For ρ near ρ_{hyp}, what can $\iota^*(\rho)$ look like?

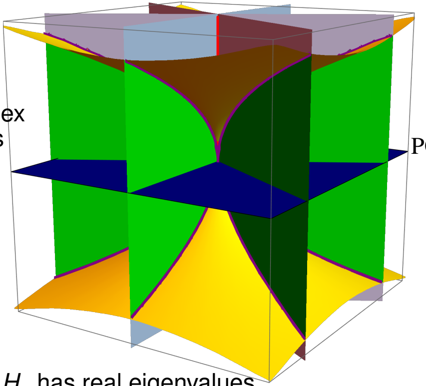
- $\mathrm{Hom}(\pi_1 \partial M, \mathrm{G}) = \mathrm{Hom}(\mathbb{Z} \times \mathbb{Z}, \mathrm{G})$ smooth near $\iota^* \rho_{hyp}$.
  dim $= 6$ up to conjugation.
- We get a good picture of $\mathrm{Hom}(\mathbb{Z} \times \mathbb{Z}, \mathrm{G})$.
  - WARNING: $\mathrm{Hom}(\mathbb{Z} \times \mathbb{Z}, \mathrm{G})/\mathrm{G}$ poorly behaved near $\iota^* \rho_{hyp}$.
  - instead, work in an explicit six-dimensional slice.

Assumption: (IPR)

The complete hyperbolic structure on M is *infinitesimally projectively rigid rel boundary*.

- $\mathrm{Hom}(\pi_1 M, \mathrm{G})$ smooth near ρ_{hyp}. dim $= 3$ up to conjugation.
- ι^* locally injective, up to conjugation.

Picture of $X(\pi_1 M, \mathrm{G})$



$H_\rho := \mathrm{hull}(\rho(\pi_1 \partial M)) < \mathrm{G}$ 2-d Lie subgroup

for Dehn fillings, need $H_\rho >$ 1-d rotation subgroup $\sim \begin{bmatrix} e^{i\theta} & & & \\ & e^{-i\theta} & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$

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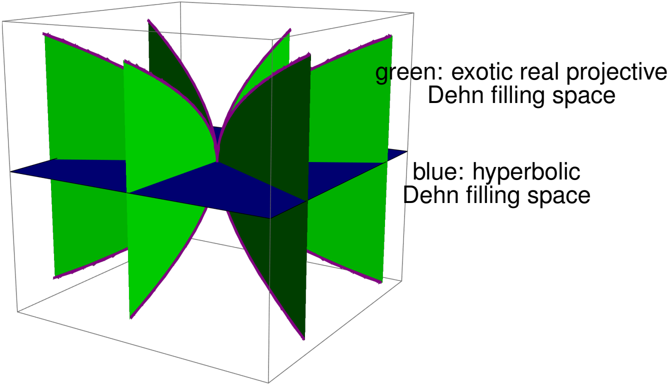

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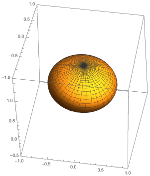
Let $M = \Gamma \backslash \mathbb{H}^3$ finite volume, one cusp.

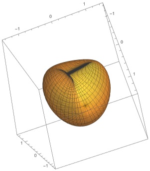
Let $\rho \in X(\pi_1 M, \mathrm{G})$ near ρ_{hyp}. $\qquad H_\rho := \mathrm{hull}(\rho(\pi_1 \partial M)) < \mathrm{G}$.

(A) H_ρ has real eigenvalues.

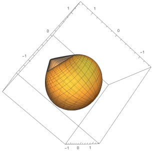
- generalized cusp group (Cooper-Long-Tillmann). 4 types (Ballas-Cooper-Leitner)



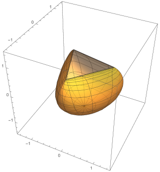
type 0 cusp ($\mathrm{PO}(3,1)$)



type 1 cusp



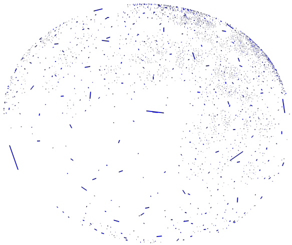
type 2 cusp



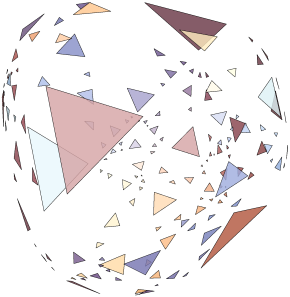
type 3 cusp

- ρ determines a convex projective structure on M with *generalized cusps*. (Cooper-Long-Tillmann)

Convex projective deformations of the figure 8 complement.



with type 1 cusp (Ballas).

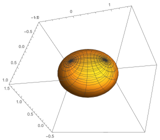


with type 3 cusp (Ballas-D-Lee).

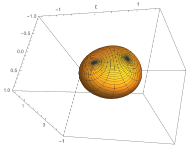
Let $\rho \in X(\pi_1 M, \mathrm{G})$ near ρ_{hyp}.
$\rho(\pi_1 \partial M)$ determines a 2-d Lie subgroup: $H_\rho = \mathrm{hull}(\rho(\pi_1 \partial M)) < \mathrm{G}$.

(B) complex eigenvalues

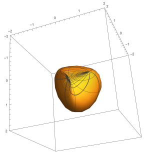
(B1) generalized loxodromic group: preserves (convex) neighborhood of line segment. rotates and translates. Can solve for (p, q)!



loxodromic, in $\mathrm{PO}(3,1)$

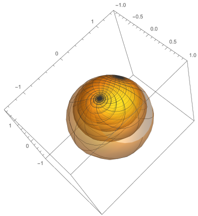


loxodromic, not in $\mathrm{PO}(3,1)$

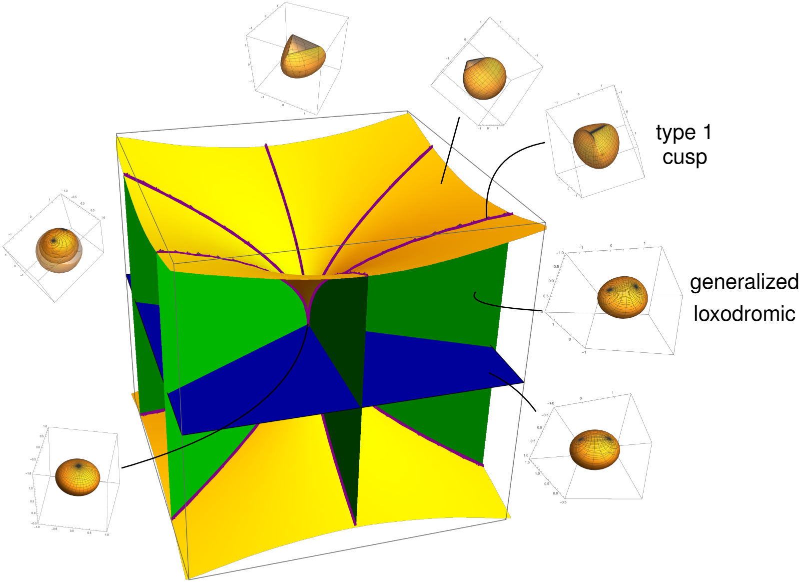


loxodromic, near type 1 cusp

(B2) no convex orbit.



A picture of $X(\pi_1 M, \mathrm{G})$ labeled by the orbits of H_ρ.



Type 1 cusps and nearby loxodromic groups

$$H_\rho = \begin{pmatrix} e^u & e^u v & \frac{1}{2}e^u(v^2+2u) & 0 \\ 0 & e^u & e^u v & 0 \\ 0 & 0 & e^u & \\ 0 & 0 & 0 & e^{-3u} \end{pmatrix}$$

- Solve

$$\rho(\alpha)^r \rho(\beta)^s = \text{ unipotent.}$$

$\mu(\rho) := \frac{r}{s} \in \mathbb{R} \cup \{\infty\}$: the *unipotent slope*.

Proposition

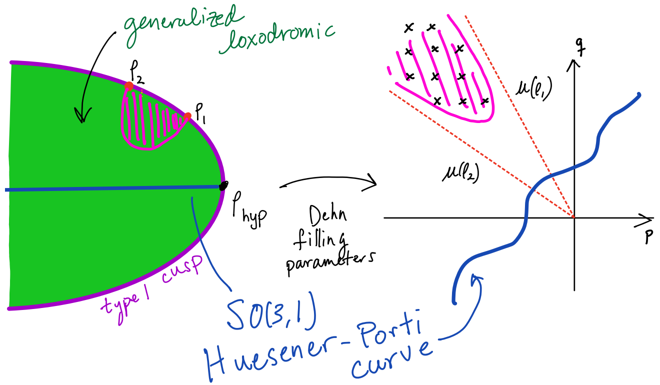
If $\rho_t \to \rho_0$ with H_{ρ_t} converging from loxodromic ($t > 0$) to type 1 cusp ($t = 0$), then

$$\frac{p_t}{q_t} \to \mu(\rho_0)$$

where $(p_t, q_t) =$ generalized Dehn surgery coordinates of ρ_t.

Assumption 2

(One of) the path(s) of deformations of *M* through convex $\mathbb{RP}^3$ structures with type 1 cusps has *non-constant* unipotent slope.



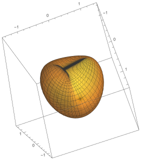
Main theorem

Then there exists $I \subset \mathbb{P}^1\mathbb{Q}$ so that for all but finitely many filling slopes $p/q \in I$, $M_{p/q}$ admits an exotic convex $\mathbb{RP}^3$ structure.

Convexity

Suppose M is an $\mathbb{RP}^3$ manifold. Is it convex?

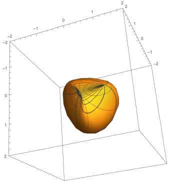
- Canonical line bundle $E \to M$.
- M convex $\Leftrightarrow$ there is a proper, strictly convex section $s : M \to E - \mathbf{0}$.



type 1 cusp

Suppose M is convex. Deform the $\mathbb{RP}^3$ structure.

- s remains strictly convex, on a compact part of M. (Koszul)
- If M not compact, must control s in the ends.
  - ends deforming as generalized cusps (Cooper-Long-Tillmann)
  - for a (p, q)-Dehn filling deformation, make a convex model of the appropriate solid torus, with section s, and glue back to convex part.



loxodromic, near type 1 cusp

Thank you!